Constant scalar curvature Kähler metrics on fibred complex surfaces

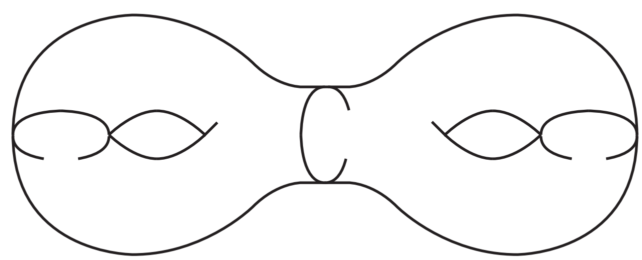
Joel Fine
Imperial College

Why look for cscK metrics?

CscK metrics and complex curves

On high genus curves, one-to-one correspondence between hyperbolic metrics and complex structures.

Geometric realisation of moduli:



$6g-6$ real parameters are lengths of $3g-3$ geodesics and size of $3g-3$ twists.

Are cscK metrics of use in the study of higher dimensional varieties? If so, important to know *which varieties admit cscK metrics.*

Kähler-Einstein metrics

The *Ricci form* ρ of a Kähler metric is determined by curvature of the canonical bundle: $\rho = -iF_K$.

A Kähler-Einstein metric is proportional to its Ricci form: $\lambda\omega = \rho$, for some constant λ.

Since $[\rho] = 2\pi c_1$, necessary condition is that c_1 is zero or definite.

Taking traces: KE $\Rightarrow$ cscK.

CscK *and* $\lambda[\omega] = 2\pi c_1 \Rightarrow$ KE. This follows from Kähler identities and Hodge theory:

- $[\Lambda, \partial] = i\bar{\partial}^*$, so cscK $\Rightarrow$ ρ harmonic;
- $\lambda[\omega] = [\rho] \Rightarrow \lambda\omega = \rho$.

CscK is generalisation of KE.

Lots of work has been done on finding and using KE metrics.

- $c_1 < 0$ (Aubin) and $c_1 = 0$ (Yau) $\Rightarrow$ KE.

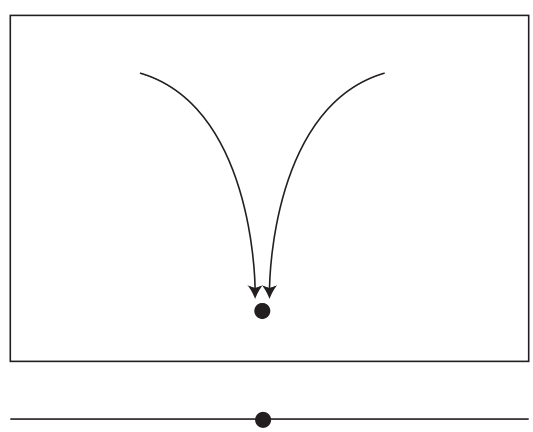
- $c_1 > 0$ much less is known. Certainly some are not KE. For a Fano surface X, X is KE $\Leftrightarrow$ Aut(X) is reductive (Tian). In higher dims this is necessary, but not sufficient condition for KE.

- Existence of KE metrics give results in algebraic geometry *e.g.* Chern number inequalities, rigidity of $\mathbb{P}^2$; use of CYs in Mirror symmetry.

Stably polarised varieties

Want to form a moduli space of polarised varieties *e.g.* consider all subvarieties of $\mathbb{P}^n$ modulo action of $\mathrm{GL}(n+1, \mathbb{C})$.

Want more than taxonomy; want geometric structure on moduli space.



Need to exclude *unstable* orbits. Moduli of stably polarised varieties has good geometry (GIT).

Leaves vital question: *when is a given variety stable?*

Conjecture. *For (X, L) a smooth polarised variety, (X, L) is K-stable if and only if $-2\pi c_1(L)$ contains cscK metric.*

Due to Yau in KE case, Tian in cscK case, Donaldson in symplectic case.

Remarkable: relates two deep problems, one entirely algebraic, the other analytic.

Fits into well studied framework: symplectic reduction vs GIT (Donaldson). *Cf.* Hitchin–Kobayashi correspondence.

Addresses question of which projective varieties *and which ample classes* admit cscK metrics.

Known that cscK $\Rightarrow$ K-stable (Chen–Tian). Converse not yet proved.

K-stability not easy to check for. New concept of slope-stability of (X, L) has been suggested which is easier to use. (Again *cf.* holomorphic bundles.)

K-stable $\Rightarrow$ SS (Ross-Thomas). Not known whether SS $\Rightarrow$ K-stable.

There exist X with some (X, L) cscK, other (X, L) not slope stable, hence not cscK (*e.g.* $\mathbb{P}^2$ blown up four times.)

Summary

If DTY is correct, should be able to study stably polarised varieties via cscK metrics. This should be the correct generalisation of situation for curves mentioned above.

One drawback is that hardly any examples of cscK metrics are known outside of KE case.

The constant scalar curvature equation

Parametrise Kähler forms in given class by open set of functions (Kähler potentials):

$$\omega_\phi = \omega + i\bar{\partial}\partial\phi.$$

Attempt to solve $\mathrm{Scal}(\omega_\phi) = \mathrm{const}$.

This is a fourth order fully non-linear PDE for ϕ. Very hard to solve in general.

Geometric input is essential to find a solution (knew this already from DTY).

If L denotes the linearisation of

$$\phi \mapsto \mathsf{Scal}(\omega_\phi),$$

then

$$L(\phi) = \mathscr{D}^* \mathscr{D}(\phi) + \nabla \,\mathsf{Scal} \cdot \nabla \phi$$

where $\mathscr{D}$ is defined by

$$\mathscr{D} = \bar{\partial} \circ \nabla \colon C^\infty \to \Omega^{0,1}(T)$$

and $\mathscr{D}^*$ is L^2 adjoint of $\mathscr{D}$.

Leading order term of L is Δ^2. So L is elliptic. Can use standard theory of elliptic partial differential equations.

CscK metrics on fibred complex surfaces

`math.DG/0401275`

X compact connected complex surface.

$$\pi\colon X \to \Sigma$$

a holomorphic submersion to a curve. Fibres have high genus ($\geq$2).

If $g(\Sigma) = 0, 1$ easy to show X is quotient of product $S \times \widetilde{\Sigma}$ by group of isometries, hence cscK. So assume high genus base.

$V \to X$ vertical tangent bundle. For any r let

$$\kappa_r = -2\pi\left(c_1(V) + rc_1(\Sigma)\right)$$

For large r these classes are ample.

Theorem. *For all large r, κ_r contains a constant scalar curvature Kähler metric.*

Outline of proof:

1. Find explicit metrics $\omega_r \in \kappa_r$ with
$$\mathrm{Scal}(\omega_r) \to \mathrm{const.}$$
as $r \to \infty$.

2. Show that, for all large r, ω_r can be perturbed to a genuinely cscK metric. Do this by applying inverse function theorem to the map
$$\phi \mapsto \mathrm{Scal}(\omega_r + i\bar{\partial}\partial\phi).$$
Parameter dependence makes things awkward.

Theorem (Quantitative IFT).

- *Let $F\colon B_1 \to B_2$ be a C^1 map of Banach spaces. Suppose that DF is an isomorphism with bounded inverse P.*

- *Let δ' be radius of ball about $0 \in B_1$ on which $F - DF$ is Lipschitz with constant $(2\|P\|)^{-1}$.*

- *Let $\delta = \delta'(2\|P\|)^{-1}$.*

Then for any $y \in B_2$ with

$$\|y - F(0)\| < \delta$$

there exists $x \in B_1$ with $F(x) = y$. x is unique solution with $\|x\| < \delta'$.

Assume that ω_r has been found with $\mathsf{Scal}(\omega_r) \to$ const. Apply inverse function theorem to the map

$$S_r \colon \phi \mapsto \mathsf{Scal}(\omega_r + i\bar{\partial}\partial\phi)$$

The map depends on r, so δ depends on r

Need $\|S_r(0) - \text{const.}\| \leq \delta_r$ to apply inverse function theorem and prove the theorem.

Assuming linearisation is an isomorphism, there are two key steps to perturbing to cscK metric:

- Need $\mathsf{Scal}(\omega_r) \to$ const. very quickly.

- Need to control $\|P_r\|$.

Derivative is an isomorphism

Recall that

$$L\phi = \mathscr{D}^*\mathscr{D}(\phi) + \nabla \,\mathsf{Scal} \cdot \nabla\phi$$

Assume $\mathsf{Scal}(\omega_r)$ is nearly constant;

$$L(\omega_r) \approx \mathscr{D}^*\mathscr{D}.$$

$\ker \mathscr{D}^*\mathscr{D} = \ker \mathscr{D}$ is functions with holomorphic gradient.

No holomorphic vector fields on X; $\ker \mathscr{D} = \mathbb{R}$.

$\mathscr{D}^*\mathscr{D}$ has index zero, hence is an isomorphism between spaces of functions with mean value zero.

This implies same is true for L. So can apply the inverse function theorem.

First order approximate solution

Hyperbolic metrics in fibres give metric in V. Let $\omega_0 = -iF_V$. Fibrewise restriction of ω_0 is hyperbolic metric on that fibre.

For large enough r,

$$\omega_r = \omega_0 + r\omega_\Sigma$$

is Kähler; $\omega_r \in \kappa_r$.

$r \to \infty$ is "stretching the base." An example of an *adiabatic limit*.

$\mathrm{Scal}(\omega_r) = -1 + O(r^{-1})$.

Adiabatic principle: *in an adiabatic limit, the local geometry is dominated by that of the fibre.*

Justified by the fact that locally (over the base) the metric converges to the product $S \times \mathbb{C}$.

Second order approximate solution

Aim to find ϕ with

$$\mathrm{Scal}(\omega_r + r^{-1} i\bar{\partial}\partial\phi) = \text{const.} + O(r^{-2})$$

To $O(r^{-2})$,

$$\mathrm{Scal}(\omega_r + r^{-1} i\bar{\partial}\partial\phi) = \mathrm{Scal}(\omega_r) + r^{-1} L(\omega_r)\phi.$$

$L(\omega_r) = L_F + O(r^{-1})$

L_F surjective onto functions with mean value zero over each fibre. Have decomposition

$$C^\infty(X) = C^\infty_0(X) \oplus C^\infty(\Sigma).$$

Can only remove C^∞_0 component of r^{-1} error by adding $r^{-1}\phi$.

Setting $C^\infty(\Sigma)$ component equal to constant gives non-linear PDE for ω_Σ similar to cscK equation for base. Solution exists.

Now solve linear PDE, $L_F\phi = \Theta$ to remove C^∞_0 component.

Higher order approximate solutions

Aim to find $\phi_1, \ldots, \phi_n$ such that

$$\text{Scal}\left(\omega_r + i\bar{\partial}\partial \sum r^{-j}\phi_j\right)$$

is $O(r^{-n-1})$ from constant.

Find the ϕ_j recursively. Each is found by solving two linear PDEs, namely the linearisations of the two non-linear PDEs mentioned above (cscK metrics on the fibre, similar PDE on the base).

Fredholm alternative for index zero operators: the solutions to the non-linear problems are "rigid," so their linearisations are surjective.

Upshot is that, for each k, there exists an approximate solution with scalar curvature converging to constant quicker than r^{-k}.

Local analysis

Need to prove various local analytic estimates (Sobolev inequalities, elliptic estimates *etc.*) hold uniformly in r.

As $r \to \infty$, Kähler structure near a fibre converges to a product $S \times \mathbb{C}$.

Any estimate which holds over $S \times \mathbb{C}$ holds uniformly over (X, ω_r).

Analogous to study of tubes $Y \times \mathbb{R}$ which occurs in Floer theory.

Key point is that the geometry of $S \times \mathbb{C}$ is uniformly bounded.

Global analysis

Upper bound on $\|P_r\|$ is global result, so local model not sufficient.

For global model, take Riemannian submersion with hyperbolic metrics on fibres, $r\omega_\Sigma$ transverse to fibres. Note that this is *not* Kähler. Much easier to calculate the r dependence, however.

Upper bound on $\|P_r\|$ is essentially the same as lower bound for first non-zero eigenvalue $\lambda_1(r)$ of $\mathscr{D}^*\mathscr{D}$. *Cf.* finite dimensional linear algebra.

Need $\lambda_1(r) \geq Cr^{-k}$ for some k

Use fact that if for all ϕ with mean value zero,

$$\|\mathscr{D}\phi\|^2 \geq c\|\phi\|^2$$

then $\lambda_1 \geq c$. (Norms are L^2; everything is with respect to metric ω_r.) Again, *cf.* finite dimensions.

Work with global model. Have explicit formulae for r dependence of every term appearing in above inequality. Gives bound

$$\lambda_1(r) \geq Cr^{-3}.$$

I suspect best bound is $\lambda_1(r) \geq Cr^{-2}$ (certainly is in the case of a product). Not relevant to the proof of the result, however.