My way to Algebraic Geometry

Varieties and Schemes from the point of view
of a PhD student

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Introduction

I began writing these notes during the last weeks of year 2002, collecting some work I had already written in my studentship. This is intended to be my personal approach to Algebraic Geometry, in particular to Hartshorne (1977), and originally it was meant to be a collection of solved exercises. But as my understanding grew stronger I began to add comments of any sort, so as to reach the level of exposition you will find. I am very pleased to hear that there is somebody out there who thinks those comments are interesting, and I really wish them to be of any help. To those students approaching the subject, wishing to learn something about the abstract concepts that arise in Algebraic Geometry and finding themselves lost into a sea of difficult to grasp material, I have to say that unfortunately there is no easy way, they will have to work a lot and pretty much alone.

A word of caution: don't take these notes for granted! I am a lone student, I'm writing as I'm learning the subject and I'm not a good nor an experienced mathematician. Chances are that some of my proofs are wrong, and be advised that also some of the statements are mine and may be wrong. For this reason when I state a result I always try to give a reference for it, in particular most of the times I will deal with statements that are actually exercises in some book and the reader may want to check their proofs carefully if not to try to work them out on his own.

We wish to learn the formalism of schemes, therefore throughout these notes our main reference will be Hartshorne (1977) which is the most celebrated book where to learn the machinery. The reader will be assumed familiar with the notations and definitions in there, however there are quite a few other books one can look at, such as Eisenbud and Harris (2000), Shafarevich (1994b) or Mumford (1999). The original work by Grothendieck is also still a good reference, although very abstract, you can obtain all the *Éleménts de géométrie algébrique* from <http://www.numdam.org/> (rigorously in French), and many other articles, included the *Séminaire de géométrie algébrique* from <http://www.grothendieck-circle.org/>. Unfortunately the concepts we are going to introduce cannot make any sense if you haven't at least a basic knowledge of Commutative Algebra. I used to study this subject in Atiyah and Macdonald (1969) and this is the main reference for these notes, but you may want to read Eisenbud (1995) or the classic textbooks by Matsumura (1989), Zariski and Samuel (1958, 1960), or Bourbaki (1998). The word "ring" will always mean "commutative ring with unit" and a morphism of rings will be always assumed to respect this structure, that is will send the unit element to the unit element.

As Eisenbud and Harris say in the introduction to their book, "the basic definitions of scheme theory appear as natural and necessary ways of dealing with a range of ordinary geometric phenomena, and the constructions in the theory take on an intuitive geometric content which makes them much easier to learn and work with." For this reason, in line with the authors, I think there is no point in learning all the machinery first, and then work out the geometry as a consequence of it. But I still think that any student has to think a little through the basic definitions before going straight to geometry, and in this respect I disagree with Eisenbud and Harris. Chapter one is my attempt to compromise, with all the basic abstract concepts introduced in a systematic way before to arrive at the very definition of algebraic variety. The reader familiar with the second chapter of Hartshorne (1977) will surely recognise where the discussion is going, and will find some useful comments helping him to familiarise with abstract definitions otherwise totally detached by the geometric content they are supposed to carry over. In fact, this approach is essentially different from any other I am aware of. Classic references as Mumford (1999), or Shafarevich (1994a,b), develop the theory quite extensively before getting into schemes, while more recent accounts such as Liu (2002) or Ueno (1999, 2001, 2003) concentrate more on the abstract machinery leaving the geometry slightly aside. Finally in Iitaka (1982) commutative algebra is developed together with algebraic geometry, which makes it the most self-contained book.

Once again let me say that these notes are not to be trusted that much, but if you are going to read them for whatever reason then let me ask you something in return. If you find any mistake, misprint or anything wrong just let me know, you can contact me by e-mail on giudice@mat.unimi.it and you can find out something more about me on my homepage at the University of Bath <http://www.maths.bath.ac.uk/~mapmlg/>, any comments or suggestions are also more than welcome.

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Chapter 1

Affine Varieties

Our goal in this chapter is to define affine varieties from an abstract point of view. The main effort required to the reader is getting accustomed to switching frequently between algebra and geometry, for what we will do is giving geometric names to purely algebraic objects. We will follow the guidelines of §I.1 in Eisenbud and Harris (2000), but we will try to keep as close as possible to standard introductions such as Chapter II in Reid (1988). In fact the author has been highly inspired by Milne (2005), where a wonderfully clear account of the basic material is accompanied with a precise description of some of the abstract concepts.

It must be said that originally this was simply a collection of solved exercises from Atiyah and Macdonald (1969), therefore some background knowledge of commutative algebra is required throughout (nothing more than a standard undergraduate course, see for example Reid, 1995). The reader will also be assumed familiar with sheaf theory, at least with what Hartshorne (1977) reads about it, but occasionally we may refer to Tennison (1975) for more advanced material.

1.1 The Spectrum of a Ring

1.1.1 Zeros of an Ideal The main object of study in Algebraic Geometry are systems of algebraic equations and their sets of solutions. Let k be a field and let R be the polynomial ring $k[x_1, \ldots, x_n]$, a *system of algebraic equations* $\mathfrak{S}$ is a collection of equations

$$\mathfrak{S} = \begin{cases} F_1(x_1, \ldots, x_n) = 0 \\ \ldots \\ F_r(x_1, \ldots, x_n) = 0 \end{cases}$$

where F_i is a polynomial in R for every i, and a *solution* of $\mathfrak{S}$ in k is an n-tuple $(\lambda_1, \ldots, \lambda_n) \in k^n$ satisfying every equation. If we denote by $\mathfrak{a}$ the ideal generated by $F_1, \ldots, F_r$, then solutions of $\mathfrak{S}$ are in one-to-one correspondence with n-tuples in k^n annihilating any polynomial in $\mathfrak{a}$, in particular any other set of generators for $\mathfrak{a}$ will define a system of algebraic equations with exactly the same set of solutions as $\mathfrak{S}$. We encode this information in our language by saying that $(\lambda_1, \ldots, \lambda_n)$ is a *zero* of $\mathfrak{a}$.

Proposition. *Let k be any field. There is a natural injective correspondence from n-tuples in k^n to maximal ideals of R, given by*

$$\lambda = (\lambda_1, \ldots, \lambda_n) \quad \mapsto \quad \mathfrak{m}_\lambda = (x_1 - \lambda_1, \ldots, x_n - \lambda_n)$$

If $\mathfrak{a} \subseteq R$ is any ideal, then λ is a zero of $\mathfrak{a}$ if and only if the maximal ideal $\mathfrak{m}_\lambda$ contains $\mathfrak{a}$.

Proof. The ideal $\mathfrak{m}_\lambda$ is maximal because it is the kernel of the surjective homomorphism $\varphi\colon k[x_1, \ldots, x_n] \to k$ defined by evaluation in λ. Indeed it is enough to show that the kernel is contained in $\mathfrak{m}_\lambda$: we can apply the division algorithm to any polynomial $F \in \ker \varphi$, to obtain $F = G + \alpha$ where $G \in \mathfrak{m}_\lambda$ and α is a constant. The reader can learn more about the division algorithm in Cox, Little, and O'Shea (1997, §2.3).

For any subset Σ of k^n define $\imath(\Sigma)$ as the ideal consisting of those polynomials vanishing at every n-tuple in Σ, while for any ideal $\mathfrak{a}$ of R define $v(\mathfrak{a})$ to be the subset of k^n consisting of the zeros of $\mathfrak{a}$. In this way for any n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ in k^n we have $\mathfrak{m}_\lambda = \imath(\lambda)$. Now we conclude that $\lambda \mapsto \mathfrak{m}_\lambda$ is injective since $v\big(\imath(\lambda)\big)$ consists of the n-tuple λ only. □

An ideal of R does not always have zeros, the obvious example is the ideal $(x^2 + 1)$ in $\mathbb{R}[\,x\,]$, but it does when the field k is algebraically closed. This is the famous *Hilbert's Nullstellensatz*, which can be found in the literature in many different guises, from the classic statement in Reid (1988, §3.10) to the very abstract one in Eisenbud (1995, Theorem 4.19).

Hilbert's Nullstellensatz. *Let k be any field. The following statements are equivalent:*

i) *k is algebraically closed;*

ii) *For any $n \in \mathbb{N}_+$, every proper ideal $\mathfrak{a} \subset k[x_1, \ldots, x_n]$ has a zero in k^n;*

iii) *For any $n \in \mathbb{N}_+$, the association $\lambda \mapsto \mathfrak{m}_\lambda$ is a bijection between n-tuples in k^n and maximal ideals of $k[x_1, \ldots, x_n]$.*

Proof. The only difficult part is how to obtain *ii*) from *i*), the rest is almost trivial. Indeed assuming *ii*) any maximal ideal $\mathfrak{m} \subseteq k[x_1, \ldots, x_n]$ will have a zero λ and therefore $\mathfrak{m}$ will be contained in $\mathfrak{m}_\lambda$. From *iii*) to show *i*) we have to prove that any polynomial $F \in k[x]$ has a root in k, and this follows immediately by considering any maximal ideal that contains (F).

We will not give the complete proof in that we are going to use the following purely algebraic result, known as *Weak Nullstellensatz* in the formulation of Atiyah and Macdonald (1969, Corollary 7.10) or as *Zariski Lemma* in the formulation of Milne (2005, Lemma 2.7)

> Let k be algebraically closed and $\mathfrak{m}$ be a maximal ideal in R, then the quotient $k[x_1, \ldots, x_n]/\mathfrak{m}$ is isomorphic to k.

With this result understood we can easily derive *ii*) from *i*). First, since every proper ideal is contained in a maximal ideal, it is enough to show that every maximal ideal $\mathfrak{m}$ has a zero in k^n, then the canonical projection defines a surjective homomorphism $\varphi\colon k[x_1, \ldots, x_n] \to k$ whose kernel is given both by $\mathfrak{m}$ and $\big(x_1 - \varphi(x_1), \ldots, x_n - \varphi(x_n)\big)$. □

1.1.2 Affine Space In our way to abstraction we have so far replaced the system of algebraic equations $\mathfrak{S}$ with the ideal $\mathfrak{a}$ and we have identified the set of solutions of $\mathfrak{S}$ with a subset of the maximal ideals of R containing $\mathfrak{a}$. In order to go a step further recall that there is a one-to-one correspondence between (maximal) ideals of $R/\mathfrak{a}$ and (maximal) ideals of R containing $\mathfrak{a}$, so that in fact we have identified the set of solutions of $\mathfrak{S}$ with a subset of the maximal ideals of $R/\mathfrak{a}$. Thus we say goodbye to n-tuples and we work directly inside the quotient ring, of which we now consider a more general set of ideals than the maximal ones only.

Definition. Let A be any ring, the *prime spectrum* of A is the set of all its prime ideals and is denoted by $\operatorname{Spec} A$; thus a *point* of $\operatorname{Spec} A$ is a prime ideal $\mathfrak{p} \subseteq A$. We will adopt the usual convention that A itself is not a prime ideal, so that $\operatorname{Spec}\{0\} = \varnothing$. Of course, the zero ideal (0) is an element of $\operatorname{Spec} A$ if and only if A is a domain.

We define *affine n-space* to be the spectrum of the ring of polynomials R, that is

$$\mathbb{A}^n_k := \operatorname{Spec} k[x_1, \ldots, x_n]$$

In view of the Proposition above $\mathbb{A}^n_k$ contains k^n in a natural way, and by the Nullstellensatz if the field k is algebraically closed this inclusion is precisely given by all the maximal ideals in R. We will refer to "the point $(\lambda_1, \ldots, \lambda_n)$" to mean the point $\mathfrak{m}_\lambda$, but of course not every point in affine space corresponds to some n-tuple, for instance the zero ideal. This oddity is one of the reasons why schemes are not so popular among mathematicians, and can be quite frustrating for the beginner.

1.1.3 Zariski Topology Let A be a ring and let X be its spectrum $\operatorname{Spec} A$. For each subset E of A, let $\mathcal{V}(E)$ denote the set of all prime ideals of A which contain E, in other words define

$$\mathcal{V}(E) = \{\mathfrak{p} \in X \mid E \subseteq \mathfrak{p}\}$$

Note that this association is inclusion-reversing, that is whenever $E' \subseteq E$ we have the opposite inclusion $\mathcal{V}(E') \supseteq \mathcal{V}(E)$. The following result shows that the sets $\mathcal{V}(E)$ satisfy the axioms for the closed sets in a topological space. The resulting topology is called the *Zariski topology*.

Proposition (Exercise I.15 in Atiyah and Macdonald, 1969)**.**

i) *if* $\mathfrak{a}$ *is the ideal generated by E, then* $\mathcal{V}(E) = \mathcal{V}(\mathfrak{a}) = \mathcal{V}(\sqrt{\mathfrak{a}})$

ii) $\mathcal{V}(0) = X, \mathcal{V}(1) = \varnothing$

iii) *if* $(E_i)_{i\in I}$ *is any family of subsets of A, then* $\mathcal{V}\left(\bigcup E_i\right) = \bigcap \mathcal{V}\left(E_i\right)$

iv) $\mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{V}(\mathfrak{a}\mathfrak{b}) = \mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b})$ *for any ideals* $\mathfrak{a}$, $\mathfrak{b}$ *of A.*

Proof. To prove these statements, at least the first three, it is in fact enough to make some remarks: a prime ideal containing the set E also contains the ideals $\mathfrak{a}$ and $\sqrt{\mathfrak{a}}$. Every prime ideal contains the zero element but doesn't contain the unit element (that's because we are assuming that prime ideals are proper). An ideal contains a union of sets if and only if it contains every set of the family. We then prove part *iv*) only.

Since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$ we have $\mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a}\mathfrak{b})$. Conversely $x^2 \in \mathfrak{a}\mathfrak{b}$ for every $x \in \mathfrak{a} \cap \mathfrak{b}$ so if a prime ideal contains $\mathfrak{a}\mathfrak{b}$ it contains the intersection $\mathfrak{a} \cap \mathfrak{b}$ also. Now we have the obvious inclusion $\mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a} \cap \mathfrak{b})$, while the other follows from Proposition 1.11 in Atiyah and Macdonald (1969): if a prime ideal contains a finite intersection of ideals then it contains one of the ideals. □

Although we have defined a closed set $\mathcal{V}(E)$ for any subset E of the ring A, it is clear by the Proposition that it is enough to consider ideals $\mathfrak{a} \subseteq A$. For this reason usually the properties above are stated as follows:

iii) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, then $\mathcal{V}(\mathfrak{a}\mathfrak{b}) = \mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b})$.

iv) If $\{\mathfrak{a}_i\}$ is any set of ideals of A, then $\mathcal{V}\left(\sum \mathfrak{a}_i\right) = \bigcap \mathcal{V}(\mathfrak{a}_i)$.

It is useful at this stage to think a little through this correspondence between ideals of A and closed sets of X. The following Lemma may help our understanding, showing that in fact we have a one-to-one correspondence between radical ideals and closed sets of X.

Lemma (II.1.6 in Liu, 2002)**.** *Let A be a ring and let $\mathfrak{a}, \mathfrak{b}$ be two ideals of A. Then:*

(*a*) *The radical $\sqrt{\mathfrak{a}}$ equals the intersection of the ideals $\mathfrak{p} \in \mathcal{V}(\mathfrak{a})$.*

(*b*) *$\mathcal{V}(\mathfrak{a}) \subseteq \mathcal{V}(\mathfrak{b})$ if and only if $\mathfrak{b} \subseteq \sqrt{\mathfrak{a}}$.*

In particular $\mathcal{V}(\mathfrak{b}) = \operatorname{Spec} A$ if and only if $\mathfrak{b} \subseteq \operatorname{Nil}(A)$, where $\operatorname{Nil}(A)$ denotes the nilradical of the ring A, the set of all nilpotent elements.

Examples. The correspondence v of §1.1.1 is given by $\mathcal{V}$ on affine space $\mathbb{A}^n_k$, more precisely for every ideal $\mathfrak{a} \subseteq R$ the set $v(\mathfrak{a})$ of the zeros of $\mathfrak{a}$ is given by the maximal ideals of the form $\mathfrak{m}_\lambda$ contained in $\mathcal{V}(\mathfrak{a})$.

If k is a field, then $\operatorname{Spec} k$ is the point-set topological space, its unique element is given by the zero ideal in k.

Let A be the local ring $k[\,t\,]_{(t)}$, obtained from $k[\,t\,]$ by localising on the maximal ideal $(\,t\,)$. The set $\operatorname{Spec} A$ consists of two points only, the ideals 0 and $(\,t\,)$, and the topology is given by $\varnothing \subseteq \{\,0\,\} \subseteq \operatorname{Spec} A$.

1.1.4 A Base for the Zariski Topology

"An open set in the Zariski topology is simply the complement of one of the sets $\mathcal{V}(E)$. The open sets corresponding to sets E with just one element will play a special role, essentially because they are again spectra of rings (see §1.4.5); for this reason they get a special name and notation. If $\alpha \in A$, we define the *distinguished* (or *basic*) open subset $D(\alpha)$ of X associated with α to be the complement of $\mathcal{V}(\alpha)$."

taken from Eisenbud and Harris (2000, §I.1.2)

The distinguished open sets form a *base* for the Zariski topology in the sense that any open set is a union of distinguished ones:

$$U = X \setminus \mathcal{V}(E) = X \setminus \left(\bigcap_{\alpha \in E} \mathcal{V}(\alpha) \right) = \bigcup_{\alpha \in E} D(\alpha)$$

Proposition (Exercise I.17 in Atiyah and Macdonald, 1969)**.** *For each $\alpha \in A$, let $D(\alpha)$ denote the distinguished open subset of $X = \operatorname{Spec} A$ associated with α. Then for any $\alpha, \beta \in A$ we have the following:*

i) $D(\alpha) \cap D(\beta) = D(\alpha\beta)$

ii) $D(\alpha) = \varnothing \Leftrightarrow \alpha$ *is nilpotent* *iii)* $D(\alpha) = X \Leftrightarrow \alpha$ *is a unit*

iv) $D(\alpha) \subseteq D(\beta) \Leftrightarrow \alpha \in \sqrt{(\beta)}$

v) *X is quasi-compact*

vi) *More generally, each $D(\alpha)$ is quasi-compact*

vii) *An open subset of X is quasi-compact if and only if it is a finite union of sets $D(\alpha)$*

Proof. Again it is enough in most cases to make some remarks: a prime ideal does not contain two elements α and β if and only if it does not contain their product $\alpha\beta$. An element α is contained in every prime ideal if and only if it is contained in the nilradical of A, the set of all nilpotent elements (Atiyah and Macdonald, 1969, Proposition 1.8). An element α is not contained in any prime ideal if and only if it is a unit (that's because otherwise the ideal (α) was contained in a maximal ideal). Statement *iv)* is a reformulation of the previous Lemma, while *vii)* follows immediately from *vi)*.

To prove *v)* it is enough to consider a covering of X by basic open sets. Indeed if $\{U_j\}$ is an open covering, we can write each U_j as a union of basic sets, then find a finite subcover consisting of basic sets. Each one of these was contained in some U_j, so we find a finite subcover of the given covering. Now note the following

$$\operatorname{Spec} A = \bigcup_{i \in I} D(\alpha_i) \iff \big(\{\alpha_i \mid i \in I\}\big) = A$$

where $\big(\{\alpha_i \mid i \in I\}\big)$ is the ideal generated by the elements α_i. Indeed we have

$$\bigcup_{i \in I} D(\alpha_i) = X \setminus \mathcal{V}\left(\{\alpha_i \mid i \in I\}\right)$$

So that the sets $D(\alpha_i)$ cover X if and only if the unit element of A can be written as a finite combination of the elements α_i. But then only this finite number is enough to cover X.

The same argument applies to each set $D(\alpha)$, for if $D(\alpha) = \bigcup D(\beta_i)$ we have $\mathcal{V}(\alpha) = \mathcal{V}(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal generated by the set of elements $\{\beta_i\}$, thus we can say $\alpha \in \sqrt{\mathfrak{a}}$ and find an expression of a power of α in terms of a finite combination of the β_i. To conclude note that $\mathcal{V}(\alpha) = \mathcal{V}(\alpha^n)$ for all n. □

1.1.5 Closed Points A point x of a topological space is said to be *closed* if the set $\{x\}$ is closed. In the next result we show that the closed points of Spec A are given by the maximal ideals in the ring A and we call this set the *maximal spectrum* of A, denoted max-Spec A.

Proposition (Exercise I.18 in Atiyah and Macdonald, 1969)**.**

i) *The set* $\{\mathfrak{p}\}$ *is closed in* Spec $A \Leftrightarrow \mathfrak{p}$ *is a maximal ideal*

ii) *The closure* $\overline{\{\mathfrak{p}\}}$ *is given by* $\mathcal{V}(\mathfrak{p})$

iii) $\mathfrak{q} \in \overline{\{\mathfrak{p}\}} \Leftrightarrow \mathfrak{p} \subseteq \mathfrak{q}$

iv) Spec A *is a* T_0*-space* (*this means that if* x, y *are distinct points of* X*, then either there is a neighborhood of* x *which does not contain* y*, or else there is a neighborhood of* y *which does not contain* x)

Proof. First observe that $\mathcal{V}(E)$ is a closed set that contains the point $\mathfrak{p}$ if and only if $E \subseteq \mathfrak{p}$. Hence we have

$$\overline{\{\mathfrak{p}\}} = \bigcap_{E \subseteq \mathfrak{p}} \mathcal{V}(E) = \mathcal{V}\left(\bigcup_{E \subseteq \mathfrak{p}} E \right) = \mathcal{V}(\mathfrak{p})$$

This proves the first three statements.

Let $\mathfrak{p}$ and $\mathfrak{q}$ be different points of X, we then have $\mathfrak{p} \neq \mathfrak{q}$ as ideals, but we still can have one contained in the other. Without loss of generality we can assume $\mathfrak{q} \not\subseteq \mathfrak{p}$ and find an element $\alpha \in \mathfrak{q}$ such that $\alpha \notin \mathfrak{p}$. Then $D(\alpha)$ is the open neighborhood of $\mathfrak{p}$ we were looking for. Note however that if $\mathfrak{p} \subseteq \mathfrak{q}$ every closed set that contains $\mathfrak{p}$ is given by $\mathcal{V}(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq \mathfrak{p}$ and therefore it also contains $\mathfrak{q}$, that is every neighborhood of $\mathfrak{q}$ does contain $\mathfrak{p}$ also. □

Examples. The space Spec $\mathbb{Z}$ contains a set of closed points in one-to-one correspondence with prime numbers and a non-closed point η given by the zero-ideal in $\mathbb{Z}$. Observe that *this point is dense in* Spec $\mathbb{Z}$ that is the closure of the set $\{\eta\}$ is the whole space.

Affine space $\mathbb{A}^n_k$ contains analogously a dense one-point set, its unique element is given by the zero ideal of R.

1.1.6 Algebraic Sets Let $\mathfrak{S}$ be a system of algebraic equations and let $\mathfrak{a}$ be the ideal in R given by the polynomials defining the equations in $\mathfrak{S}$. The set of solutions of $\mathfrak{S}$ is, strictly speaking, given by the zeros of $\mathfrak{a}$, which are particular closed points inside the closed subset $\mathcal{V}(\mathfrak{a})$ of $\mathbb{A}^n_k$, or all the closed points when k is algebraically closed.

Definition. An *algebraic set* is a closed subset of $\mathbb{A}^n_k$, in other words it is given by $\mathcal{V}(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$. It consists of all the prime ideals in R that contain $\mathfrak{a}$.

Following Eisenbud (1995) we will call *affine k-algebra* any finitely generated algebra over the field k (not necessarily algebraically closed), when it will not be necessary to refer explicitly to k, we will say simply *affine ring*. We will soon see to what extent affine rings and affine varieties are in fact different names for the same object, here we are interested in pointing out that an algebraic set is naturally given by the spectrum of an affine ring, but first we need the following result.

Lemma. *Let A be a ring and let $X = \operatorname{Spec} A$. A closed subset $V = \mathcal{V}(\mathfrak{a})$ of X is naturally homeomorphic to* $\operatorname{Spec} A/\mathfrak{a}$.

Proof. It is well known that for any ring A there is a one-to-one correspondence between ideals of $A/\mathfrak{a}$ and ideals of A that contain $\mathfrak{a}$. Such a correspondence defines the homeomorphism of the statement as follows. Every closed subset of V is in fact a closed subset in X, therefore it is given by $\mathcal{V}(\mathfrak{b})$ for some ideal $\mathfrak{b}$ of A; thus we have $\mathcal{V}(\mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a})$ and we have seen in §1.1.3 that this is equivalent to say $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$. □

The spectrum of an affine ring $R/\mathfrak{a}$ is therefore naturally homeomorphic to the algebraic set $V = \mathcal{V}(\mathfrak{a})$ in $\mathbb{A}^n_k$, note however that the same algebraic set can be regarded as the spectrum of a ring in many different ways, namely as many as the ideals $\mathfrak{b}$ such that $\sqrt{\mathfrak{b}} = \sqrt{\mathfrak{a}}$. For this reason we sometimes denote $\mathcal{I}(V)$ the unique radical ideal corresponding to V.

1.2 Topological Properties

We are going to examine more closely the topology of an algebraic set, but first let us recall what we already know about it. An algebraic set is the spectrum of an affine ring A, and as such it is always quasi-compact and it always satisfies the T_0 separation axiom. Further we have a one-to-one correspondence between radical ideals in A and closed subsets of $\operatorname{Spec} A$, and we know in general how a base for the Zariski topology looks like. We proceed now analysing some specific topological properties related to specific properties of the ring, the beginner will face some difficulty in realising how far the Zariski topology is from the usual Euclidean topology.

1.2.1 Irreducible Spaces A non-empty topological space X is *irreducible* if it cannot be expressed as the union $X = X_1 \cup X_2$ of two proper subsets, each one of which is closed in X. The empty set is not considered to be irreducible.

Lemma. *For a non-empty topological space X, the following conditions are equivalent*

i) *X is irreducible;*

ii) *Every non-empty open set is dense in X;*

iii) *Every pair of non-empty open sets in X intersect;*

iv) *if $X = C_1 \cup C_2$ where C_i is a closed subset of X then $C_i = X$ for some $i = 1, 2$.*

Proof. Assume X is irreducible and let $U \subseteq X$ be a non-empty and non total open subset, write $X = \overline{U} \cup (X \setminus U)$; if we assume $\overline{U} \neq X$ we have a decomposition of X into two distinct and proper closed subsets. But this is not possible since X is irreducible, so that *ii*) holds. Let now $U, V \subseteq X$ be non-empty open sets; if the intersection $U \cap V$ was empty then U would be contained in the proper closed set $X \setminus V$ and hence $\overline{U}$ couldn't be the whole of X. But assuming *ii*) this is not possible, hence *ii*) implies *iii*). To show that *iii*) implies *iv*) let $X = C_1 \cup C_2$ where C_i is a closed subset, and let U_i be the complement of C_i. Note that $U_1 \cap U_2$ is the complement of $C_1 \cup C_2$ and hence is empty. By *iii*) this is only possible if one of the two open sets is empty, that is if one of the two closed sets is the whole space. Finally *iv*) implies *i*) by definition. □

Example (Points are the only irreducible subsets of a Hausdorff space). In a Hausdorff space X points are in particular closed and therefore irreducible, conversely let Y be any irreducible subset of X and assume it contains at least two points P_1 and P_2. Since X is a Hausdorff space there exist two open sets U_1 and U_2 with $P_i \in U_i$ and such that $U_1 \cap U_2 = \varnothing$. We have

$$(U_1 \cap Y) \cap (U_2 \cap Y) = \varnothing$$

note that $U_i \cap Y$ is not empty since it contains P_i, and is open in the induced topology on Y. Taking complements we obtain

$$\big(Y \setminus (U_1 \cap Y\big) \cup \big(Y \setminus (U_2 \cap Y)\big) = Y$$

which is a decomposition of Y into proper closed subsets, contradicting Y is irreducible.

Proposition (Exercise I.19 in Atiyah and Macdonald, 1969)**.** *Let A be a ring. Then* Spec *A is irreducible if and only if the nilradical of A is a prime ideal.*

Proof. In §1.1.4 we have seen that $D(\alpha) \cap D(\beta) = D(\alpha\beta)$ and also

$$D(\alpha) \cap D(\beta) = \varnothing \iff \alpha\beta \in \mathrm{Nil}(A)$$

where $\mathrm{Nil}(A)$ denotes the nilradical of the ring A. Hence if $\mathrm{Nil}(A)$ is prime every pair of non-empty distinguished open sets in $\operatorname{Spec} A$ intersect. Conversely let $\alpha\beta \in \mathrm{Nil}(A)$ then $D(\alpha) \cap D(\beta) = \varnothing$, but we are assuming $\operatorname{Spec} A$ is irreducible, then one of the two open sets is empty, and this is equivalent to say that one between α and β is an element of $\mathrm{Nil}(A)$. □

Corollary. *Let A be a ring and $\mathfrak{a}$ an ideal. Then the closed subset of* $\operatorname{Spec} A$ *defined by $\mathfrak{a}$ is irreducible if and only if the radical of $\mathfrak{a}$ is prime.*

Proof. Recall that the closed subset $\mathcal{V}(\mathfrak{a})$ is homeomorphic to $\operatorname{Spec} A/\mathfrak{a}$ (see §1.1.6). Now apply the Proposition, knowing that the nilradical of $A/\mathfrak{a}$ corresponds to $\sqrt{\mathfrak{a}}$. □

Remark (Points and Irreducible Subsets). Let A be a ring and $X = \operatorname{Spec} A$ its spectrum, an interesting way of rephrasing the above Corollary is the following. A closed subset V of X is irreducible if and only if it is the closure $\overline{\{\mathfrak{p}\}}$ of a uniquely determined point $\mathfrak{p}$.

Example. Let k be a field and R be the polynomial ring $k[x_1, \dots, x_n]$. Since R is a domain, affine space $\mathbb{A}^n_k$ is irreducible

Proposition (Exercise I.1.6 in Hartshorne, 1977)**.** *Any non-empty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.*

Proof. Let $U \subseteq X$ be a non-empty open subset. We have already seen U is dense, to show that U is irreducible suppose it isn't and write $U = C_1 \cup C_2$ where C_i is closed in U, non-empty and non total; then $A_i = U \setminus C_i$ is open both in U and X. We now obtain

$$(X \setminus A_1) \cup (X \setminus A_2) = X \setminus (A_1 \cap A_2) = X \setminus \big(U \setminus (C_1 \cup C_2)\big) = X \setminus \varnothing = X$$

where $X \setminus A_i$ is a non total and non-empty closed subset of X. This is a contradiction since X is irreducible.

Let now Y be irreducible in its induced topology and write $\overline{Y} = Y_1 \cup Y_2$ where Y_i is closed both in $\overline{Y}$ and X. We have

$$Y = \overline{Y} \cap Y = (Y_1 \cup Y_2) \cap Y = (Y_1 \cap Y) \cup (Y_2 \cap Y)$$

Since Y is irreducible there exists $i = 1, 2$ such that $Y_i \cap Y = Y$, so that $Y_i \supseteq \overline{Y}$. Hence $Y_i = \overline{Y}$ and $\overline{Y}$ is irreducible. □

1.2.2 Linear Subspaces – lines and planes Let k be a field and R be the polynomial ring $k[x_1, \ldots, x_n]$. The first geometric objects we want to define inside $\mathbb{A}_k^n$ are *lines*, but to do this we need to go back to basic geometry. In a classical set-up an affine space is a set of points directed by a vector space and a line is just an affine subspace of dimension one (see for example Audin, 2003), however a vector space, such as k^n has a natural structure of affine space for which it is very easy to define lines in terms of linear algebra. To this purpose I would like to take the chance of naming another really beautiful book, Artin (1991).

A *linear subspace* of $\mathbb{A}_k^n$ is the set of solutions of a linear system of equations, which we call $\mathfrak{S}$; in other words it is an algebraic set $\mathcal{V}(\mathfrak{a})$ where the ideal $\mathfrak{a}$ is generated by linear polynomials $L_1(x_1, \ldots, x_n), \ldots, L_r(x_1, \ldots, x_n)$. But this definition is useless if we are not able to accompany it with a notion of dimension, so firstly let us agree that the dimension of $\mathbb{A}_k^n$ is n.

Lemma. *The algebraic set $\mathcal{V}(\mathfrak{a})$ is not empty if and only if the linear system $\mathfrak{S}$ has a solution in k.*

Proof. Indeed the linear system doesn't have any solution if and only if the ideal $\mathfrak{a}$ contains an element of k. The proof is given by the following remark: if we form a matrix from the coefficients of the system of generators for $\mathfrak{a}$ we can assume it to be in row echelon form. To see this, we have to convince ourselves that reducing that matrix in row echelon form actually gives us another system of generators for $\mathfrak{a}$, and this follows from two key observations: the first is that such a transformation involve linear combinations of the rows, that is of the polynomials, and the second is that this process is invertible. □

Given this Lemma, assuming a linear subspace is non-empty we can perform Gauss-Jordan elimination to solve the system $\mathfrak{S}$. If ρ is the rank of $\mathfrak{S}$, we will be able in this way to write $x_{i_1}, \ldots, x_{i_\rho}$ as a linear expression of all the other indeterminates, actually defining a surjective morphism of k-algebras

$$k[x_1, \ldots, x_n] \longrightarrow k[y_1, \ldots, y_s]$$

where $s = n - \rho$. It is easy to see that $\mathfrak{a}$ is the kernel of this morphism, thus it is a prime ideal. Putting all things together we have seen that any nonempty linear subspace of $\mathbb{A}_k^n$ is an affine space of dimension s, in particular it is irreducible. Linear subspaces of dimension one are *lines* while linear subspaces of dimension two are *planes*.

1.2.3 Irreducible Components Any topological space is a union of irreducible ones. If X is Hausdorff this is equivalent to saying that X is the union of its points and is not particularly interesting, while from our point of view this statement is not only interesting but also not immediately obvious.

Lemma (Exercise I.20 in Atiyah and Macdonald, 1969)**.** *Let X be a topological space.*

i) Every irreducible subspace of X is contained in a maximal irreducible subspace.

ii) The maximal irreducible subspaces of X are closed and cover X. They are called the irreducible components *of X.*

Proof. Part i) is a standard application of Zorn's Lemma. Indeed if $\{S_i\}_{i\in I}$ is a chain of irreducible subspaces then the union $S = \bigcup_{i\in I} S_i$ is also irreducible.

A maximal irreducible subspace S must be closed, otherwise its closure would be irreducible and strictly bigger (see above). Note that for each $x \in X$ the closure $\overline{\{x\}}$ is irreducible, hence every point of X is contained in some (maximal) irreducible subspace. □

Let A be a ring $\neq 0$. Then the set of prime ideals of A has minimal elements with respect to inclusion. To see this we can apply Zorn's Lemma to the family of all prime ideals in A, indeed if $\{\mathfrak{p}_i\}_{i\in I}$ is a chain of prime ideals then the intersection $\mathfrak{a} = \bigcap_{i\in I} \mathfrak{p}_i$ is also a prime ideal. The argument goes as follows: if $xy \in \mathfrak{a}$ suppose that neither $x \in \mathfrak{a}$ nor $y \in \mathfrak{a}$, then there exist i_1 and i_2 such that $x \notin \mathfrak{p}_{i_1}$ and $y \notin \mathfrak{p}_{i_2}$. But $\{\mathfrak{p}_i\}_{i\in I}$ is a chain so either $\mathfrak{p}_{i_1} \subseteq \mathfrak{p}_{i_2}$ or $\mathfrak{p}_{i_2} \subseteq \mathfrak{p}_{i_1}$ and in any case we arrive to the contradiction $xy \notin \mathfrak{a}$.

Proposition (Exercise I.20 in Atiyah and Macdonald, 1969)**.** *Let A be a ring and let X be the topological space* Spec *A. The irreducible components of X are the closed sets $\mathcal{V}(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A.*

Proof. A maximal irreducible subspace of Spec A must be closed, that is of the form $\mathcal{V}(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A. Since it is irreducible we can assume the ideal to be prime, so it is $\mathcal{V}(\mathfrak{p})$ for some $\mathfrak{p} \subset A$. Now $\mathfrak{p}$ is minimal because $\mathcal{V}(\mathfrak{p})$ is maximal. □

1.2.4 Noetherian Spaces The rings we are most interested in are affine k-algebras, which are of course rather special rings. The first point of distinction is represented by Hilbert's Basis Theorem, which asserts that they are all Noetherian rings, it is explained in Cox, Little, and O'Shea (1997). This has some interesting consequences on the geometry of the spectrum of an affine ring, so interesting to deserve a purely topological definition.

Definition. A topological space X is called *Noetherian* if it satisfies the descending chain condition for closed subsets: for any descending sequence $Y_1 \supseteq Y_2 \supseteq \ldots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = \ldots$

If the ring A is Noetherian and $\mathfrak{a} \subseteq A$ is an ideal, it is a well known result that among the primes of A containing $\mathfrak{a}$ there are only finitely many that are minimal with respect to inclusion, these are usually called the *minimal primes of* $\mathfrak{a}$. It can be proved directly quite easily as in Eisenbud (1995, Exercise 1.2). In geometric terms this means that in Spec A every non-empty closed subset has only a finite number of irreducible components, indeed a closed irreducible subset containing $\mathcal{V}(\mathfrak{a})$ is of the form $\mathcal{V}(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$, and it is maximal if and only if $\mathfrak{p}$ is minimal (as in §1.2.3). This property holds in general for Noetherian topological spaces.

Proposition (I.1.5 in Hartshorne, 1977)**.** *In a Noetherian topological space X, every non-empty closed subset Y can be expressed as a finite union of irreducible closed subsets $Y = Y_1 \cup \ldots \cup Y_r$. If we require that $Y_i \not\supseteq Y_j$ for $i \neq j$, then the Y_i are uniquely determined.*

Example (Exercise I.1.3 in Hartshorne, 1977). Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. We have the following equalities of ideals in $k[x, y, z]$

$$\begin{aligned}(x^2 - yz, xz - x) &= (x^2 - yz, x) \cap (x^2 - yz, z - 1)\\ &= (x, yz) \cap (x^2 - y, z - 1)\\ &= (x, y) \cap (x, z) \cap (x^2 - y, z - 1)\end{aligned}$$

Therefore Y is the union of three irreducible components, two of them are lines and the third is a plane curve.

The next example is the evidence that dealing with these general concepts can be quite misleading. It is in fact obviously true since the very definition that the spectrum of a Noetherian ring is a Noetherian topological space but the converse is not true.

Example (In which Spec A is Noetherian while A is not). Let $\mathfrak{m}$ be the maximal ideal in $R = k[\{x_i\}_{i \in \mathbb{N}}]$ generated by the set of indeterminates. Localise R in $\mathfrak{m}$ and consider $A = R_{\mathfrak{m}}/\mathfrak{p}$ where $\mathfrak{p}$ is the prime ideal

$$\mathfrak{p} = (x_1 - x_2^2,\ x_2 - x_3^3,\ x_3 - x_4^4,\ \ldots)$$

It happens that in R there are no prime ideals between $\mathfrak{p}$ and $\mathfrak{m}$, which would correspond to prime ideals in A other than the zero ideal and the maximal ideal. Indeed in the quotient $R/\mathfrak{p}$ every polynomial F can be written as a polynomial in only one indeterminate x_i, then if F has to be in $\mathfrak{m}$ one can factorise a power of x_i and conclude that a prime ideal containing F that is contained in $\mathfrak{m}$ has to contain x_i also. But then it is $\mathfrak{m}$ already. Therefore Spec A consists of only two points and in particular is a Noetherian topological space. But A is not a Noetherian ring, since we have the strictly ascending chain of ideals $(x_1) \subsetneq (x_2) \subsetneq \dots$

1.2.5 Characterisation of Noetherianity The definition of Noetherian topological space is just a chain condition, like the ones described in Chapter VI of Atiyah and Macdonald (1969). Not surprisingly then we have the following characterisation in terms of maximal and minimal conditions also.

Lemma (Exercise I.1.7 in Hartshorne, 1977)**.** *The following conditions are equivalent for a topological space X:*

i) *X is Noetherian;*

ii) *every non-empty family of closed subsets has a minimal element;*

iii) *X satisfies the ascending chain condition for open subsets;*

iv) *every non-empty family of open subsets has a maximal element.*

Proof. The equivalences *i*) $\Leftrightarrow$ *iii*) and *ii*) $\Leftrightarrow$ *iv*) are trivial (it is enough to consider the complements). To complete the proof we then show the equivalence *i*) $\Leftrightarrow$ *ii*). If *ii*) was false we could find inductively a strictly descending chain of closed subsets. Conversely a descending chain of closed subsets is a particular family of closed subsets, so it must have a minimal element, which proves that it is stationary. □

To be able to prove a more useful (and less immediate) criterion for Noetherianity we need to understand better such a topology. First we can make the following remark.

- A Noetherian topological space is quasi-compact.

To see this, we want to use Zorn's Lemma. Let $\{A_i\}_{i\in I}$ be an open covering of X, i.e. $X = \bigcup_{i\in I} A_i$. Let S be the following family of open subsets of X, ordered by inclusion

$$S = \left\{ \bigcup_{j\in J} A_j \;\middle|\; J \text{ is finite} \right\}$$

Since X is Noetherian every ascending chain of open subsets in X is stationary, hence in S every chain has an upper bound and by Zorn's Lemma S has a maximal element $B = \bigcup_{j \in J_0} A_j$ (where J_0 is some finite subset of I). In particular B is an open set of X which contains any open set of the form $\bigcup_{j \in J} A_j$ where J is a finite subset of I. Then $B = X$ and $\{A_j\}_{j \in J_0}$ is the finite subcover we were looking for. Next we examine the subsets of a Noetherian topological space, and we conclude the following.

- Any subset of a Noetherian topological space is Noetherian in its induced topology.

Let $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \ldots$ be a descending chain of closed subsets of $Y \subseteq X$. Consider the descending chain of closed subsets of X given by $\overline{Y_0} \supseteq \overline{Y_1} \supseteq \overline{Y_2} \supseteq \ldots$, where $\overline{Y_i}$ is the closure in X of Y_i. Since X is Noetherian the latter chain is stationary, hence also the first is and Y is Noetherian.

Example. A Noetherian space which is also Hausdorff must be a finite set with the discrete topology. Indeed let X be a Hausdorff and Noetherian space, then $Y = X \setminus \{x\}$ is also Noetherian. In particular Y is compact (by the first remark above) hence is closed (compact in a Hausdorff space). We conclude that $\{x\}$ is open and X has the discrete topology. To see that X is also finite assume that there exists an injective sequence $(x_n)_{n \in \mathbb{N}}$ and let $A_n = \{x_1, \ldots, x_n\}$. Since X is Noetherian every ascending chain of open subsets of X is stationary, but we have found a strictly ascending chain $A_1 \subsetneq A_2 \subsetneq \ldots$. This is a contradiction thus X is finite.

Proposition (Exercise II.2.13 in Hartshorne, 1977)**.** *A topological space X is Noetherian if and only if every open subset $U \subseteq X$ is quasi-compact.*

Proof. If X is a Noetherian topological space, any open subset is again Noetherian in its induced topology, and therefore it is quasi-compact. Conversely let $U_1 \subseteq U_2 \subseteq \ldots$ be an ascending chain of open subsets. Let U be the union of all the opens in the chain. Then U is open and therefore is quasi-compact. Hence U is the union of only a finite number of the open sets, and since these were on a chain we can conclude that U is one of them. This proves that $U_1 \subseteq U_2 \subseteq \ldots$ is stationary. □

1.3 The Structure Sheaf

Any algebraic set is given by the spectrum of an affine ring, but we have seen that different affine rings can define the same algebraic set. The need to remove this ambiguity is at the heart of the following development, which eventually will lead us to the definition of Affine Algebraic Variety. In complete generality now that we have learned how to construct a topological space given any ring, we are going to tighten this correspondence, adding some extra structure in order to be able to recover the ring given the "space" associated with it. The right instrument turns out to be sheaf theory, given a ring A we will now endow the topological space $\operatorname{Spec} A$ with a sheaf of rings. A quick look at our references highlights two different approaches, the more geometric and in fact closer to our intuition of Hartshorne (1977) and the really algebraic one in Eisenbud and Harris (2000). They are essentially equivalent, although this is not immediately obvious, but they show different aspects of the theory that is useful to keep in mind, therefore we are going to describe them both.

Definition. "Let A be a ring, we define a sheaf of rings $\mathscr{O}$ on $\operatorname{Spec} A$. For each prime ideal $\mathfrak{p} \subseteq A$, let $A_\mathfrak{p}$ denote the localisation of A in $\mathfrak{p}$. For an open set $U \subseteq \operatorname{Spec} A$, we define $\mathscr{O}(U)$ to be the set of functions $s\colon U \to \coprod_{\mathfrak{p}\in U} A_\mathfrak{p}$, such that $s(\mathfrak{p}) \in A_\mathfrak{p}$ for each $\mathfrak{p}$ and such that s is locally a quotient of elements of A: to be precise, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$, contained in U, and elements $\alpha, \beta \in A$, such that for each $\mathfrak{q} \in V$ we have that $\beta \notin \mathfrak{q}$, and $s(\mathfrak{q}) = \alpha/\beta$ in $A_\mathfrak{q}$."

taken from Hartshorne (1977, §II.2)

In other contexts such as Topology or Differential Geometry the structure sheaf of a manifold consists of functions. By analogy the definition just explained usually fits intuition best, but on the other hand one has to work a bit to make it useful. Not unexpectedly it turns out that there is a strong connection between the sheaf $\mathscr{O}$, and the ring A.

Proposition (II.2.2 in Hartshorne, 1977)**.** *Let A be a ring, and $(\operatorname{Spec} A, \mathscr{O})$ its prime spectrum.*

(*a*) *For any $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $\mathscr{O}_\mathfrak{p}$ is isomorphic to the local ring $A_\mathfrak{p}$.*

(*b*) *For any $\alpha \in A$, the ring $\mathscr{O}\big(D(\alpha)\big)$ is isomorphic to the localised ring A_α.*

(*c*) *In particular, $\Gamma(\operatorname{Spec} A, \mathscr{O}) \cong A$*

The really hard part in the proof of this proposition is statement (b), which asserts that the obvious homomorphism $A_\alpha \to \mathscr{O}\big(D(\alpha)\big)$ is an isomorphism. As we mentioned above a more algebraic approach is possible, and it goes precisely the other way round. The starting point is to define $\mathscr{O}\big(D(\alpha)\big)$ to be the localised ring A_α for any distinguished open subset $D(\alpha)$, recalling that distinguished open subsets form a base $\mathscr{B}$ for the Zariski topology in $\operatorname{Spec} A$. If $D(\beta) \subseteq D(\alpha)$ then $\beta \in \sqrt{(\alpha)}$ (see §1.1.4), therefore $\alpha/1$ is invertible in A_β

and we can define the restriction map as the unique homomorphism $A_\alpha \to A_\beta$ such that the following diagram commutes (universal property of localisation)

$$\begin{array}{ccc} A & \longrightarrow & A_\beta \\ \downarrow & \nearrow & \\ A_\alpha & & \end{array}$$

Of course this doesn't define a sheaf yet, but according to Eisenbud and Harris (2000, Proposition I.18) it defines a *$\mathscr{B}$-sheaf* on Spec A.

Let X be a topological space and let $\mathscr{B}$ be a base for the topology of X. "We say that a collection of groups $\mathscr{F}(U)$ for open sets $U \in \mathscr{B}$ and maps

$$res^U_V \colon \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$$

for $V \subseteq U$ form a $\mathscr{B}$-sheaf if they satisfy the sheaf axiom with respect to inclusions of basic open sets in basic open sets and coverings of basic open sets by basic open sets. (The condition in the definition that sections of $U_i, U_j \in \mathscr{B}$ agree on $U_i \cap U_j$ must be replaced by the condition that they agree on any basic open set $V \in \mathscr{B}$ such that $V \subseteq U_i \cap U_j$)."

taken from Eisenbud and Harris (2000, §I.1.3)

Theorem. *Let X be a topological space and let $\mathscr{B}$ be a base for the topology of X. There is a one-to-one correspondence between $\mathscr{B}$-sheaves and sheaves over X. In particular every sheaf is a $\mathscr{B}$-sheaf and every $\mathscr{B}$-sheaf extends uniquely to a sheaf.*

In Eisenbud and Harris (2000, §I.1.3) the reader will find the construction for the extension of a $\mathscr{B}$-sheaf, here we prove the converse.

Lemma. *Let X be a topological space and let $\mathscr{B}$ be a base for the topology of X. Then every sheaf $\mathscr{F}$ over X is a $\mathscr{B}$-sheaf. More generally the gluing condition on basic open subsets applies for any open subset.*

Proof. Let $U \subseteq X$ be any open set, cover U by basic open sets, say $U = \bigcup_{i \in I} U_i$, and let $\{f_i \in \mathscr{F}(U_i)\}$ be a family of sections such that for any basic open set $V \subseteq U_i \cap U_j$ the restrictions to V of f_i and f_j are equal. We claim that there exists a unique element $f \in \mathscr{F}(U)$ such that $res^U_{U_i}(f) = f_i$.

It suffices to prove that f_i and f_j agree on $U_i \cap U_j$, for this means that the family is coherent for the sheaf $\mathscr{F}$ and hence defines a unique element in $\mathscr{F}(U)$ as required. We are now in the following situation: U is an open set of X and for any open basic set $V \subseteq U$ we are given a section $f_V \in \mathscr{F}(V)$. If $V_1 \subseteq V_2$ is an inclusion of two such open basic sets with clear meaning of notation we have $res^2_1(f_2) = f_1$.

We must show that there is a unique section $f \in U$ such that $res_V^U(f) = f_V$ for any open basic set $V \subseteq U$, to this purpose we will use Zorn's Lemma. Define Σ to be the set of couples (W, f) such that

$W \subseteq U$ is an open set

$f \in \mathscr{F}(W)$ is a section

$res_V^W(f) = f_V$ for any open basic set $V \subseteq W$

Σ is not empty for it contains all the couples (V, f_V) where V is a basic open set contained in U. Now we order Σ as follows: we say $(W_1, f_1) \leq (W_2, f_2)$ if $W_1 \subseteq W_2$ and $res_1^2(f_2) = f_1$. To apply Zorn's Lemma we must show that every chain in Σ has an upper bound in Σ; let $(W_1, f_1) \leq (W_2, f_2) \leq \ldots$ be such a chain, set $W = \bigcup_{i \geq 1} W_i$ and observe the following: $\{W_i\}$ is an open covering for W and the family $\{f_i\}$ is a coherent family for the sheaf $\mathscr{F}$ (since $W_i \cap W_j = W_j$ if $j \leq i$). Hence there exists a unique section $f \in \mathscr{F}(W)$ such that $res_i^W(f) = f_i$. The couple (W, f) will be an upper bound for the given chain if we prove that for any open basic set $V \subseteq W$ we have $res_V^W(f) = f_V$.

We have an open covering of V consisting of open basic sets A such that $A \subseteq V \cap W_i$ for some W_i. On this open covering we can define the coherent family $\{res_A^W(f)\}$ given by the restriction of f, this defines the section $res_V^W(f) \in \mathscr{F}(V)$. Observe that this coherent family is given by $\{f_A\}$ so that the unique corresponding section in $\mathscr{F}(V)$ is given by f_V and hence $res_V^W(f) = f_V$.

We can now consider a maximal element (W, f) in Σ and what remains to prove is that $W = U$. Assume this is not the case, hence there exists $x \in U$ such that $x \notin W$. Let V be a basic open set such that $x \in V$ and consider the open set $W \cup V$. We have an open covering of it made up by V, W and all the basic open sets $A \subseteq W \cap V$, define the coherent family given by f, f_V and by the set of sections $\{res_A^V(f_V) = f_A\}$. This defines a unique element $g \in \mathscr{F}(W \cup V)$ such that the couple $(W \cup V, g)$ is an element of Σ contradicting the maximality of (W, f). □

Given a ring A we are now able to construct a *ringed space*, that is a couple $(\operatorname{Spec} A, \mathscr{O})$ consisting of a topological space and a sheaf of rings on it, more about ringed spaces can be found in Tennison (1975, Chapter 4). The sheaf $\mathscr{O}$ is called the *structure sheaf* or the *sheaf of functions* of Spec A.

When A is an affine ring, the structure sheaf of Spec A is in fact a sheaf of k-algebras, meaning that there exists a natural homomorphism $k \to \mathscr{O}(U)$ for any open set U. To see this observe that any element $\alpha \in A$ defines a section in $\mathscr{O}(U)$, as it defines a global section, in particular any $\alpha \in k$. Thus we have in fact a composition of homomorphisms $k \to \Gamma(\operatorname{Spec} A, \mathscr{O}) \to \mathscr{O}(U)$, where the external one is just the restriction of the sheaf $\mathscr{O}$.

Examples. This is in fact Exercise I.20 in Eisenbud and Harris (2000), we want to describe the points and the structure sheaf of each of the following spaces.

(a) $X_1 = \operatorname{Spec} \mathbb{C}[\,x\,]/(x^2)$ (b) $X_2 = \operatorname{Spec} \mathbb{C}[\,x\,]/(x^2 - x)$

(c) $X_3 = \operatorname{Spec} \mathbb{C}[\,x\,]/(x^3 - x^2)$ (d) $X_4 = \operatorname{Spec} \mathbb{R}[\,x\,]/(x^2 + 1)$

As a topological space, points of X_1 corresponds to prime ideals in $\mathbb{C}[\,x\,]$ that contain x^2, hence that contain x. Since (x) is a maximal ideal we can conclude that $X_1 = \{(x)\}$. Open sets are only $\varnothing \subseteq X_1$ and the structure sheaf is obvious

$$\mathscr{O}(\varnothing) = 0 \quad \mathscr{O}(X_1) = \mathbb{C}[\,x\,]/(x^2)$$

Observe that $\mathscr{O}(X_1)$ is a local ring.

The same argument brings to the conclusion that

$$X_2 = \{(x), (x-1)\} = \{P, Q\}$$

and both P and Q are closed points. The topology is discrete and we have the structure sheaf

$$\mathscr{O}(\varnothing) = 0, \quad \mathscr{O}(X_2) = \mathbb{C}[\,x\,]/(x^2 - x), \quad \mathscr{O}(P) = \mathbb{C}, \quad \mathscr{O}(Q) = \mathbb{C}$$

Indeed $\{P\} = D(x-1)$, and we have

$$\begin{aligned}\big[\mathbb{C}[\,x\,]/\big(x(x-1)\big)\big]_{x-1} &= \mathbb{C}[\,x\,]_{x-1}/\big(x(x-1)\big)\\ &= \mathbb{C}[\,x\,]_{x-1}/(x) = \big[\mathbb{C}[\,x\,]/(x)\big]_{x-1} = \mathbb{C}\end{aligned}$$

Observe that the restriction maps are different; from X_2 to P you send (the equivalence class of) a polynomial $f(x)$ to $f(0)$, while from X_2 to Q you send the same polynomial to $f(1)$.

As a topological space X_3 is the same as X_2, but this time we have the following structure sheaf

$$\begin{aligned}&\mathscr{O}(\varnothing) = 0, && \mathscr{O}(X_3) = \mathbb{C}[\,x\,]/(x^3 - x^2),\\ &\mathscr{O}(P) = \mathbb{C}[\,x\,]/(x^2), && \mathscr{O}(Q) = \mathbb{C}\end{aligned}$$

Finally observe that $\mathbb{R}[\,x\,]/(x^2 + 1) \cong \mathbb{C}$.

At this stage we have all the instruments to define affine varieties, but unfortunately here is where the real trouble begins. There is not a unique and widely accepted definition of algebraic variety, there are instead several slightly different ones. Depending on the field of research, each mathematician will choose his favourite notion and will work with it. As a result, here is a brief summary of the confusion reigning already among our references. In Hartshorne (1977) and Shafarevich (1994b) an affine algebraic variety is the spectrum of an affine *domain*. In Milne (2005) it is the spectrum of a *reduced* affine ring. In Ueno (2003) and Liu (2002) it is the spectrum of any affine ring, but be careful because in Liu (2002) the field of definition is not assumed to be algebraically closed. In Eisenbud and Harris (2000) there is not even a definition of algebraic variety! To deal with this confusion we will follow the guidelines of Fulton (1998), which contains probably the most reasonable set of definitions.

Definition. An *affine scheme* is the spectrum of a ring (any ring), while an *affine algebraic scheme* is the spectrum of an affine ring, that is, a finitely generated algebra over a field k. We don't make any assumption on the base field k, assuming that whether it will be algebraically closed or not will depend on the context. An *affine algebraic variety* will be the spectrum of an affine domain.

1.4 Functoriality

1.4.1 The Functor Spec Let $\varphi\colon A \to B$ be a ring homomorphism. For any $\mathfrak{q} \in \operatorname{Spec} B$, by taking $\varphi^{-1}(\mathfrak{q})$ we construct a prime ideal of A, that is a point of $\operatorname{Spec} A$. Hence φ induces a function

$$\operatorname{Spec}\varphi\colon\ \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$

This is a continuous function and Spec turns out to be a contravariant functor between the category of Rings and the category of Topological Spaces. The reader not familiar with category theory shouldn't worry too much because we will use only very basic material, it can be convenient anyway having a look at Mac Lane (1998) or Berrick and Keating (2000) just for reference.

Proposition (Exercise I.21 in Atiyah and Macdonald, 1969)**.** *Let $\varphi\colon A \to B$ be a ring homomorphism. Then the induced map $\Phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ has the following properties:*

i) If $\alpha \in A$ then $\Phi^{-1}\big(D(\alpha)\big) = D\big(\varphi(\alpha)\big)$, hence Φ is continuous.

ii) If $\mathfrak{a}$ is an ideal of A, then $\Phi^{-1}\big(\mathcal{V}(\mathfrak{a})\big) = \mathcal{V}(\mathfrak{a}^e)$.

iii) *If* $\mathfrak{b}$ *is an ideal of* B*, then* $\overline{\Phi\big(\mathcal{V}(\mathfrak{b})\big)} = \mathcal{V}(\mathfrak{b}^c)$.

iv) *If* φ *is surjective, then* Φ *is a homeomorphism of* $\operatorname{Spec} B$ *onto the closed subset* $\mathcal{V}(\ker\varphi) \subseteq \operatorname{Spec} A$.

v) *If* φ *is injective, then* $\Phi(\operatorname{Spec} B)$ *is dense in* $\operatorname{Spec} A$*. More precisely, the image* $\Phi(\operatorname{Spec} B)$ *is dense in* $\operatorname{Spec} A$ *if and only if* $\ker\varphi \subseteq \operatorname{Nil}(A)$*, where* $\operatorname{Nil}(A)$ *is the nilradical of* A.

vi) *Let* $\psi\colon B \to C$ *be another ring homomorphism, and let* $\operatorname{Spec}\psi$ *be the corresponding induced map. Then*

$$\operatorname{Spec}(\psi\circ\varphi) = \operatorname{Spec}\varphi \circ \operatorname{Spec}\psi$$

Proof. Statements *i)* and *ii)* are easily worked out by writing their meaning in terms of sets, while *vi)* is trivial. Assuming *iii)* we have

$$\overline{\Phi(\operatorname{Spec} B)} = \overline{\Phi\big(\mathcal{V}(0)\big)} = \mathcal{V}(\ker\varphi)$$

which implies *v)*, since $\mathcal{V}(\ker\varphi)$ is the whole space if and only if $\ker\varphi$ is contained in the nilradical of A. About *iv)*, injectivity of Φ follows immediately by a direct argument, further observe the following:

> If φ is surjective and $\mathfrak{p} \subseteq A$ is a prime ideal such that $\ker\varphi \subseteq \mathfrak{p}$, then $\mathfrak{p} = \Phi(\mathfrak{p}^e)$. Indeed this amounts to say that $\mathfrak{p} = \mathfrak{p}^{ec}$, and the only thing to prove is the inclusion $\mathfrak{p}^{ec} \subseteq \mathfrak{p}$. This is easily done by contradiction.

assuming *iii)* this proves that Φ is a closed map with image $\mathcal{V}(\ker\varphi)$.

We now prove statement *iii)*. Observe first

$$\Phi\big(\mathcal{V}(\mathfrak{b})\big) = \big\{\mathfrak{p} \subseteq A \mid \mathfrak{p} = \varphi^{-1}(\mathfrak{q}) \text{ with } \mathfrak{b} \subseteq \mathfrak{q}\big\}$$

This set is contained in $\mathcal{V}(\mathfrak{b}^c)$ so the inclusion $\overline{\Phi\big(\mathcal{V}(\mathfrak{b})\big)} \subseteq \mathcal{V}(\mathfrak{b}^c)$ is given. To prove the converse we need to show that for each $\mathfrak{p} \in \mathcal{V}(\mathfrak{b}^c)$ every open neighborhood of $\mathfrak{p}$ intersects $\Phi\big(\mathcal{V}(\mathfrak{b})\big)$. Let $\mathfrak{p} \in \mathcal{V}(\mathfrak{b}^c)$ and let $D(\alpha)$ be a basic open set which contains it. By definition $\alpha \notin \mathfrak{p}$ and in particular $\alpha \notin \mathfrak{b}^c$, hence $\varphi(\alpha) \notin \mathfrak{b}$. We claim that there is a point $\mathfrak{q} \in \mathcal{V}(\mathfrak{b})$ such that $\varphi(\alpha) \notin \mathfrak{q}$. Indeed if $\mathcal{V}(\mathfrak{b}) \subseteq \mathcal{V}\left(\varphi(\alpha)\right)$ then $\varphi(\alpha) \in \sqrt{\mathfrak{b}}$, that is $\varphi(\alpha)^t \in \mathfrak{b}$. But then $\alpha^t \in \mathfrak{b}^c \subseteq \mathfrak{p}$ which contradicts α not being in $\mathfrak{p}$. Now $\varphi^{-1}(\mathfrak{q}) \in D(\alpha) \cap \Phi\big(\mathcal{V}(\mathfrak{b})\big)$. $\square$

1.4.2 Connectedness So far we have described functoriality precisely but only when we regard the spectrum of a ring as a topological space, forgetting about the structure sheaf. Nevertheless what we have seen can be useful on its own, for instance it allows us to characterise those rings A for which $\operatorname{Spec} A$ is connected. First we need a preliminary algebraic Lemma.

Lemma. *Let $A = A_1 \times A_2$. Then every prime ideal of A is of the form $\mathfrak{p} \times A_2$ or $A_1 \times \mathfrak{q}$ where $\mathfrak{p} \subseteq A_1$ and $\mathfrak{q} \subseteq A_2$ are prime ideals.*

Proof. Every ideal of the claimed form is clearly prime, conversely if $\mathfrak{a} \subseteq A$ is a prime ideal then $p_1(\mathfrak{a}) \subseteq A_1$ and $p_2(\mathfrak{a}) \subseteq A_2$ are prime ideals (where p_i is the projection). Assume that both of them are proper and take elements $a \notin p_1(\mathfrak{a})$ and $b \notin p_2(\mathfrak{a})$, then $(a, 0)$ and $(0, b)$ are not in $\mathfrak{a}$ but their product is (it is in fact the zero element of the ring). Therefore we can assume without loss of generality that $p_2(\mathfrak{a}) = A_2$, and we have to prove that $\mathfrak{a}$ is in fact $p_1(\mathfrak{a}) \times A_2$.

We have the obvious inclusion $\mathfrak{a} \subseteq p_1(\mathfrak{a}) \times A_2$, so we must show the converse. If $a \in p_1(\mathfrak{a})$ then there exists $b \in A_2$ such that $(a, b) \in \mathfrak{a}$, write (a, b) as $(a, 1) \cdot (1, b)$ and observe that $(1, b)$ is not an element of $\mathfrak{a}$, so that $(a, 1)$ is. Now for any $c \in A_2$ we have $(a, c) = (a, 1) \cdot (1, c)$, so that $\mathfrak{a} = p_1(\mathfrak{a}) \times A_2$. □

Proposition (Exercise I.22 in Atiyah and Macdonald, 1969)**.** *Let A be the direct product of rings $\prod_{i=1}^{n} A_i$. Then* Spec *A is the disjoint union of open (and closed) subspaces X_i, where X_i is canonically homeomorphic with* Spec *A_i.*
Conversely, let A be any ring. Then the following statements are equivalent:

i) $X = \operatorname{Spec} A$ *is disconnected.*

ii) $A \cong A_1 \times A_2$ *where neither of the rings A_1, A_2 is the zero ring.*

iii) *A contains an idempotent $\neq 0, 1$.*

The same is Exercise II.2.19 in Hartshorne (1977), *where iii) is more precise: there exist non-zero elements $e_1, e_2 \in A$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$ (these elements are called* orthogonal idempotents*).*

Proof. Let $p_j \colon A \to A_j$ be the projection. Since p_j is surjective, the induced map $\operatorname{Spec} p_j$ is a homeomorphism of $\operatorname{Spec} A_j$ onto the closed subset $X_j = \mathcal{V}(\ker p_j)$ of $\operatorname{Spec} A$. Observe that $\ker p_j = A_1 \times \cdots \times A_{j-1} \times \{0\} \times A_{j+1} \times \cdots \times A_n$, and recall how prime ideals are made in A, which was discussed above, then:

- $\mathcal{V}(\ker p_j) = D(e_j)$ where e_j is the n-tuple $(a_1, \ldots, a_n)$ with $a_i = 0$ if $i \neq j$ and $a_j = 1$. Hence X_j is open and closed in $\operatorname{Spec} A$.

- $\operatorname{Spec} A = X_1 \cup \ldots \cup X_n$.

- $X_i \cap X_j = \varnothing$. Indeed $X_i \cap X_j = \mathcal{V}(\ker p_i) \cap \mathcal{V}(\ker p_j) = \mathcal{V}(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal generated by $\ker p_i \cup \ker p_j$. Note that the unit element of A is contained in $\mathfrak{a}$.

To prove the equivalence of the three statements note first that *ii*) $\Rightarrow$ *i*) is the first part of the Proposition. Now assume *iii*) and let e be an idempotent $\neq 0, 1$, then $(1-e)$ is idempotent as well and the two ideals eA and $(1-e)A$ are nonzero commutative rings with unit element (namely, their generator). Observe that if $ae = b(1-e)$ then $b = (a+b)e$; multiply by e and get $ae = 0$. Hence $eA \cap (1-e)A = 0$. Define

$$\varphi\colon eA \times (1-e)A \longrightarrow A$$

sending (α, β) to $\alpha + \beta$. This is a ring homomorphism that is an isomorphism; indeed for all $a \in A$ we can write $a = ae + a(1-e)$, while if $ae + b(1-e) = ce + d(1-e)$ we get $(a-c)e = (d-b)(1-e)$ from which follows $ae = ce$ and $b(1-e) = d(1-e)$. This proves *ii*); it remains to prove *i*) $\Rightarrow$ *iii*).

Let $X = E_1 \cup E_2$ where E_i is both open and closed and $E_1 \cap E_2 = \varnothing$. We can write $E_1 = \mathcal{V}(\mathfrak{a})$ and $E_2 = \mathcal{V}(\mathfrak{b})$, so that $E_1 \cap E_2 = \mathcal{V}(\mathfrak{a} + \mathfrak{b})$. Since the intersection is empty we have $1 \in \mathfrak{a} + \mathfrak{b}$, that is $1 = a + b$ with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Note that for each $\alpha \in \mathfrak{a}$ it is $D(\alpha) \subseteq E_2$, indeed $\mathcal{V}(\mathfrak{a}) \subseteq \mathcal{V}(\alpha)$ hence $D(\alpha) \subseteq X \setminus \mathcal{V}(\mathfrak{a}) = E_2$. Assume $D(a) \subsetneq E_2$. Let $\mathfrak{p} \in E_2$ such that $a \in \mathfrak{p}$, then since $\mathfrak{p} \in E_2$ we have $\mathfrak{b} \subseteq \mathfrak{p}$, but since $a = 1 - b$ we have $1 \in \mathfrak{p}$, which is a contradiction. Then we have $E_2 = D(a)$ and $E_1 = D(b)$. Again the intersection is empty, that is ab is nilpotent.

Let $t > 0$ be the least integer such that $(ab)^t = 0$ and observe $D(a^t) = E_2$ and $D(b^t) = E_1$. Again we have $X = E_1 \cup E_2 = D(a^t) \cup D(b^t) = X \setminus \mathcal{V}(\mathfrak{c})$ where $\mathfrak{c}$ is the ideal generated by a^t and b^t. It follows that $1 \in \mathfrak{c}$ that is $\alpha a^t + \beta b^t = 1$ for some $\alpha, \beta \in A$. Let $e_1 = \alpha a^t$ and $e_2 = \beta b^t$, then $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$. □

1.4.3 Locally Ringed Spaces Given a ring A we have endowed the topological space Spec A with a sheaf of rings, the complete construction is therefore giving rise to something more than simply a topological space. To investigate functorial properties we need a suitable category, which turns out to be the category of *locally ringed spaces*.

Definition. "A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism* of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a continuous map $f\colon X \to Y$ and a map $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y."

"The ringed space $(X, \mathscr{O}_X)$ is a *locally ringed space* (or also a *geometric space*) if for each point $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring. A *morphism* of locally ringed spaces is a morphism $(f, f^\#)$ of ringed spaces, such that for each point $x \in X$, the induced map (see below) of local rings $f^\#_x \colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a *local homomorphism* of local rings."

"We explain this last condition. First of all, given a point $x \in X$, the morphism of sheaves $f^\# \colon \mathscr{O}_Y \to f_* \mathscr{O}_X$ induces a homomorphism of rings $\mathscr{O}_Y(V) \to \mathscr{O}_X\big(f^{-1}(V)\big)$, for every open set V in Y. As V ranges over all open neighborhoods of $f(x)$, $f^{-1}(V)$ ranges over a subset of the neighborhoods of x. Taking direct limits, we obtain a map

$$\mathscr{O}_{Y,f(x)} = \varinjlim_{f(x) \in V} \mathscr{O}_Y(V) \longrightarrow \varinjlim_{f(x) \in V} \mathscr{O}_X\big(f^{-1}(V)\big),$$

and the latter limit maps to the stalk $\mathscr{O}_{X,x}$. Thus we have an induced homomorphism $f^\#_x \colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$. We require that this be a local homomorphism. If A and B are local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively, a homomorphism $\varphi \colon A \to B$ is called a *local homomorphism* if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$ (or equivalently if $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$)."

taken from Hartshorne (1977, §II.2)

Not surprisingly the construction of a Locally Ringed Space from a Ring turns out to be functorial, but in fact we have something more, this functor is *fully faithful*. This means that if A and B are rings, and $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are the corresponding locally ringed spaces then A and B are isomorphic if and only if so are X and Y. The precise statement is the following.

Proposition (II.2.3 in Hartshorne, 1977)**.**

(a) *If $\varphi \colon A \to B$ is a morphism of rings, then φ induces a natural morphism of locally ringed spaces*

$$(f, f^\#) \colon \operatorname{Spec} B \to \operatorname{Spec} A$$

(b) *If A and B are rings, then any morphism of locally ringed spaces from* Spec B *to* Spec A *is induced by a homomorphism of rings $\varphi \colon A \to B$ as in* (a).

1.4.4 The Induced Morphism of Sheaves The morphism $(f, f^\#)$ consists of a continuous map $f \colon X \to Y$ which is the induced map $\operatorname{Spec} \varphi$ defined above, and a sheaf homomorphism $f^\# \colon \mathscr{O}_Y \to f_* \mathscr{O}_X$. This homomorphism of sheaves plays a very important role in the theory and should not be ignored, it is defined by a ring homomorphism

$$f^\#_V \colon \mathscr{O}_Y(V) \longrightarrow \mathscr{O}_X\big(f^{-1}(V)\big)$$

for any open set $V \subseteq Y$. It is in other words a homomorphism of sheaves over Y, and like any other homomorphism of sheaves it induces a morphism on the stalks for any $y \in Y$

$$f_y^\#\colon \mathscr{O}_{Y,y} \longrightarrow [f_*\mathscr{O}_X]_y$$

This should not be confused with the induced local homomorphism of local rings, defined for any $x \in X$ between stalks over Y and stalks over X

$$f_x^\#\colon \mathscr{O}_{Y,f(x)} \longrightarrow \mathscr{O}_{X,x}$$

Further, when V is an open basic subset $D(\alpha) \subseteq \operatorname{Spec} A$ for some $\alpha \in A$, the homomorphism $f_V^\#$ is of the form

$$f_\alpha^\#\colon \mathscr{O}_Y\big(D(\alpha)\big) \longrightarrow \mathscr{O}_X\left(D\big(\varphi(\alpha)\big)\right)$$

since $f^{-1}\big(D(\alpha)\big) = D\big(\varphi(\alpha)\big)$ (we have seen it in §1.4.1). But we know that $\mathscr{O}_Y\big(D(\alpha)\big)$ is given by the localised ring A_α, it is therefore reasonable to ask whether $f_\alpha^\#$ is given by the localised homomorphism φ_α. The precise statement is the following.

Proposition. *Let $\varphi\colon A \to B$ be a morphism of rings, and let $(f, f^\#)$ be the corresponding morphism of locally ringed spaces as above. Then the following statements hold.*

i) For any $\alpha \in A$ the homomorphism $f_\alpha^\#$ is given by the localised map

$$\varphi_\alpha\colon A_\alpha \longrightarrow B_{\varphi(\alpha)}$$

In particular, on global sections $f^\#$ coincides with φ.

ii) For any $x \in X$ let $\mathfrak{q}$ be the corresponding prime ideal of B, then the homomorphism $f_x^\#$ is given by the localised map

$$\varphi_\mathfrak{q}\colon A_{\varphi^{-1}(\mathfrak{q})} \longrightarrow B_\mathfrak{q}$$

iii) For any $y \in Y$ let $\mathfrak{p}$ be the corresponding prime ideal of A, then the homomorphism $f_y^\#$ is given by the localised map

$$\varphi_\mathfrak{p}\colon A_\mathfrak{p} \longrightarrow S^{-1}B$$

where $S = \varphi(A \setminus \mathfrak{p})$.

Proof. We leave the reader to work out the proof on its own. In fact, depending on the preferred point of view for the definition of the sheaf $\mathscr{O}$ statement $i)$ is either the definition of $f^\#$ or an immediate consequence of Proposition II.2.2 in Hartshorne (1977). □

1.4.5 Basic Sets as Spectra We mentioned in §1.1.4 that open basic subsets are again spectra of rings. This is true in the strongest sense, namely open basic subsets equipped with the obvious structure sheaf are locally ringed spaces isomorphic to the spectrum of a ring.

Proposition. *Let A be a ring. Then for any $\alpha \in A$ the distinguished open subset $D(\alpha)$ of* Spec A *equipped with the restriction of the structure sheaf $\mathscr{O}|_{D(\alpha)}$ is isomorphic to the locally ringed space* Spec A_α.

Proof. Let $j\colon A \to A_\alpha$ be the localisation map. Then we have the induced continuous function

$$\operatorname{Spec} j\colon \operatorname{Spec} A_\alpha \longrightarrow \operatorname{Spec} A$$

It is well known that for any multiplicatively closed subset S of A there is a one-to-one correspondence between ideals of $S^{-1}A$ and ideals of A that don't meet S. In this case S is given by the powers of α, therefore any prime ideal that doesn't meet S also doesn't contain α and viceversa. This proves that Spec j is an injective function whose image is $D(\alpha)$. Moreover it is a closed map; indeed if $\mathfrak{b} \subseteq A_\alpha$ is any ideal we have $\operatorname{Spec} j\left(\mathcal{V}(\mathfrak{b})\right) = \mathcal{V}(\mathfrak{b}^c) \cap D(\alpha)$. Hence $\operatorname{Spec} j\colon \operatorname{Spec} A_\alpha \to D(\alpha)$ is a homeomorphism.

Let now $\beta \in A$ such that $D(\beta) \subseteq D(\alpha)$ and observe that, according to the discussion above, the morphism of sheaves is given over $D(\beta)$ by the homomorphism

$$\operatorname{Spec} j^\#_\beta\colon A_\beta \longrightarrow (A_\alpha)_{\beta/1}$$

But in this case $\beta \in \sqrt{(\alpha)}$ (we discussed this in §1.1.4) and therefore this homomorphism is in fact an isomorphism. □

1.4.6 Injective and Surjective Homomorphisms Let $\varphi\colon A \to B$ be a homomorphism of rings, and let $f\colon X \to Y$ be the induced morphism of locally ringed spaces. Any ideal of A defines a closed subset of Y, and we have seen in §1.4.1 that when φ is surjective it induces the topological injection of $\mathcal{V}(\ker\varphi)$. The converse is not true, namely a homomorphism can induce a topological injection of a closed subset without being surjective.

Example (In which f is a homeomorphism but φ is not surjective). We let Y be the spectrum of the ring $A = \mathbb{C}[x,y]/(xy)$, that is the union of two lines, while X will be the spectrum of $B = \mathbb{C}[s,t]/(s^2)$. This is an affine scheme whose underling topological space is a line, but among the global sections of the structure sheaf there are nilpotent elements. Now we consider the following homomorphism of k-algebras

$$\varphi\colon \mathbb{C}[x,y]/(xy) \longrightarrow \mathbb{C}[s,t]/(s^2)$$

defined by $\varphi(x) = s^2$ and $\varphi(y) = t$. Then φ is not surjective, since s is not in the image, but it defines a homeomorphism $f: X \to \mathcal{V}(x^2)$.

The problem here is that φ doesn't induce an *isomorphism of schemes* between X and the spectrum of $A/\ker\varphi$. In fact, the main content of the next result is that this happens if and only if the homomorphism is surjective.

Theorem (Exercise II.2.18 in Hartshorne, 1977)**.** *Let $\varphi: A \to B$ be a homomorphism of rings, and let $f: X \to Y$ be the induced morphism of locally ringed spaces.*

(a) *φ is injective if and only if the map of sheaves $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective. Furthermore in that case f is* dominant, *i.e. $f(X)$ is dense in Y.*

(b) *If φ is surjective, then f is a homeomorphism of X onto a closed subset of Y, and the morphism of sheaves $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.*

(c) *The converse to (b) is also true, namely, if $f: X \to Y$ is a homeomorphism onto a closed subset, and $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, then φ is surjective.*

Proof. Assume $\varphi: A \to B$ is injective. Let $D(\alpha) \subseteq \operatorname{Spec} A$ be a basic open set, and consider the map $f^\#_\alpha$ which is given as in §1.4.4 by

$$\varphi_\alpha: A_\alpha \longrightarrow B_{\varphi(\alpha)}$$

Let $\delta/\alpha^r \in A_\alpha$ and assume $\varphi_\alpha(\delta/\alpha^r) = \varphi(\delta)/\varphi(\alpha)^r = 0$. Then there exists an integer $t \in \mathbb{N}$ such that $\varphi(\alpha)^t\varphi(\delta) = 0$ that is $\varphi(\alpha^t\delta) = 0$. Since φ is injective this means $\alpha^t\delta = 0$ and this is equivalent to say that $\delta/\alpha^r = 0$ in A_α. So $f^\#$ is injective over the basic open sets. Now let $V \subseteq Y$ be any open set, let $f^\#_V: \mathcal{O}_Y(V) \to f_*\mathcal{O}_X(V)$ and let $s \in \mathcal{O}_Y(V)$ such that $f^\#_V(s) = 0$. Then for any basic open set $D(\alpha) \subseteq V$ we have $f^\#_\alpha\left(s|_{D(\alpha)}\right) = f^\#_V(s)|_{D(\alpha)} = 0$ and so $s|_{D(\alpha)} = 0$. This is enough to say that $s = 0$ and hence that $f^\#$ is injective.

The converse follow by the more general fact that if a map of sheaves is injective then the induced map on global sections (that in our case is φ) is injective. We have already seen that in this case f is dominant in §1.4.1.

In part (b) the only new result for us is that $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. In fact we have that $f^\#_\alpha: A_\alpha \longrightarrow B_{\varphi(\alpha)}$ is surjective for any $\alpha \in A$. To see it, let $b/\varphi(\alpha)^t \in B_{\varphi(\alpha)}$, then since φ is surjective, there exists $\gamma \in A$ such that $b = \varphi(\gamma)$ and we have

$$\frac{b}{\varphi(\alpha)^t} = \frac{\varphi(\gamma)}{\varphi(\alpha^t)} = f^\#_\alpha\left(\frac{\gamma}{\alpha^t}\right)$$

This is enough to say that $f^\#$ induces surjective homomorphisms on the stalks, and hence $f^\#$ is surjective.

To prove (c), consider the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\pi}\downarrow & \nearrow{\scriptstyle\psi} & \\ A/\ker\varphi & & \end{array}$$

It is well known that φ is surjective if and only if ψ is an isomorphism, moreover ψ is an isomorphism if and only if it induces an isomorphism of locally ringed spaces between $X = \operatorname{Spec} B$ and $Y' = \operatorname{Spec} A/\ker\varphi$. We will prove that ψ induces an isomorphism. First consider the diagram induced on topological spaces

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g}\searrow & \uparrow{\scriptstyle p} \\ & & Y' \end{array}$$

Since π is surjective, p is a homeomorphism onto $\mathcal{V}(\ker\pi) = \mathcal{V}(\ker\varphi)$ and this closed subset contains $f(X)$, more precisely we have $\overline{f(X)} = \mathcal{V}(\ker\varphi)$ (see §1.4.1). It follows that g is defined as the restriction of f to Y', and therefore it is a homeomorphism onto a closed subset of Y'. But since ψ is injective, by part (a) of this theorem g is dominant, that is $\overline{g(X)} = Y'$, hence g must be a homeomorphism.

Consider now $g^{\#}\colon \mathscr{O}_{Y'} \to g_*\mathscr{O}_X$; by part (a) of this theorem, since ψ is injective, this is an injective morphism of sheaves. It remains to prove that it is surjective. For any $y \in Y$, that is for any prime ideal $\mathfrak{p} \subseteq A$ containing $\ker\varphi$ we have the diagram

$$\begin{array}{ccc} A_{\mathfrak{p}} & \xrightarrow{\varphi_{\mathfrak{p}}} & S^{-1}B \\ {\scriptstyle\pi_{\mathfrak{p}}}\downarrow & \nearrow{\scriptstyle\psi_{\mathfrak{p}}} & \\ (A/\ker\varphi)_{\overline{\mathfrak{p}}} & & \end{array}$$

where $S = \varphi(A \setminus \mathfrak{p})$. We know from §1.4.6 that for any $y \in Y'$ the morphism $g^{\#}_y$ is given by $\psi_{\mathfrak{p}}$ and we are assuming $\varphi_{\mathfrak{p}}$ to be surjective. Hence $g^{\#}_y$ is also surjective. $\square$

Chapter 2

General Properties of Schemes

After the introduction to affine schemes of the first chapter, we are now going to introduce schemes in general. The definition will appear at this stage extremely simple, but some of the properties we are going to describe can result somewhat surprising. The very basic idea that we want to convey is the parallel with the classical construction of projective space. There are essentially two possible ways to define $\mathbb{P}^n_k$, we can glue together $n + 1$ copies of affine n-space or we can start with a polynomial ring in $n + 1$ indeterminates and regard it as a *graded ring*. Both this ideas are encoded in the theory of schemes, the first is a fundamental tool called *gluing Lemma*, and the second is a general construction analogous to Spec, the homogeneous spectrum of a graded ring.

Given the scheme theoretic definition of projective space, it will hopefully be clear how special a scheme this is. Here we will concentrate on topological properties, describing them in the greatest generality but always keeping in mind that these are good characteristics shared by all our main examples. We will meet again the duality between algebra and geometry, in the definition of a Noetherian scheme as well as throughout the chapter.

2.1 Schemes

2.1.1 The Category of Schemes In analogy with the definition of a manifold in Differential Geometry, a scheme will be given by gluing together several "local" patches, or in other words it will be defined to have a distinguished open covering satisfying some basic properties. Affine algebraic varieties, or more generally affine algebraic schemes, are the "local charts" of Algebraic Geometry, but as before it is not necessary at this level to assume our rings to be k-algebras therefore we drop the word "algebraic" and call *affine scheme* the spectrum of any ring.

Definition. "An *affine scheme* is a locally ringed space which is isomorphic (as a locally ringed space) to the spectrum of a ring. A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ in which every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathscr{O}_X|_U$ is an affine scheme. We refer to the ringed space $(U, \mathscr{O}_X|_U)$ as an *open affine subset* of X. We call X the *underlying topological space* of the scheme $(X, \mathscr{O}_X)$, and $\mathscr{O}_X$ its *structure sheaf*. By abuse of notation we will often write simply X for the scheme $(X, \mathscr{O}_X)$. If we wish to refer to the underlying topological space without its scheme structure, we write $\mathrm{sp}(X)$, read "space of X". A *morphism* of schemes is a morphism as locally ringed spaces."

taken from Hartshorne (1977, §II.2)

Remark (On the composition of morphisms). Starting with two morphisms of schemes, $(f, f^\#)\colon X \to Y$ and $(g, g^\#)\colon Y \to Z$, we want to define another morphism $X \to Z$ to be the composition of these two. On the underlying topological spaces take

$$g \circ f\colon \mathrm{sp}(X) \longrightarrow \mathrm{sp}(Z)$$

which is obviously a continuous map. Now we have to define a morphism of sheaves

$$(g \circ f)^\#\colon \mathscr{O}_Z \longrightarrow (g \circ f)_* \mathscr{O}_X$$

using the two that are given

$$g^\#\colon \mathscr{O}_Z \longrightarrow g_* \mathscr{O}_Y$$
$$f^\#\colon \mathscr{O}_Y \longrightarrow f_* \mathscr{O}_X$$

there is only one thing we can do. For each open set $W \subseteq Z$ define $(g \circ f)^\#_W$ as the following composition

$$\mathscr{O}_Z(W) \xrightarrow{g^\#_W} \mathscr{O}_Y(g^{-1}(W)) \xrightarrow{f^\#_V} \mathscr{O}_X\left(f^{-1}\left(g^{-1}(W)\right)\right)$$

where $V = g^{-1}(W)$, that is define $(g \circ f)^\#_W = f^\#_{g^{-1}(W)} \circ g^\#_W$. Now it is not difficult to check (essentially one has to verify that a big diagram commutes) that for all $x \in X$ we have

$$(g \circ f)^\#_x\colon \mathscr{O}_{Z,gf(x)} \longrightarrow \mathscr{O}_{X,x}$$

is obtained by the composition $f^\#_x \circ g^\#_{f(x)}$ and it is in fact a local homomorphism of local rings. All this discussion requires X, Y and Z to be locally ringed spaces and nothing more, in accordance with the definition of a morphism of schemes.

2.1.2 Generic Points and Open Immersions If X is a scheme then $\operatorname{sp}(X)$ is a T_0-space, which means that if x, y are distinct points of X, then either there is a neighborhood of x which does not contain y, or else there is a neighborhood of y which does not contain x. Indeed we can cover X by affine open sets, then either x and y are contained in the same affine subset of X or there is an open affine neighborhood of x which does not contain y. Now we already know that an affine scheme is a T_0-space from §1.1.5.

Lemma (Exercise II.2.2 in Hartshorne, 1977)**.** *On any scheme X open affine sets form a base for the topology. In particular for every open set $U \subseteq X$, the pair $(U, \mathscr{O}_X|_U)$ is again a scheme.*

Proof. Let $U \subseteq X$ be any open set. For all $x \in U$ there exists an open affine neighborhood W of x. Then $U \cap W$ is an open set of W in its induced topology, and since $(W, \mathscr{O}_X|_W)$ is the spectrum of a ring we can cover $U \cap W$ with *distinguished* open sets, which are affine open sets (see §1.4.5). Doing this for all $x \in U$ we find an open covering of U consisting of affine open sets. □

Definition. Let $U \subseteq X$ be any open subset, we refer to $(U, \mathscr{O}_X|_U)$ as an *open subscheme* of X. An *open immersion* is a morphism of schemes $f : X \to Y$ which induces an isomorphism of X with an open subscheme of Y.

Example (In which there is an open subscheme of an affine scheme which is not affine). Let k be an algebraically closed field and let X be the affine plane $\mathbb{A}^2_k = \operatorname{Spec} k[x_1, x_2]$. If we remove the origin from the plane we obtain a non-affine scheme $U = X \setminus \mathcal{V}(x_1, x_2)$.

First we show that the restriction map

$$\Gamma(X, \mathscr{O}_X) = k[x_1, x_2] \longrightarrow \mathscr{O}_X(U)$$

is an isomorphism. Injectivity follows immediately, since $k[x_1, x_2]$ is an integral domain. Now we take the open covering $U = D(x_1) \cup D(x_2)$ and an element $s \in \mathscr{O}_X(U)$. If s is given by β/x_1^t on $D(x_1)$ and by γ/x_2^r on $D(x_2)$ we then have $\beta/x_1^t = \gamma/x_2^r$ in $D(x_1x_2)$, which means $x_2^r\beta = x_1^t\gamma$. This leads eventually to the conclusion $\beta = x_1^t\alpha$ and $\gamma = x_2^r\alpha$ for some $\alpha \in k[x_1, x_2]$, that is the restriction map is surjective.

Now the open set U cannot be affine, indeed if this was the case it would be given by $\operatorname{Spec} \mathscr{O}_X(U)$ and the open immersion $U \to \mathbb{A}^2_k$ would be induced by the restriction morphism above. But the restriction morphism above can only induce an isomorphism while the open immersion is not even a bijection of topological spaces.

Proposition (Exercise II.2.9 in Hartshorne, 1977)**.** *If X is a topological space, and Z an irreducible closed subset of X, a* generic point *for Z is a point ζ such that $Z = \{\zeta\}^-$. If X is a scheme, every (nonempty) irreducible closed subset has a unique generic point.*

Proof. We first show that if such a point exists then it is necessarily unique. Assume that there exists two different such points ζ_1 and ζ_2, since X is a T_0-space there exists an open set U such that $\zeta_1 \in U$ and $\zeta_2 \notin U$, which is absurd for $\zeta_1 \in \{\zeta_2\}^-$ and hence every neighborhood of ζ_1 must contain ζ_2 also.

To prove the existence let X be affine first. Then $Z = \mathcal{V}(\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$, since it is closed. But Z is also irreducible, then $\sqrt{\mathfrak{a}} = \mathfrak{p}$ is a prime ideal. Since $Z = \mathcal{V}(\mathfrak{p})$ we have $\mathfrak{p} \in Z$ and $Z = \{\mathfrak{p}\}^-$. Now let X be any scheme. There exists an open affine set $U \subseteq X$ such that $Z \cap U \neq \varnothing$. Hence $Z \cap U$ is a closed and irreducible subset of the affine scheme U, and there exists a point $\zeta \in Z \cap U$ such that the closure in U of the set $\{\zeta\}$ is $Z \cap U$. Consider now $\{\zeta\}^-$ in X, this is the intersection of all closed sets containing the point ζ, in particular Z is one of those closed sets, hence $\{\zeta\}^- \subseteq Z$. Now observe that we can write Z as the following union of closed subsets

$$Z = \{\zeta\}^- \cup \big((X \setminus U) \cap Z\big)$$

being Z irreducible, and different from $\big((X \setminus U) \cap Z\big)$, we can conclude that $Z = \{\zeta\}^-$. □

2.1.3 Distinguished Open Subsets Let s be a global section of the sheaf $\mathscr{O}_X$. The *distinguished open subset* of X defined by s, denoted X_s, is the subset of points $p \in X$ such that the stalk s_p of s at p is not contained in the maximal ideal $\mathfrak{m}_p$ of the local ring $\mathscr{O}_p$. Observe that when X is affine, say $X = \operatorname{Spec} A$, the global section s is just an element of the ring A and the distinguished open subset of X defined by s is just $D(s)$, thus the terminology is consistent. The next result will clarify the reason why we call X_s *open*.

Proposition (Exercise II.2.16 in Hartshorne, 1977)**.** *If $U = \operatorname{Spec} B$ is an open affine subscheme of X, and if $\bar{s}$ is the restriction of s to $B = \Gamma(U, \mathscr{O}_X|_U)$, then the intersection $U \cap X_s = D(\bar{s})$. In particular X_s is an open subset of X.*

Proof. Indeed each point of U is given by a prime ideal $\mathfrak{p}$ in B, and the local ring $\mathscr{O}_\mathfrak{p}$ is isomorphic to the localisation $B_\mathfrak{p}$. Hence $\mathfrak{p} \in U \cap X_s$ if and only if $\bar{s} \notin \mathfrak{p}$, that is $\mathfrak{p} \in D(\bar{s})$. In particular $X_s \cap U$ is an open set in U, therefore in X also, for any open affine subset of X. Recalling that open affine subsets form a base for the topology in X we conclude that X_s is a union of open subsets hence it is open. □

Definition. Let X be a scheme. For any $x \in X$ let $\mathscr{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the *residue field* of X on x to be the field $k(x) = \mathscr{O}_x/\mathfrak{m}_x$.

Remark. With a section $s \in \Gamma(X, \mathscr{O}_X)$ one can define a sort of function, sending every point $x \in X$ to the value $\overline{s_x}$ of s in the residue field $k(x)$ on x. Even if this function takes values on fields that vary from point to point, we can still give sense at least to the locus where it vanishes. The set X_s is the complement of this locus, so what we have proved is that this locus is closed.

Lemma. *Let $(f, f^\#)\colon X \to Y$ be a morphism of schemes, and let $\sigma \in \Gamma(Y, \mathscr{O}_Y)$. Then $f^{-1}(Y_\sigma) = X_s$ where $s = f_Y^\#(\sigma)$ is the image of σ on the global sections of $\mathscr{O}_X$.*

Proof. For every $x \in X$ we have the following commutative diagram

$$\begin{array}{ccc} \Gamma(Y, \mathscr{O}_Y) & \xrightarrow{f_Y^\#} & \Gamma(X, \mathscr{O}_X) \\ \downarrow & & \downarrow \\ \mathscr{O}_{Y,f(x)} & \xrightarrow{f_x^\#} & \mathscr{O}_{X,x} \end{array}$$

where vertical arrows are localisation maps. Now $f(x) \in Y_\sigma$ if and only if the stalk $\sigma_{f(x)}$ of σ at $f(x)$ is not contained in the maximal ideal $\mathfrak{m}_{f(x)}$ of the local ring $\mathscr{O}_{Y,f(x)}$. Since $f_x^\#$ is a local homomorphism this is equivalent to say that $f_x^\#(\sigma_{f(x)}) = f_Y^\#(\sigma)_x$ is not in $\mathfrak{m}_x$, which is precisely the condition for $x \in X_s$. $\square$

2.1.4 The Adjoint Property of Spec In §1.4 we have already observed that Spec defines a contravariant functor from the category of Commutative Rings to the category of Schemes, now the reader can convince himself that taking global sections defines a contravariant functor Γ going in the opposite direction, we call it the *global sections functor*. From this point of view, the next Theorem asserts that Spec and Γ are adjoint functors (see Mac Lane, 1998, Ch.IV for more information about adjoint functors).

Theorem (I-40 in Eisenbud and Harris, 2000)**.** *Let A be a ring and let $(X, \mathscr{O}_X)$ be a scheme. Given a morphism $f\colon X \to \operatorname{Spec} A$, we have an associated map $f^\#$ on sheaves. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathscr{O}_X)$. Thus there is a natural map*

$$\varepsilon\colon \operatorname{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec} A) \longrightarrow \operatorname{Hom}_{\mathfrak{Rings}}\left(A, \Gamma(X, \mathscr{O}_X)\right)$$

This map ε is bijective.

Proof. We describe the inverse association. Set $Y = \operatorname{Spec} A$, and consider a ring homomorphism $\varphi\colon A \to \Gamma(X, \mathscr{O}_X)$. If $x \in \operatorname{sp}(X)$ is a point, the preimage of the maximal ideal under the composition $A \to \Gamma(X, \mathscr{O}_X) \to \mathscr{O}_{X,x}$ is a prime ideal, so that φ induces a map of sets

$$\eta[\varphi]\colon \operatorname{sp}(X) \longrightarrow \operatorname{sp}(Y),$$

For any $\alpha \in A$ the preimage $\eta[\varphi]^{-1}(D(\alpha))$ is the set $X_{\varphi(\alpha)}$, which we have seen above is open. Thus $\eta[\varphi]$ is continuous. Next we have to define a map of sheaves $\eta[\varphi]^\#$ and we can do it over a base for the topology. So let $U = D(\alpha)$ and define the ring homomorphism $\eta[\varphi]^\#_U\colon \mathscr{O}_Y(U) \to \eta[\varphi]_*\mathscr{O}_X(U)$ to be the composition

$$A_\alpha \longrightarrow \Gamma(X, \mathscr{O}_X)_{\varphi(\alpha)} \longrightarrow \mathscr{O}_X\big(\eta[\varphi]^{-1}(U)\big)$$

obtained by localising φ (note that, over $\mathscr{O}_X(X_{\varphi(\alpha)})$, $\varphi(\alpha)$ defines an invertible element). Localising further, we see that if $\eta[\varphi](x) = \mathfrak{p}$, then $\eta[\varphi]^\#$ defines a local map of local rings $A_\mathfrak{p} \to \mathscr{O}_{X,x}$, and thus $(\eta[\varphi], \eta[\varphi]^\#)$ is a morphism of schemes. Clearly, the induced map satisfies $\varepsilon\left(\eta[\varphi]\right) = \eta[\varphi]^\#_Y = \varphi$ (just set $\alpha = 1$), to complete the proof let $f\colon X \to Y$ be any morphism of schemes and do the above construction with $\varphi = f^\#_Y$. We need to show that in this way we find again the morphism f, in other words that $\eta\left[\varepsilon(f)\right] = f$. First let $x \in X$ be any point and let U be any affine neighborhood of x. We have the following diagram (which commutes by definition)

$$\begin{array}{ccccc} A & \longrightarrow & \Gamma(X, \mathscr{O}_X) & \longrightarrow & \mathscr{O}_X(U) \\ & & & \searrow & \downarrow \\ & & & & \mathscr{O}_{X,x} \end{array}$$

Recalling how morphisms of affine schemes are defined we can conclude that $\eta\left[\varepsilon(f)\right] = f$ at least as continuous maps. Again we take $\alpha \in A$ and consider $U = D(\alpha)$, since $f^\#$ is a morphism of sheaves we have also the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f^\#_Y = \varphi} & \Gamma(X, \mathscr{O}_X) \\ {\scriptstyle r_Y}\downarrow & & \downarrow{\scriptstyle r_X} \\ A_\alpha & \xrightarrow{f^\#_U} & \mathscr{O}_X\big(f^{-1}(U)\big) \end{array}$$

Hence $\eta\left[\varepsilon(f)\right]^\#_U = (r_X \circ f^\#_Y)_\alpha = (r_X)_{\varphi(\alpha)} \circ (f^\#_Y)_\alpha = f^\#_U$, and we are done. □

Example (Initial and final object). The scheme $\operatorname{Spec}\mathbb{Z}$ is a final object for the category of schemes. This follows from the Theorem, recalling that $\mathbb{Z}$ is an initial object for the category of rings.

"This is very important because it shows us that every scheme X is a kind of fibered object, with one fibre for each prime p, and one over the generic point of Spec $\mathbb{Z}$. More concretely, this fibering is given by the function

$$x \longmapsto \operatorname{char} k(x)$$

associating to each x the characteristic of its residue field."

taken from Mumford (1999, §II.2)

The scheme $\operatorname{Spec}(0)$ is given by the empty set equipped with the sheaf $\underline{0}$ sending $\varnothing \mapsto 0$. This is an initial object for the category of schemes. Indeed the empty set is an initial object for topological spaces, and the unique continuous map $i\colon \varnothing \to \operatorname{sp}(X)$ is equipped with the unique morphism of sheaves $i^{\#}\colon \mathscr{O}_X \to \underline{0}$.

2.1.5 A Criterion for Affineness We devote this subsection to the proof of the following theorem, it will establish a criterion to decide when a scheme X is affine.

Theorem (Exercise II.2.17 in Hartshorne, 1977)**.** *A scheme X is affine if and only if there exists a finite set of global sections $s_1 \dots, s_r \in \Gamma(X, \mathscr{O}_X)$, which generates the unit ideal, such that for each $i = 1, \dots, r$ the distinguished open subset X_{s_i} is affine.*

Proof. We only need to prove the "if" part. By the adjointness of Spec and global sections above, the identity map of $A = \Gamma(X, \mathscr{O}_X)$, which for psychological reasons we will denote $\varphi\colon A \to \Gamma(X, \mathscr{O}_X)$, induces a morphism of schemes

$$(f, f^{\#})\colon (X, \mathscr{O}_X) \longrightarrow \operatorname{Spec} A$$

for which $f^{-1}\big(D(s)\big) = X_s$ for any $s \in A$. Since $s_1 \dots, s_r$ generate the unit ideal we can conclude that $X_{s_1}, \dots, X_{s_r}$ cover X, further for each $i = 1, \dots, r$ we know that $X_{s_i} = \operatorname{Spec} A^{(i)}$ is affine. Let s be any element of $\Gamma(X, \mathscr{O}_X)$ and consider the localisation of the restriction map given by

$$\rho_s\colon \Gamma(X, \mathscr{O}_X)_s \longrightarrow \mathscr{O}_X(X_s)$$

We want to prove that it is an isomorphism, remember that the intersection $X_s \cap \operatorname{Spec} A^{(i)}$ is given by $D(\bar{s}) \subseteq \operatorname{Spec} A^{(i)}$. To check injectivity take any global section $a \in \Gamma(X, \mathscr{O}_X)$ and assume that $\rho_s(a/1) = 0$. The restriction of a to $X_s \cap \operatorname{Spec} A^{(i)}$ vanishes, so that there exists an integer t_i such that $\bar{s}^{t_i} a = 0$.

If we now define t to be the greatest among $t_1, \ldots, t_r$ we can conclude that $s^t a = 0$ in $\Gamma(X, \mathscr{O}_X)$, that is $a/1 = 0$. To see that ρ_s is surjective take any b in $\mathscr{O}_X(X_s)$, and let b_{ij} be the restriction of b to $X_s \cap \operatorname{Spec} A^{(i)} \cap \operatorname{Spec} A^{(j)}$, so that $b_{ij} = a_{ij}/\bar{s}^t$ in $A^{(i)}_{s_j s} = A^{(j)}_{s_i s}$ where the integer t is fixed because the covering is finite. In the same ring we then have $\bar{s}^t b_{ij} = a_{ij}$, and by construction the family $a_{ii} \in \mathscr{O}_X(X_{s_i})$ is coherent, so that there exists an element $a \in \Gamma(X, \mathscr{O}_X)$ such that the restriction of a to X_{s_i} is a_{ii}. Now b is given by $\rho_s(a/s^t)$.

Observe now that f defines a morphism $f_i \colon X_{s_i} \longrightarrow \operatorname{Spec} A_{s_i}$ that is induced by ρ_{s_i}, hence it is an isomorphism, and this proves that f itself is an isomorphism. □

2.1.6 Closed Immersions A *closed immersion* is a morphism $f \colon Z \to X$ of schemes such that f induces a homeomorphism of $\operatorname{sp}(Z)$ onto a closed subset of $\operatorname{sp}(X)$, and furthermore the induced map $f^\# \colon \mathscr{O}_X \to f_* \mathscr{O}_Z$ of sheaves on X is surjective. A *closed subscheme* of a scheme X is an equivalence class of closed immersions, where we say $f \colon Z \to X$ and $f' \colon Z' \to X$ are equivalent if there is an isomorphism $i \colon Z' \to Z$ such that $f' = f \circ i$.

To better understand this definition we can look at the affine case, so let $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} B$ be affine schemes, and $f \colon Z \to X$ be induced by a morphism of rings $\varphi \colon A \to B$. In this situation we have already seen in §1.4.6 that f is a closed immersion if and only if φ is surjective. This in particular implies that for any ideal $\mathfrak{a}$ of A, the projection $\pi \colon A \to A/\mathfrak{a}$ induces a closed immersion and that furthermore any affine closed subscheme of X arises in this way. However consider the situation in which $f \colon Z \to \mathbb{A}^n_k$ is a closed immersion, but we don't have any further information on Z. We clearly expect Z to be given by some ideal $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$, in particular we expect Z to be affine, and this is the content of the next result.

Proposition (Exercise II.3.11 in Hartshorne, 1977)**.** *Let $X = \operatorname{Spec} A$ be affine, and let Z be any scheme. If $f \colon Z \to \operatorname{Spec} A$ is a closed immersion then Z is affine.*

Proof. If $U \subseteq Z$ is an open affine subset, say $U = \operatorname{Spec} B$ then, since f is a homeomorphism, there exists an open set $V \subseteq X$ such that $f^{-1}(V) = U$. Let $D(s)$ be an open basic subset of X contained in V, then $f^{-1}(D(s)) \subseteq U$ and as in §2.1.3 it is given by the open basic subset of $\operatorname{Spec} B$ defined by the image of s, in particular it is an affine open subset. Therefore we have proved the following: any open affine subset of Z contains another open affine subset which is given by the preimage of a basic subset of X, in other words there exist an open covering of Z given by open affine subsets of the form $f^{-1}(D(s_i))$.

By adding some more s_i with $f^{-1}(D(s_i)) = \varnothing$ if necessary, we may assume further that the $D(s_i)$ cover $\operatorname{Spec} A$, which is quasi-compact. Hence a finite number of these open subsets will be enough to cover both X and Z, furthermore we know by §1.1.4 that the elements $s_i \in A$ generate the unit ideal.

By adjunction f is induced by $\varphi\colon A \to \Gamma(Z, \mathscr{O}_Z)$ and $f^{-1}\big(D(s_i)\big) = Z_{\varphi(s_i)}$, hence we have a finite set of elements $\varphi(s_i)$ in $\Gamma(Z, \mathscr{O}_Z)$ that generates the unit ideal and such that the open subsets $Z_{\varphi(s_i)}$ are affine. Now we can apply the criterion above. □

2.2 The Projective Spectrum of a Graded Ring

We begin from the classic construction of *projective space* (over k), going back for a moment to where we started. On the set of $(n+1)$-tuples $(a_0, \ldots, a_n)$ of elements of k, not all zero, we define an equivalence relation $\sim$ by setting

$$(a_0, \ldots, a_n) \sim (a'_0, \ldots, a'_n)$$

if there is a nonzero element $\lambda \in k$ such that $(a_0, \ldots, a_n) = \lambda(a'_0, \ldots, a'_n)$. The set of equivalence classes of $\sim$ in $k^{n+1} \setminus \{0\}$ is the space we are interested in, this can be described more geometrically as the set of all the lines through the origin in k^{n+1}. In our setup a line is a closed irreducible subset of the affine space $\mathbb{A}_k^{n+1}$ and the origin is the maximal ideal $(x_0, \ldots, x_n)$; every line is uniquely determined by its generic point and the generic point of a line through the origin is a prime ideal, contained in $(x_0, \ldots, x_n)$ and generated by linear forms. In particular it is generated by *homogeneous* polynomials, it turns out that this property is the right one to retain for this construction to give rise to a scheme.

2.2.1 Definition of Proj – Zariski Topology A *graded ring* is a ring S, together with a decomposition $S = \bigoplus_{d \geq 0} S_d$ of S into a direct sum of abelian groups S_d, such that for any $d, e \geq 0$, $S_d S_e \subseteq S_{d+e}$. An element of S_d is called *homogeneous element of degree d*. Thus every element of S can be written uniquely as a finite sum of homogeneous elements. An ideal $\mathfrak{a} \subseteq S$ is called *homogeneous ideal* if $\mathfrak{a} = \bigoplus_{d \geq 0}(\mathfrak{a} \cap S_d)$. As an example the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ is a graded ring by taking S_d to be the set of all linear combinations of monomials of total weight d. If S is a graded ring we denote by S_+ the ideal $\bigoplus_{d > 0} S_d$, observe that it is a prime ideal.

We define the set $\operatorname{Proj} S$ to be the set of all homogeneous prime ideals $\mathfrak{p}$ which do not contain all of S_+, and introduce in this set the topology induced by the inclusion $\operatorname{Proj} S \subset \operatorname{Spec} S$. Thus a closed subset of $\operatorname{Proj} S$ will be given by the intersection $\mathcal{V}_h(E) = \mathcal{V}(E) \cap \operatorname{Proj} S$, where E is any subset of S; in other words it will consists of homogeneous prime ideals containing the set E. We can therefore assume any element of E to be homogeneous because, by definition, if a homogeneous ideal contains an element then it contains all of its homogeneous components.

Proposition. *With notations and definitions as above, let E be any subset of S consisting of homogeneous elements and let $\mathfrak{a}$ be the homogeneous ideal generated by E. Then*

i) $\mathcal{V}_h(E) = \mathcal{V}_h(\mathfrak{a}) = \mathcal{V}_h(\sqrt{\mathfrak{a}})$*, moreover* $\mathcal{V}_h(\mathfrak{a}) = \mathcal{V}_h(\mathfrak{a}_+)$ *where* $\mathfrak{a}_+ = \mathfrak{a} \cap S_+$

ii) $\mathcal{V}_h(0) = \operatorname{Proj} S$, $\mathcal{V}_h(S_+) = \varnothing$

iii) if $(E_i)_{i\in I}$ *is any family of subsets of S as above, then* $\mathcal{V}_h\left(\bigcup E_i\right) = \bigcap \mathcal{V}_h\left(E_i\right)$

iv) $\mathcal{V}_h(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{V}_h(\mathfrak{a}\mathfrak{b}) = \mathcal{V}_h(\mathfrak{a}) \cup \mathcal{V}_h(\mathfrak{b})$ *for any homogeneous ideals* $\mathfrak{a}$, $\mathfrak{b}$ *of A.*

Proof. Given the analogy with §1.1.3 we only have to show the second part of statement *i*). Since $\mathfrak{a}_+ \subseteq \mathfrak{a}$ we have the inclusion $\mathcal{V}_h(\mathfrak{a}) \subseteq \mathcal{V}_h(\mathfrak{a}_+)$. It remains to prove the other. Let $\mathfrak{p} \in \operatorname{Proj} S$ be a prime ideal that contains $\mathfrak{a}_+$. There is an element α in S_+ such that $\alpha \notin \mathfrak{p}$, therefore for any homogeneous element of degree zero $\beta \in \mathfrak{a}$ we have $\alpha\beta \in \mathfrak{a}_+ \subseteq \mathfrak{p}$ and we can conclude that $\beta \in \mathfrak{p}$. This proves that $\mathfrak{a} \subseteq \mathfrak{p}$. □

2.2.2 Homogeneous Basic Open Sets The open sets corresponding to sets E with just one element again will play a special role, observe however that by the Proposition above we can assume E to be contained in S_+. So for any homogeneous $\alpha \in S_+$, we define the *distinguished* (or *basic*) open subset $D_h(\alpha)$ of $X = \operatorname{Proj} S$ associated with α to be $D(\alpha) \cap \operatorname{Proj} S$ or equivalently to be the complement of $\mathcal{V}_h(\alpha)$. The distinguished open sets form a base for the Zariski topology in the sense that any open set is a union of distinguished ones.

$$U = X \setminus \mathcal{V}_h(E) = X \setminus \left(\bigcap_{\alpha\in E} \mathcal{V}_h(\alpha)\right) = \bigcup_{\alpha\in E} D_h(\alpha)$$

Proposition. *For each homogeneous $\alpha \in S_+$, let $D_h(\alpha)$ denote the distinguished open subset of $X = \operatorname{Proj} S$ associated with α. Then for any homogeneous $\alpha, \beta \in S_+$ we have the following.*

(a) $D_h(\alpha) \cap D_h(\beta) = D_h(\alpha\beta)$

(b) $D_h(\alpha) = \varnothing \Leftrightarrow \alpha$ *is nilpotent*

(c) $D_h(\alpha) \subseteq D_h(\beta) \Leftrightarrow \alpha \in \sqrt{(\beta)}$

This result follows easily from the analog in §1.1.4, however it can be useful to remind the reader the following purely algebraic result concerning homogeneous nilpotent elements.

Lemma. *The nilradical* $\operatorname{Nil}(S)$ *of a graded ring is the intersection of all* homogeneous *prime ideals of S. In particular* $\operatorname{Nil}(S)$ *is a homogeneous ideal.*

Proof. A nilpotent element of S is contained in every prime ideal of S, in particular in every homogeneous prime ideal. Conversely if $\alpha \in S$ is not nilpotent we can find a homogeneous prime ideal that doesn't contain it applying Zorn's Lemma to the inductively ordered set

$$\Sigma = \left\{ \mathfrak{a} \subseteq S \;\middle|\; \begin{array}{c} \mathfrak{a} \text{ is a homogeneous ideal} \\ \alpha^n \notin \mathfrak{a} \;\; \forall n \in \mathbb{N} \end{array} \right\}$$

Note that Σ is not empty for it contains the zero ideal, and that the union of a chain of homogeneous ideals is again a homogeneous ideal. □

2.2.3 Algebra vs Topology In general there is not an homogeneous element $\alpha \in S_+$ such that $D_h(\alpha) = \operatorname{Proj} S$, in other words in general $\operatorname{Proj} S$ is not a basic open set, but at least we can characterise when it is the empty set.

Proposition (Exercise II.2.14 in Hartshorne, 1977)**.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be homogeneous ideals of a graded ring S, then the following properties hold.*

- $\mathcal{V}_h(\mathfrak{a}) \subseteq \mathcal{V}_h(\mathfrak{b})$ *if and only if* $\mathfrak{b}_+ \subseteq \sqrt{\mathfrak{a}}$*;*
- $\operatorname{Proj} S = \varnothing$ *if and only if every element of S_+ is nilpotent;*
- $\mathcal{V}_h(\mathfrak{a}) = \varnothing$ *if and only if* $S_+ \subseteq \sqrt{\mathfrak{a}}$*.*

In particular if E is any subset of S consisting of homogeneous elements and $\mathfrak{a}$ is the homogeneous ideal generated by E, we have

- $\bigcup_{\alpha \in E} D_h(\alpha) = \operatorname{Proj} S$ *if and only if* $S_+ \subseteq \sqrt{\mathfrak{a}}$*.*

Proof. The first statement is a consequence of the Lemma above, more precisely we have

$$\sqrt{\mathfrak{a}} = \bigcap \{\mathfrak{p} \in \mathcal{V}_h(\mathfrak{a})\}$$

while the other two follow from taking into account that $\mathcal{V}_h(0) = \operatorname{Proj} S$ and $\mathcal{V}_h(S_+) = \varnothing$ as in §2.2.1. To obtain the final remark instead, just recall that $\bigcup_{\alpha \in E} D_h(\alpha) = \operatorname{Proj} S \setminus \mathcal{V}_h(\mathfrak{a})$. □

2.2.4 The Scheme Structure over Proj – Projective Space Our goal now is to define the structure of a scheme over Proj S, to this purpose we will show first that we have defined a topological space locally homeomorphic to the spectrum of a ring. The precise statement is the following, where for any homogeneous element $\alpha \in S_+$ we denote by $S_{(\alpha)}$ the subring of elements of degree zero in the localised ring S_α.

Lemma (II.3.36 in Liu, 2002). *Let S be a graded ring and $\alpha \in S_+$ be a homogeneous element of degree r. Then*

(a) *there exists a canonical homeomorphism $\theta\colon D_h(\alpha) \to \operatorname{Spec} S_{(\alpha)}$;*

(b) *if $D_h(\beta) \subseteq D_h(\alpha)$ then $\theta\big(D_h(\beta)\big) = D(m)$ where $m = \beta^r/\alpha^{\deg\beta} \in S_{(\alpha)}$;*

(c) *the canonical homomorphism $S_{(\alpha)} \to S_{(\beta)}$ induces an isomorphism between the localised rings $\big(S_{(\alpha)}\big)_m \cong S_{(\beta)}$.*

The proof of this Lemma is based on the following remark: Proj S is a subset of Spec S and for any homogeneous $\alpha \in S_+$ the basic open subset $D_h(\alpha)$ is equal to the intersection $\operatorname{Proj} S \cap D(\alpha)$. Therefore we can define a continuous function $\theta\colon D_h(\alpha) \to \operatorname{Spec} S_{(\alpha)}$ as the restriction of the canonical map induced by the inclusion $S_{(\alpha)} \to S_\alpha$.

Definition. Let S be a graded ring and let T be a multiplicative system consisting of homogeneous elements of S. We define the *homogeneous localisation* of S in T as the subring of elements of degree zero in the localised ring $T^{-1}S$. For each $\mathfrak{p} \in \operatorname{Proj} S$, we define the ring $S_{(\mathfrak{p})}$ to be the homogeneous localisation of S in the multiplicative system consisting of all *homogeneous* elements of S which are not in $\mathfrak{p}$.

Proposition. *Let S be a graded ring and let A be the degree zero part of S. Then the following properties define on $X = \operatorname{Proj} S$ a unique sheaf of rings $\mathscr{O}_X$ such that the resulting ringed space $(X, \mathscr{O}_X)$ is a scheme.*

(a) *For any homogeneous $\alpha \in S_+$, we have $\mathscr{O}_X\big(D_h(\alpha)\big) = S_{(\alpha)}$,*

(b) *For any $\mathfrak{p} \in \operatorname{Proj} S$, the stalk $\mathscr{O}_{X,\mathfrak{p}}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.*

Further the scheme Proj S *comes endowed with a natural morphism $X \to \operatorname{Spec} A$.*

Proof. Among our references we find again two different ways of constructing $\mathscr{O}_X$. The direct approach in Liu (2002, Proposition II.3.38) consists in observing that property (a) above characterises a sheaf as we have seen in §1.3 (the

Theorem in there), provided that the gluing condition is satisfied. The more geometric definition of Hartshorne (1977) is instead the following.

For any open subset $U \subseteq \operatorname{Proj} S$, we define $\mathscr{O}(U)$ to be the set of functions $s\colon U \to \coprod_{\mathfrak{p}\in U} S_{(\mathfrak{p})}$, such that $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ for each $\mathfrak{p}$ and such that s is locally a quotient of elements of S: for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$, contained in U, and elements $\alpha, \beta \in S$, of the same degree, such that for all $\mathfrak{q} \in V$ we have that $\beta \notin \mathfrak{q}$, and $s(\mathfrak{q}) = \alpha/\beta$ in $S_{(\mathfrak{p})}$.

Observe that elements of A give rise in a natural way to global sections of $\mathscr{O}_X$, this defines a homomorphism $A \to \Gamma(X, \mathscr{O}_X)$ that in turns defines the natural morphism $X \to \operatorname{Spec} A$ by the adjunction of §2.1.4. □

Let A be any ring, we define *projective n-space* over A to be the projective spectrum of the ring of polynomials $A[x_0, \dots, x_n]$, that is

$$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$$

In view of the Proposition there is a natural affine covering of $\mathbb{P}^n_A$ given by $n+1$ copies of affine n-space over A, as follows

$$\mathbb{P}^n_A = D_h(x_0) \cup \dots \cup D_h(x_n)$$

Where for $i = 0, \dots, n$ we have

$$D_h(x_i) = \operatorname{Spec} A\left[x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i\right]$$

2.2.5 Functorial Properties of Proj Now that we have learned how to construct a scheme from a graded ring we are going to investigate possible functorial properties. Let $\varphi\colon S \to T$ be a graded homomorphism of graded rings (preserving degrees) and let $U \subseteq \operatorname{Proj} T$ be the set of all homogeneous prime ideals in T that do not contain the whole of $\varphi(S_+)$. Note that U is the complement of $\mathcal{V}_h\left(\varphi(S_+)\right)$, in particular it is an open set of $\operatorname{Proj} T$.

Proposition (Exercise II.2.14 in Hartshorne, 1977)**.** *With notations and definitions as above, φ determines a natural morphism $(f, f^\#)\colon U \to \operatorname{Proj} S$. Further the continuous function f satisfies*

i) If $s \in S_+$ then $f^{-1}\big(D_h(s)\big) = D_h\big(\varphi(s)\big)$.

ii) If $\mathfrak{a}$ is an homogeneous ideal of S, then $f^{-1}\big(\mathcal{V}_h(\mathfrak{a})\big) = \mathcal{V}_h(\mathfrak{a}^e) \cap U$.

Proof. We start with the continuous function $g\colon \operatorname{Spec} T \to \operatorname{Spec} S$ defined by φ between the spectra, and observe that U is precisely the open set $g^{-1}(\operatorname{Proj} S)$.

Then we define f to be the restriction of g, from U to $\operatorname{Proj} S$. For any homogeneous $s \in S_+$ we have the obvious inclusion $g^{-1}\big(D_h(s)\big) \subseteq U$, therefore

$$f^{-1}\big(D_h(s)\big) = g^{-1}\big(D(s)\big) \cap \operatorname{Proj} T = D_h\big(\varphi(s)\big)$$

Now we have to define a morphism of sheaves $f^\#\colon \mathscr{O}_Y \to f_*\mathscr{O}_X|_U$ where $Y = \operatorname{Proj} S$ and $X = \operatorname{Proj} T$. For any homogeneous $s \in S_+$ consider the basic open set $D_h(s)$ and using $i)$ above define $f^\#_s\colon S_{(s)} \to T_{(\varphi(s))}$ to be the localisation of φ. □

Unfortunately Proj doesn't satisfy all the good functorial properties of Spec, it is already worrying that the induced morphism f is not defined on the whole of $\operatorname{Proj} T$ but that's not the only reason of concern. For instance f can be an isomorphism even when φ is not.

Lemma (Exercise II.2.14 in Hartshorne, 1977)**.** *Let S and T be two graded rings and let $\varphi\colon S \to T$ be a graded homomorphism of graded rings (preserving degrees). Assume that $\varphi_d\colon S_d \to T_d$ is an isomorphism for all $d \geq d_0$, where d_0 is an integer. Then $U = \operatorname{Proj} T$ and the induced morphism $f\colon \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.*

Proof. The open set U is given by the complement of $\mathcal{V}_h\left(\varphi(S_+)\right)$, and $\varphi(S_+)$ contains the ideal $\mathfrak{a} = \bigoplus_{d \geq d_0} T_d$. We know from §2.2.1 that $\mathcal{V}_h(\mathfrak{a})$ is the same as $\mathcal{V}_h(\sqrt{\mathfrak{a}})$, but $\sqrt{\mathfrak{a}}$ contains T_+, indeed for any homogeneous $t \in T_+$ there is an integer n such that the degree of t^n is greater than d_0. In other words we have the following, which proves that $U = \operatorname{Proj} T$.

$$\mathcal{V}_h\left(\varphi(S_+)\right) \subseteq \mathcal{V}_h(\mathfrak{a}) = \mathcal{V}_h(\sqrt{\mathfrak{a}}) \subseteq \mathcal{V}_h(T_+) = \varnothing$$

Now for any homogeneous $s \in S_+$ we have seen in the Proposition above that $f^{-1}\big(D_h(s)\big) = D_h\big(\varphi(s)\big)$ and the localisation of φ induces a morphism of affine schemes, namely the restriction

$$f|^{D_h(s)}\colon \operatorname{Spec} T_{(\varphi(s))} \longrightarrow \operatorname{Spec} S_{(s)}$$

But in our hypotheses the localisation $S_{(s)} \to T_{(\varphi(s))}$ is an isomorphism, therefore the restriction of f to any basic open set is an isomorphism, and so f itself is an isomorphism. □

2.2.6 Surjective Homomorphisms and Closed Immersions Let A be a ring and let $R = A[x_0, \dots, x_n]$. Consider the case of a surjective homomorphism of graded rings $\varphi\colon R \to R/I$ where I is a homogeneous ideal of R. People working in Algebraic Geometry are mainly interested in objects like $\operatorname{Proj} R/I$, which are called *projective schemes*. The reason for the name is that the homomorphism φ induces a closed immersion into projective space. More precisely we have the following result.

Proposition (Exercise II.3.12 in Hartshorne, 1977)**.** *Let $\varphi\colon S \to T$ be a surjective graded homomorphism of graded rings (preserving degrees). Then the natural morphism $f\colon \operatorname{Proj} T \to \operatorname{Proj} S$ determined by φ is a closed immersion.*

Proof. We have seen how f is defined in §2.2.5 above, but clearly we have to check that actually the open set U is the whole of $\operatorname{Proj} T$. So let $\mathfrak{q} \in \operatorname{Proj} T$, then there exists an element $t \in T_+$ such that $t \notin \mathfrak{q}$; since φ is surjective we have $t = \varphi(s)$ and this shows that $\varphi(S_+) \not\subseteq \mathfrak{q}$, that is $\mathfrak{q} \in U$. Next we observe that, with the same notations as in §2.2.5, g is a homeomorphism onto the closed subset $\mathcal{V}(\ker\varphi)$ of $\operatorname{Spec} T$ and since f is just the restriction of g we can conclude immediately that f is bijective. Moreover for any homogeneous ideal $\mathfrak{b} \subseteq T$ we have

$$f\big(\mathcal{V}_h(\mathfrak{b})\big) = g\big(\mathcal{V}(\mathfrak{b})\big) \cap \operatorname{Proj} S = \mathcal{V}\left(g^{-1}(\mathfrak{b})\right) \cap \operatorname{Proj} S$$

which proves f is closed. Now we have to prove that the morphism of schemes $f^{\#}$ is surjective, but in fact for any $s \in S$ we have $f^{\#}_s\colon S_{(s)} \to T_{(\varphi(s))}$ is surjective. □

Remark (Projective Schemes and Homogeneous Ideals). According to the result just proved any homogeneous ideal $I \subseteq A[x_0, \ldots, x_n]$ defines a projective scheme, but we have to be careful here because different homogeneous ideals can give rise to the same projective scheme. If for example we define the ideal I' to be the sum $\bigoplus_{d \geq d_0} I_d$ where d_0 is an integer, the projection $R/I' \to R/I$ defines an isomorphism in all large enough degrees. We can then apply the Lemma above and conclude that $\operatorname{Proj} R/I'$ is in fact isomorphic to $\operatorname{Proj} R/I$.

Let $X = \operatorname{Proj} T$ and $Y = \operatorname{Proj} S$ be projective schemes. We may ask under which conditions a homomorphism of graded rings $\varphi\colon S \to T$ defines a morphism of schemes $f\colon X \to Y$ on the whole of X. With the notations of §2.2.5 we are asking for U to be the whole of $\operatorname{Proj} T$. We have seen above that this is the case when φ_d is an isomorphism for all large enough d (in which case f is an isomorphism), or also when φ is surjective (in which case f is a closed immersion). Another obvious possibility is to have the homomorphism φ_d to be surjective for all large enough d, in which case f will be a closed immersion. Indeed we have the factorisation $S \to \operatorname{Im}\varphi \to T$ where the first homomorphism induces a closed immersion and the second an isomorphism. It turns out that there are not so many other possibilities, we have in fact the following characterisation.

Lemma. *Let S and T be two graded rings, with T Noetherian, and let $\varphi\colon S \to T$ be a graded homomorphism of graded rings (preserving degrees). Then the induced morphism of schemes f is defined on the whole of $\operatorname{Proj} T$ if and only if $T_+^r \subseteq S_+^e$ for some integer r*

Proof. We only have to recall some earlier results. The morphism f is defined on the whole of Proj T if and only if $\mathcal{V}_h\left(\varphi(S_+)\right) = \varnothing$, and this happens if and only if T_+ is contained in $\sqrt{\mathfrak{a}}$ where $\mathfrak{a} \subseteq T$ is the ideal generated by $\varphi(S_+)$. Since T is Noetherian the latter is possible if and only if $T_+^r \subseteq \mathfrak{a}$ for some integer r. □

2.3 Gluing Constructions

The idea of gluing together different schemes is very natural, after all by definition a scheme is obtained by gluing together affine schemes. Nevertheless it turns out to be quite difficult to grasp, mainly because it involves a complete change of point of view: so far we had a scheme and we gradually restricted our attention to more specific subschemes, now we are "only" given the subschemes. In other words it seems as we were neglecting to understand the scheme as a whole. Of course this is not the case, but it will take some time for the beginner to be aware of it, for now let him think through the definitions and the constructions that we are about to describe. We need to be precise, in particular we need a couple of general results to be applied in the greatest possible variety of applications, so we take nothing for granted and we begin with sheaves.

2.3.1 Gluing Sheaves Temporarily we let X be a topological space, we fix an open covering $\{U_i\}$ of X and we assume to be given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism

$$\varphi_{ij}\colon \mathscr{F}_i|_{U_i\cap U_j} \longrightarrow \mathscr{F}_j|_{U_i\cap U_j}$$

To fix notations we recall that φ_{ij} is given by a collection of isomorphisms, one for each open set $W \subseteq U_i \cap U_j$

$$\varphi_{ij_W}\colon \mathscr{F}_i(W) \longrightarrow \mathscr{F}_j(W)$$

therefore it makes sense to speak about the restriction of φ_{ij} to the open set $U_i \cap U_j \cap U_k$, this is given by the collection φ_{ij_W} for $W \subseteq U_i \cap U_j \cap U_k$. We will denote it again with φ_{ij}.

Now we need to make a couple of assumptions on the isomorphisms φ_{ij}: first we want φ_{ii} to be the identity of the sheaf $\mathscr{F}_i$ for each i, and secondly on the intersection to $U_i \cap U_j \cap U_k$, we want to have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$, for each i, j, k. The latter is called *cocycle condition*, from cohomology theory, its meaning is better expressed by a diagram that we assume to be commutative

$$\begin{array}{ccc} \mathscr{F}_i|_{U_i\cap U_j\cap U_k} & \xrightarrow{\varphi_{ik}} & \mathscr{F}_k|_{U_i\cap U_j\cap U_k} \\ \varphi_{ij}\Big\downarrow & \nearrow_{\varphi_{jk}} & \\ \mathscr{F}_j|_{U_i\cap U_j\cap U_k} & & \end{array}$$

Proposition (Exercise II.1.22 in Hartshorne, 1977)**.** *Let X be a topological space and let $\{U_i\}$ be an open covering of X. Assume that we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism*

$$\varphi_{ij} \colon \mathscr{F}_i|_{U_i\cap U_j} \longrightarrow \mathscr{F}_j|_{U_i\cap U_j}$$

such that for each i, φ_{ii} is the identity of the sheaf $\mathscr{F}_i$ and for each i, j, k, on the intersection $U_i \cap U_j \cap U_k$ we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$.

Then there exists a unique sheaf $\mathscr{F}$ on X, together with isomorphisms

$$\psi_i \colon \mathscr{F}|_{U_i} \to \mathscr{F}_i$$

such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathscr{F}$ is obtained by gluing *the sheaves $\mathscr{F}_i$ via the isomorphisms φ_{ij}.*

Proof. If such a sheaf exists it is necessarily unique up to isomorphism, indeed assume we have two such sheaves $\mathscr{F}$ and $\mathscr{H}$, equipped with isomorphisms $\psi_i \colon \mathscr{F}|_{U_i} \to \mathscr{F}_i$ and $\eta_i \colon \mathscr{H}|_{U_i} \to \mathscr{F}_i$, in order to define an isomorphism $\theta \colon \mathscr{F} \to \mathscr{H}$ we consider first the isomorphisms

$$\theta_i = \eta_i^{-1} \circ \psi_i \colon \mathscr{F}|_{U_i} \longrightarrow \mathscr{H}|_{U_i}$$

now for any open subset $U \subseteq X$, and for any section $s \in \mathscr{F}(U)$, the family $\{\theta_{i U\cap U_i}(s|_{U\cap U_i})\}$ is a coherent family for the sheaf $\mathscr{H}$ and therefore it defines a unique element $\theta(s)$ in $\mathscr{H}(U)$. To check coherence observe that on $U_i \cap U_j$ we have

$$\begin{aligned} \theta_i = \eta_i^{-1} \circ \psi_i &= \eta_i^{-1} \circ \varphi_{ij}^{-1} \circ \varphi_{ij} \circ \psi_i \\ &= (\varphi_{ij} \circ \eta_i)^{-1} \circ (\varphi_{ij} \circ \psi_i) = \eta_j^{-1} \circ \psi_j = \theta_j \end{aligned}$$

The morphism θ is in fact an isomorphism, indeed we can define its inverse $\rho \colon \mathscr{H} \to \mathscr{F}$ simply considering the family of isomorphisms $\rho_i = \psi_i^{-1} \circ \eta_i$.

Now we come to the actual construction of a sheaf $\mathscr{F}$. For each open subset $V \subseteq X$, define $\mathscr{F}(V)$ as the following inverse limit

$$\mathscr{F}(V) = \varprojlim_{i} \mathscr{F}_i(V \cap U_i)$$

recall that this is defined to be the set of all the families $(s_i) \in \prod_i \mathscr{F}_i(V \cap U_i)$ such that

$$\varphi_{ij}{}_{V\cap U_i\cap U_j}(s_i|_{V\cap U_i\cap U_j}) = s_j|_{V\cap U_i\cap U_j}$$

It is straightforward to prove that $\mathscr{F}$ is a sheaf. Next for each open subset $V \subseteq U_i \subseteq X$ define $\psi_{iV} \colon \mathscr{F}(V) \to \mathscr{F}_i(V)$ as the natural projection map from the inverse limit. It is clear that in this way we define a morphism of sheaves $\psi_i \colon \mathscr{F}|_{U_i} \to \mathscr{F}_i$, and also that $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. In order to show that ψ_i is an isomorphism it is enough to prove that ψ_{iV} is an isomorphism, that is injective and surjective. Take any section $s \in \mathscr{F}_i(V) = \mathscr{F}_i(V \cap U_i)$ and define the family $(s_j) \in \prod_j \mathscr{F}_j(V \cap U_j)$ as $s_j = \varphi_{ij}{}_{V\cap U_j}(s|_{V\cap U_j})$; this is an element $\mathbf{z}$ of $\mathscr{F}(V)$ such that $\psi_{iV}(\mathbf{z}) = s$. Hence ψ_{iV} is surjective it remains to prove it is injective. Let $\mathbf{z} = (s_j) \in \mathscr{F}(V)$ such that $\psi_{iV}(\mathbf{z}) = 0$, that is $s_i = 0$. Then for each $j \neq i$ we have $V \cap U_i \cap U_j = V \cap U_j$, so that $s_j = \varphi_{ij}(s_i) = \varphi_{ij}(0) = 0$, hence $\mathbf{z} = 0$. □

The reader will have noticed that the process of gluing sheaves consists almost exclusively in some clever use of the sheaf axiom. Bearing this in mind will help to understand every gluing construction, since sheaves play a central role in the theory of schemes.

2.3.2 Gluing Morphisms What we want to do now is to glue morphisms of schemes. We start with a couple of trivial but really important remarks: given a morphism of schemes $f \colon X \to Y$, for any open set U of X we can define the *restriction of* f simply as the composition

$$U \overset{i}{\hookrightarrow} X \overset{f}{\longrightarrow} Y$$

that is $f|_U = f \circ i$, where i is the open immersion of $(U, \mathscr{O}_X|_U)$ in $(X, \mathscr{O}_X)$. This is completely obvious when speaking about continuous functions, and the reader may find it as well obvious in dealing with schemes, that is considering the extra structure given by the sheaf of rings, but it is in fact really useful to keep in mind.
There is also another way of restricting a function f, this time starting with any open set V of Y. It is again trivial to observe that there exists a unique function $f|^V \colon f^{-1}(V) \to V$ such that the following diagram commutes

$$\begin{array}{ccc} f^{-1}(V) & \xrightarrow{f|^V} & V \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array}$$

That is $f|_{f^{-1}(V)} = j \circ f|^{V}$. But again I would like to stress that we are dealing with schemes, and invite the reader to think about what happens to the structure sheaf.

Proposition. *Let X and Y be schemes, and let $\{U_i\}$ be an open covering of X. Assume to be given a family of morphisms $f_i \colon U_i \to Y$ such that the restrictions of f_i and f_j to $U_i \cap U_j$ are the same for any i, j. Then there exists a unique morphism of schemes $f \colon X \to Y$ such that $f|_{U_i} = f_i$.*

Proof. We want to define a morphism $(f, f^{\#}) \colon X \to Y$ such that the following diagram is commutative

$$\begin{array}{ccc} U_i & \xrightarrow{f_i} & Y \\ {\scriptstyle s_i}\big\downarrow & \nearrow_{f} & \\ X & & \end{array}$$

First define $f \colon \mathrm{sp}(X) \to \mathrm{sp}(Y)$ as follows: for any $x \in X$ there exists i such that $x \in U_i$, set $f(x) = f_i(x)$. This is the only possible definition compatible with the diagram and in view of our assumptions f is in fact a well defined continuous function.

For any open set $W \subseteq Y$ we have now to define a ring homomorphism $f^{\#}_W \colon \mathscr{O}_Y(W) \to \mathscr{O}_X\big(f^{-1}(W)\big)$ such that the induced diagram of sheaves over Y is commutative

$$\begin{array}{ccc} \mathscr{O}_Y(W) & \xrightarrow{f^{\#}_W} & \mathscr{O}_X\big(f^{-1}(W)\big) \\ & {\scriptstyle f^{\#}_{iW}}\searrow & \big\downarrow {\scriptstyle s^{\#}_{i\,f^{-1}(W)}} \\ & & \mathscr{O}_X(f^{-1}(W) \cap U_i) \end{array}$$

remember that $s^{\#}_{i\,f^{-1}(W)}$ is the restriction map, since s_i is an inclusion, and for any i we have

$$f^{\#}_{iW} \colon \mathscr{O}_Y(W) \longrightarrow \mathscr{O}_X\big(f_i^{-1}(W)\big) = \mathscr{O}_X\big(f^{-1}(W) \cap U_i\big)$$

Since the restrictions of f_i and f_j to $U_i \cap U_j$ are the same for any i, j we have the following commutative diagram of morphisms of schemes

$$\begin{array}{ccc} U_i \cap U_j & \hookrightarrow & U_i \\ \big\downarrow & & \big\downarrow {\scriptstyle f_i} \\ U_j & \xrightarrow{f_j} & Y \end{array}$$

which induces the following commutative diagram on sections of the structure sheaves

$$\begin{array}{ccc} \mathscr{O}_Y(W) & \xrightarrow{f_{iW}^{\#}} & \mathscr{O}_X(f^{-1}(W)\cap U_i) \\ {\scriptstyle f_{jW}^{\#}}\downarrow & & \downarrow{\scriptstyle res} \\ \mathscr{O}_X(f^{-1}(W)\cap U_j) & \xrightarrow{res} & \mathscr{O}_X(f^{-1}(W)\cap U_i\cap U_j) \end{array}$$

Therefore for any element $s \in \mathscr{O}_Y(W)$ the set

$$\{f_{iW}^{\#}(s) \in \mathscr{O}_X(f^{-1}(W)\cap U_i)\}$$

is a coherent family of sections and hence defines a unique element $f_W^{\#}(s)$ in $\mathscr{O}_X\big(f^{-1}(W)\big)$. Again this is the only possible definition for $f_W^{\#}$ compatible with the diagram and it is now immediate to check that it produces in fact a homomorphism. □

Corollary. *Let X and Y be schemes. Assume to be given a family of morphisms $f_i\colon U_i \to V_i$ such that*

(1) *$\{U_i\}$ is an open covering of X and $\{V_i\}$ is an open covering of Y;*

(2) *$f_i^{-1}(V_i \cap V_j) = U_i \cap U_j$ for any i and j;*

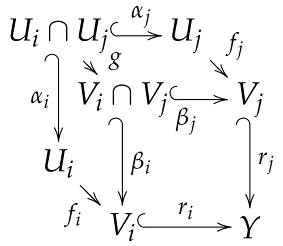
(3) *$f_i|^{V_i\cap V_j} = f_j|^{V_i\cap V_j}$.*

Then there exists a unique morphism of schemes $f\colon X \to Y$ such that $f|^{V_i} = f_i$.

Proof. We want to define a morphism $f\colon X \to Y$ such that the following diagram is commutative

$$\begin{array}{ccc} U_i & \xrightarrow{f_i} & V_i \\ {\scriptstyle s_i}\downarrow & & \downarrow{\scriptstyle r_i} \\ X & \overset{f}{\dashrightarrow} & Y \end{array}$$

that is $r_i \circ f_i = f \circ s_i = f|_{U_i}$, hence it is enough to glue together the morphisms $r_i \circ f_i$. We have to check that these morphisms agree on the overlaps $U_i \cap U_j$, so consider the following diagram where $g = f_i|^{V_i\cap V_j} = f_j|^{V_i\cap V_j}$



this is commutative by assumptions (2) and (3), hence

$$(r_i \circ f_i)|_{U_i \cap U_j} = r_i \circ f_i \circ \alpha_i = r_j \circ f_j \circ \alpha_j = (r_j \circ f_j)|_{U_i \cap U_j}$$

so the family of morphisms $r_i \circ f_i$ verifies the hypotheses of the Proposition above. □

2.3.3 The Gluing Lemma We are now arriving to the central construction, the actual Gluing Lemma. We assume to be given a family of schemes $\{X_i\}$ (possible infinite) that we want to use to construct a scheme X, more precisely we want each X_i to be identified with an open subscheme of X, in other words we want an open immersion $\psi_i \colon X_i \to X$ for each i. First, we need to understand what this means in terms of topological spaces: since we want our object to be endowed with a sort of injection maps we will have to consider the sum of the topological spaces $\mathrm{sp}(X_i)$, that is the disjoint union $\coprod_i \mathrm{sp}(X_i)$. This embeds each X_i as an open set but leaves them disjoint, while in general we will want the image of X_i to intersect that of X_j. For this reason we will need to know also something about the local structure on X, in other words on each X_i we want to find an open set U_{ij} to play in some sense the role of $X_i \cap X_j$.

Lemma (Exercise II.2.12 in Hartshorne, 1977). *Let $\{X_i\}$ be a family of schemes (possible infinite). On any of these schemes X_i, suppose given for each j an open subscheme $U_{ij} \subseteq X_i$ and an isomorphism of schemes $\varphi_{ij} \colon U_{ij} \to U_{ji}$ such that*

(*a*) *$U_{ii} = X_i$ and φ_{ii} is the identity morphism;*

(*b*) *for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$;*

(*c*) *for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$;*

(*d*) *$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.*

Then there exists a unique scheme X, equipped with an open immersion $\psi_i \colon X_i \to X$ for each i, such that the open sets $\psi_i(X_i)$ cover X, for any j the image $\psi_i(U_{ij})$ is the intersection $\psi_i(X_i) \cap \psi_j(X_j)$ and $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

We say that X is obtained by gluing the schemes X_i along the isomorphisms φ_{ij}.

Proof. First we prove that if such a scheme exists it is necessarily unique up to isomorphism. So assume there exist two schemes X and Y, equipped with an open immersion $\psi_i \colon X_i \to X$ and $\xi_i \colon X_i \to Y$ for each i. We restrict these two immersions on the codomain to obtain two isomorphisms $\widetilde{\psi}_i \colon X_i \to U_i$ and

$\widetilde{\xi}_i\colon X_i \to V_i$ where obviously we have denoted $U_i = \psi_i(X_i)$ and $V_i = \xi_i(X_i)$. Now we define a family of isomorphisms $f_i\colon U_i \to V_i$ as the composition

$$U_i \xrightarrow{\widetilde{\psi}_i^{\,-1}} X_i \xrightarrow{\widetilde{\xi}_i} V_i$$

This family satisfies the hypotheses of the Corollary in §2.3.2 and therefore glue together into an isomorphism $f\colon X \to Y$. Indeed $\{U_i\}$ is an open covering of X and $\{V_i\}$ is an open covering of Y, further for any i and j we have

$$\begin{aligned} f_i^{-1}(V_i \cap V_j) &= f_i^{-1}\big(\xi_i(X_i) \cap \xi_j(X_j)\big) = f_i^{-1}\big(\xi_i(U_{ij})\big) \\ &= \widetilde{\psi}_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j) = U_i \cap U_j \end{aligned}$$

now we know that on U_{ij} we have both $\psi_i = \psi_j \circ \varphi_{ij}$ and $\xi_i = \xi_j \circ \varphi_{ij}$, hence the following diagram is commutative

$$\begin{tikzcd} \psi_i(X_i) \cap \psi_j(X_j) \arrow[r, "\widetilde{\psi}_i^{\,-1}"] \arrow[d, "\widetilde{\psi}_j^{\,-1}"'] & U_{ij} \arrow[d, "\widetilde{\xi}_i"] \arrow[dl, "\varphi_{ij}"'] \\ U_{ji} \arrow[r, "\widetilde{\xi}_j"] & \xi_i(X_i) \cap \xi_j(X_j) \end{tikzcd}$$

And this is equivalent to say $f_i|^{V_i \cap V_j} = f_j|^{V_i \cap V_j}$.

Now we come to the actual construction of a scheme X such as in the statement, at first we construct a topological space and then we endow it with a sheaf of rings. So let Y be the disjoint union of the topological spaces X_i and define on Y the equivalence relation

$$x \sim y \iff \varphi_{ij}(x) = y$$

Let X be the quotient space $Y/\sim$ and let $\pi\colon Y \to X$ be the projection. A subset $U \subseteq X$ is open if and only if $\pi^{-1}(U)$ is open in Y. Moreover $\pi^{-1}(U)$ is open in Y if and only if $\pi^{-1}(U) \cap X_i$ is open in X_i for all i. In particular $\pi(X_i)$ is an open set of X, since

$$\pi^{-1}\big(\pi(X_i)\big) \cap X_j = U_{ji} \text{ for all } j$$

it follows that the composition of the natural injection of X_i in the disjoint union Y with the projection to the quotient π is a homeomorphism of X_i into an open subset of X, in other words we define $\psi_i\colon X_i \to X$ as

$$X_i \hookrightarrow \textstyle\coprod_j X_j \xrightarrow{\pi} Y/\sim$$

In this way the open sets $\psi_i(X_i)$ cover X, for any j the image $\psi_i(U_{ij})$ is the intersection $\psi_i(X_i) \cap \psi_j(X_j)$ and $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

Now we set $W_i = \psi_i(X_i)$ and consider the homeomorphism $\widetilde{\psi}_i \colon X_i \to W_i$. We define a sheaf of rings $\mathscr{F}_i$ on W_i as $\widetilde{\psi}_{i*}\mathscr{O}_{X_i}$, in other words we define for any $U \subseteq W_i$

$$\mathscr{F}_i(U) = \mathscr{O}_{X_i}\big(\psi_i^{-1}(U)\big)$$

For any open set V in X_j contained in U_{ji} we have an isomorphism

$$\varphi^{\#}_{ij\,V} \colon \mathscr{O}_{X_j}(V) \longrightarrow \mathscr{O}_{X_i}\big(\varphi_{ij}^{-1}(V)\big)$$

Observe that $\widetilde{\psi}_j^{\,-1}(U) \subseteq U_{ji}$ for any $U \subseteq W_i \cap W_j$, moreover we have

$$\varphi_{ij}^{-1}\big(\widetilde{\psi}_j^{\,-1}(U)\big) = (\widetilde{\psi}_j \circ \varphi_{ij})^{-1}(U) = \widetilde{\psi}_i^{\,-1}(U)$$

hence for any $U \subseteq W_i \cap W_j$ we have an isomorphism of rings

$$\varphi^{\#}_{ij\,\widetilde{\psi}_j^{-1}(U)} \colon \mathscr{O}_{X_j}\big(\widetilde{\psi}_j^{\,-1}(U)\big) \longrightarrow \mathscr{O}_{X_i}\big(\widetilde{\psi}_i^{\,-1}(U)\big)$$

So that $\varphi^{\#}_{ij}$ induces an isomorphism of sheaves

$$\sigma_{ji} \colon \mathscr{F}_j|_{W_i \cap W_j} \longrightarrow \mathscr{F}_i|_{W_i \cap W_j}$$

Now the isomorphisms σ_{ij} satisfy the following properties: for each i, σ_{ii} is the identity because $\varphi^{\#}_{ii}$ is the identity, and for any three indices i, j, k, on the intersection $W_i \cap W_j \cap W_k$ we have $\sigma_{ik} = \sigma_{jk} \circ \sigma_{ij}$, that is for any open subset $U \subseteq W_i \cap W_j \cap W_k$ the following diagram is commutative

$$\begin{array}{ccc}
\mathscr{F}_i(U) & \xrightarrow{\sigma_{ij\,V}} & \mathscr{F}_j(U) \\
& \searrow^{\sigma_{ik\,V}} & \downarrow{\scriptstyle \sigma_{jk\,V}} \\
& & \mathscr{F}_k(U)
\end{array}$$

Indeed writing down what this means in terms of φ_{ij}, and denoting by V the open set $\psi_i^{-1}(U)$ we obtain the diagram

$$\begin{array}{ccccc}
\mathscr{O}_{X_i}\big(\psi_i^{-1}(U)\big) & \xrightarrow{\varphi^{\#}_{ji\,V}} & \mathscr{O}_{X_j}\big(\psi_j^{-1}(U)\big) & = & \mathscr{O}_{X_j}\Big(\varphi_{ji}^{-1}\big(\psi_i^{-1}(U)\big)\Big) \\
& \searrow^{\varphi^{\#}_{ki\,V}} & \downarrow{\scriptstyle \varphi^{\#}_{kj}} & & \\
& & \mathscr{O}_{X_k}\big(\psi_k^{-1}(U)\big) & = & \mathscr{O}_{X_j}\Big(\varphi_{ki}^{-1}\big(\psi_i^{-1}(U)\big)\Big)
\end{array}$$

which is commutative since we are assuming $\varphi_{ki} = \varphi_{ji} \circ \varphi_{kj}$ as morphisms of schemes; observe that $\varphi_{kj}^{\#}$ is defined over $\varphi_{ji}^{-1}\big(\psi_i^{-1}(U)\big)$.

Thus the sheaves $\mathscr{F}_i$ verify the gluing conditions, and therefore there exists a unique sheaf $\mathscr{O}_X$ on X, together with isomorphisms $\psi_i^{\#}\colon \mathscr{F}|_{U_i} \to \mathscr{F}_i$, such that for each i, j, $\psi_j^{\#} = \sigma_{ij} \circ \psi_i^{\#}$ on $U_i \cap U_j$.

The locally ringed space $(X, \mathscr{O}_X)$ is obviously a scheme (for any point there is an open neighborhood which is a scheme) and moreover it is endowed with an open immersion $\psi_i\colon X_i \to X$ for each i, such that the open sets $\psi_i(X_i)$ cover X, for any j the image $\psi_i(U_{ij})$ is the intersection $\psi_i(X_i) \cap \psi_j(X_j)$ and $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}. □

2.3.4 Another construction of Projective Space Let k be an algebraically closed field. We will construct the scheme $\mathbb{P}_k^n$ by means of a gluing argument. Start with the ring $k[x_0, \dots, x_n]$ and form the localisation

$$A = k[x_0, x_0^{-1}, \dots, x_n, x_n^{-1}]$$

Recall that the ring A has a natural grading, each grade being spanned by monomial fractions of total degree n, we are interested only in the degree zero part, that is we are interested in the subalgebra spanned by fractions of monomials of the same degree, which we call B. Now we want to glue together $n+1$ copies of affine n-space, we will do it by taking $n+1$ subalgebras of B, each one given by a polynomial ring in n indeterminates.

For any $i = 0, \dots, n$ consider the subalgebra B_i spanned by the monomials

$$x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i$$

where the hat denotes as usual an element omitted from the list. Then B_i is a polynomial ring in n indeterminates and $\operatorname{Spec} B_i$ is affine n-space. Further for $i \neq j$ we have

$$B_i\big[(x_j/x_i)^{-1}\big] = B_j\big[(x_i/x_j)^{-1}\big]$$

as subsets of B. If we now use the identity maps as gluing maps the compatibility conditions for the Gluing Lemma are obvious. We leave to the reader to convince himself that this construction gives rise to the same scheme (up to isomorphism) as in §2.2.4.

2.3.5 Gluing Affine Lines Let k be an algebraically closed field. We start with two copies of the affine line $\mathbb{A}_k^1$, which we distinguish by setting $X_1 = \operatorname{Spec} k[s]$ and $X_2 = \operatorname{Spec} k[t]$. Let $U_{12} \subset X_1$ be the basic open set $D(s)$ and let $U_{21} \subset X_2$ be $D(\,t\,)$. Let $\varphi_{12}\colon U_{12} \to U_{21}$ be induced by the isomorphism of rings

$$k[t,t^{-1}] \longrightarrow k[s,s^{-1}]$$

sending t to s, and let τ_{12} be induced by the isomorphism sending t to s^{-1}. We denote by X the scheme obtained by gluing together X_1 and X_2 along the isomorphisms φ_{ij} and Y the scheme obtained by gluing along τ_{ij} instead.

By the Gluing Lemma these two schemes are equipped with open immersions $\psi_i\colon X_i \to X$ and $\xi_i\colon X_i \to Y$ such that $X = \psi_1(X_1) \cup \psi_2(X_2)$ and $Y = \xi_1(X_1) \cup \xi_2(X_2)$. They are not isomorphic, in fact Y is the projective line $\mathbb{P}^1_k$ while X is the affine line with a doubled origin.

To verify rigorously that X and Y are not isomorphic we compute the global sections of their structure sheafs. An element $\alpha \in \Gamma(X, \mathscr{O}_X)$ is given by a coherent family

$$\big\{\big(F(\,s\,), \psi_1(X_1)\big), \big(G(\,t\,), \psi_2(X_2)\big)\big\}$$

where coherence means compatibility with the isomorphism φ_{12} above, so that we must have $G(\,s\,) = F(\,s\,)$. Therefore α is uniquely determined as a polynomial in one variable, that is $\Gamma(X, \mathscr{O}_X) \cong k[\,s\,]$. Analogously an element $\beta \in \Gamma(Y, \mathscr{O}_Y)$ is given by two polynomials F and G, but this time we must have $G(\,s^{-1}\,) = F(\,s\,)$ and this is only possible if F and G are the same element of k. Hence $\Gamma(Y, \mathscr{O}_Y) \cong k$.

X is the first example of a scheme which is not *separated* over k.

2.3.6 Gluing over a Base for the Topology In §2.3.2 we have learned how to glue morphisms, we were given an open covering $\{U_i\}$ of X and morphisms $f_i\colon U_i \to Y$ agreeing on the intersections $U_i \cap U_j$. Nevertheless we have also learned in §1.3 that on a scheme it is often possible to define structures starting with a base for the topology, only in this case agreement must be required on every open basic set contained in the intersection. The next results will clarify what we actually need.

Proposition. *Let X and Y be schemes, and let $\mathscr{B}$ be a base for the topology on* $\mathrm{sp}(X)$. *Assume to have a family of morphisms $\varphi_{[U]}\colon U \to Y$, one for each open basic set $U \in \mathscr{B}$, such that if $V \in \mathscr{B}$ is contained in U we have*

$$\varphi_{[U]}|_V = \varphi_{[V]}.$$

Then there exists a unique morphism of schemes $\varphi\colon X \to Y$ such that $\varphi|_U = \varphi_{[U]}$

Proof. On topological spaces define $\varphi\colon \mathrm{sp}(X) \to \mathrm{sp}(Y)$ as follows: for any $x \in X$ there exists an open neighborhood $U \in \mathscr{B}$ of x, set $\varphi(x) = \varphi_{[U]}(x)$, in view of our assumptions this definition makes sense and gives rise to a continuous function.

We have now to define a morphism of sheaves $\varphi^{\#}\colon \mathscr{O}_Y \to \varphi_*\mathscr{O}_X$ so let $W \subseteq Y$ be any open set, we are looking for a homomorphism

$$\varphi^{\#}_W\colon \mathscr{O}_Y(W) \longrightarrow \mathscr{O}_X\big(\varphi^{-1}(W)\big).$$

For any basic open set U of X such that $U \subseteq \varphi^{-1}(W)$ we have a morphism

$$\varphi_{[U]}{}^{\#}_W\colon \mathscr{O}_Y(W) \longrightarrow \mathscr{O}_X|_U\big(\varphi_{[U]}^{-1}(W)\big) = \mathscr{O}_X\big(\varphi^{-1}(W) \cap U\big) = \mathscr{O}_X(U)$$

And any time $V \subseteq U$ we have the following commutative diagram

$$\begin{array}{ccc} \mathscr{O}_Y(W) & \xrightarrow{\varphi_{[U]}{}^{\#}_W} & \mathscr{O}_X(U) \\ & {\scriptstyle \varphi_{[V]}{}^{\#}_W}\searrow & \big\downarrow{\scriptstyle res} \\ & & \mathscr{O}_X(V) \end{array}$$

which is induced on sheaves by the relation $\varphi_{[V]} = \varphi_{[U]}|_V = \varphi_{[U]} \circ i_{[V]}$, where $i_{[V]}\colon V \hookrightarrow U$ is the inclusion. Now for any element $s \in \mathscr{O}_Y(W)$ the family of sections

$$\big\{\varphi_{[U]}{}^{\#}_W(s) \in \mathscr{O}_X(U) \,\big|\, U \in \mathscr{B},\, U \subseteq \varphi^{-1}(W)\big\}$$

defines a unique element of $\mathscr{O}_X\big(\varphi^{-1}(W)\big)$. It is now immediate to check that this definition is in fact a homomorphism. □

Corollary. *Let X and Y be schemes, let $\mathscr{B}$ be a base for the topology on $\mathrm{sp}(X)$ and let $\{U_W\}_{W\in\mathscr{B}}$ be an open covering of Y such that $U_V \subseteq U_W$ whenever $V \subseteq W$. Assume to have a family of morphisms $f_{[W]}\colon W \to U_W$, one for each open basic set $W \in \mathscr{B}$, such that if $V \in \mathscr{B}$ is contained in W we have*

i) $f_{[W]}^{-1}(U_V) = V$

ii) $f_{[W]}|^{U_V} = f_{[V]}$

Then there exists a unique morphism of schemes $f\colon X \to Y$ such that $f|^{U_W} = f_{[W]}$.

Proof. For any $W \in \mathscr{B}$ let i_W denote the inclusion $U_W \hookrightarrow Y$. The reader should convince himself that the family $g_{[W]} = i_W \circ f_{[W]}$ satisfy the hypotheses of the Proposition above. □

2.4 Reduced, Integral and Noetherian Schemes

2.4.1 Reduced Schemes A scheme $(X, \mathscr{O}_X)$ is *reduced* if for every open set $U \subseteq X$, the ring $\mathscr{O}_X(U)$ is reduced, that is it has no nilpotent elements. We want to prove that this is equivalent to the stalk $\mathscr{O}_{X,x}$ being a reduced local ring for any $x \in X$, but first we need to recall some commutative algebra.

Lemma. *For a ring A, the following are equivalent*

i) A is a reduced ring;

ii) $A_\mathfrak{p}$ is a reduced ring for all prime ideal $\mathfrak{p} \subset A$;

iii) $A_\mathfrak{m}$ is a reduced ring for all maximal ideal $\mathfrak{m} \subset A$.

Proof. This is an immediate consequence of Corollary 3.12 in Atiyah and Macdonald (1969), which states that if $\operatorname{Nil} A$ is the nilradical of A, the nilradical of $S^{-1}A$ is $S^{-1} \operatorname{Nil} A$ for any multiplicatively closed set $S \subset A$. Bearing in mind this result there is only to prove that *iii*) implies *i*).

Let $a \in A$ and let $\mathfrak{a} = \operatorname{ann}(a) = \{x \in A \mid xa = 0\}$. Let $\mathfrak{m}$ a maximal ideal containing $\mathfrak{a}$ and assume that $a^n = 0$ for some n. Then in $A_\mathfrak{m}$ we have that a is nilpotent, that is $a = 0$. Hence there exist $b \notin \mathfrak{m}$ such that $ba = 0$, but in particular $b \notin \mathfrak{a}$ and we can conclude $a = 0$. □

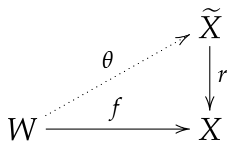
Proposition (Exercise II.2.3 in Hartshorne, 1977)**.** *The scheme $(X, \mathscr{O}_X)$ is reduced if and only if for every $x \in X$, the local ring $\mathscr{O}_{X,x}$ has no nilpotent elements.*

Proof. First observe that if $\{U_\alpha\}$ is a base for the topology in X then $(X, \mathscr{O}_X)$ is reduced if and only if $\mathscr{O}_X(U_\alpha)$ is a reduced ring for all α. Now affine open sets are a base for the topology of any scheme, hence $(X, \mathscr{O}_X)$ is reduced if and only if $\mathscr{O}_X(U)$ is a reduced ring for any affine open set U and this, by the previous lemma, happens if and only if $\mathscr{O}_{X,x}$ is a reduced ring for every $x \in U$. □

2.4.2 Associated Reduced Scheme To any scheme X we can associate a reduced scheme $\widetilde{X}$, having the same underlying topological space, and equipped with a closed immersion $\widetilde{X} \to X$, we call it the *reduced scheme* associated to X. For instance when $X = \operatorname{Spec} A$ is affine, we can consider $\widetilde{X} = \operatorname{Spec} A / \operatorname{Nil} A$, and the projection morphism $A \to A/\operatorname{Nil} A$ will define the required closed immersion.

Proposition (Exercise II.2.3 in Hartshorne, 1977)**.** *Let X be any scheme. Then there exists a reduced scheme $\widetilde{X}$, having the same underlying topological space as*

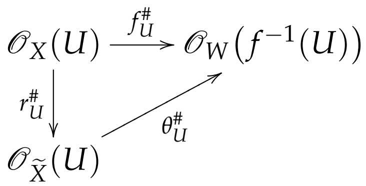
X, equipped with a closed immersion $r\colon \widetilde{X} \to X$ and with the following universal property



For any morphism $f\colon W \to X$, with W reduced, there exists a unique morphism $\theta\colon W \to \widetilde{X}$ such that f is obtained by the composition $r \circ \theta$.

Proof. The construction goes as follows. On the topological space $\operatorname{sp}(X)$, let $\mathscr{O}_{\widetilde{X}}$ be the sheaf associated to the $\mathscr{B}$-sheaf $(U = \operatorname{Spec} A) \mapsto A/\operatorname{Nil} A$, where $\mathscr{B}$ is the base for the topology of X given by all the open affine subsets. Then $(\operatorname{sp}(X), \mathscr{O}_{\widetilde{X}})$ is a scheme, indeed it is enough to observe that $\operatorname{Spec} A/\operatorname{Nil} A$ is naturally homeomorphic to $\operatorname{Spec} A$ for any ring A. Observe now that the family of morphisms defined on any open affine subset of $\operatorname{sp}(X)$ by the projection $A \to A/\operatorname{Nil} A$, satisfies the gluing hypotheses of §2.3.6 (the Corollary) and therefore gives rise to a closed immersion r as required.

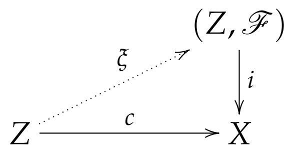
Given any morphism $f\colon W \to X$ we have for each open affine subset $U \subseteq X$ an induced homomorphism $f^\#_U\colon \mathscr{O}_X(U) \to \mathscr{O}_W\big(f^{-1}(U)\big)$, whose kernel contains the nilradical of $\mathscr{O}_X(U)$ since W is reduced. Hence there exist a unique $\theta^\#_U$ such that



This defines a unique morphism of sheaves $\theta^\#$ and eventually a unique morphism of schemes θ as required. □

2.4.3 Reduced Induced Subscheme Structure Let $(X, \mathscr{O}_X)$ be any scheme and let $Z \subseteq X$ be a closed subset. We can define over Z a sheaf of rings $\mathscr{O}_Z$ which makes it into a reduced scheme, moreover $(Z, \mathscr{O}_Z)$ will come equipped with a closed immersion $c\colon Z \to X$, we call this the *reduced induced subscheme structure* on Z. If $X = \operatorname{Spec} A$ is affine then $Z = \mathcal{V}(\mathfrak{a})$ for some ideal $\mathfrak{a}$ and is naturally homeomorphic to $\operatorname{Spec} A/\mathfrak{a}$ (see §1.1.6), furthermore we can take the ideal $\mathfrak{a}$ to be the intersection of all the prime ideals contained in Z, which we denote $\mathcal{I}(Z)$, as in §1.1.3. The affine scheme $\operatorname{Spec} A/\mathcal{I}(Y)$ is a reduced scheme whose underlying topological space is Z, and it is endowed with a closed immersion induced by the projection homomorphism $A \to A/\mathcal{I}(Y)$.

Proposition (Exercise II.3.11 in Hartshorne, 1977)**.** *Let $(X, \mathscr{O}_X)$ be any scheme and let $Z \subseteq X$ be a closed subset. Then there exists a sheaf of rings $\mathscr{O}_Z$ over Z which makes it into a reduced scheme, endowed with a closed immersion $c \colon Z \to X$, and satisfying the following universal property*



For any scheme $(Z, \mathscr{F})$, with underlying topological space Z, endowed with a closed immersion i, there exists a unique closed immersion ξ such that $i\xi = c$.

Proof. If X is any scheme we can repeat the previous construction on any open affine subset U_i of X, this defines an affine scheme $Z_i = Z \cap U_i$ and there is to prove that the schemes Z_i glue together, which can be found in Hartshorne (1977, Example II.3.2.6).

Observe now that when $X = \operatorname{Spec} A$ is affine, $(Z, \mathscr{F})$ is the spectrum of $A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ and $Z = \operatorname{Spec} A/\sqrt{\mathfrak{a}}$, therefore ξ is necessarily induced by the projection homomorphism $A/\mathfrak{a} \to A/\sqrt{\mathfrak{a}}$. This proves in particular that the reduced induced subscheme structure on Z coincides with the structure of reduced scheme associated to $(Z, \mathscr{F})$, and that ξ coincides with the closed immersion r of the previous Proposition. □

2.4.4 Integral Schemes A scheme X is *integral* if for every open set $U \subseteq X$, the ring $\mathscr{O}_X(U)$ is an integral domain, or equivalently if for every $x \in X$, the local ring $\mathscr{O}_{X,x}$ is an integral domain.

According to Hartshorne (1977, Proposition II.3.1) a scheme is integral if and only if it is both reduced and irreducible, where we say X is *irreducible* if its topological space is irreducible. We can then use the results in §2.1.2 to conclude that an integral scheme has a unique generic point ξ, this is a point such that $X = \{\xi\}^-$. For any other point $x \in X$ there is a canonical homomorphism $\mathscr{O}_{X,x} \to \mathscr{O}_{X,\xi}$, which is given by the localisation in a prime ideal. Indeed let $U = \operatorname{Spec} A$ be any open affine neighborhood of x, then U contains ξ also and both the points correspond to prime ideals, $\mathfrak{p}_\xi$ and $\mathfrak{p}_x$, of A such that $\mathfrak{p}_\xi \subseteq \mathfrak{p}_x$.

Proposition (II.4.18 in Liu, 2002)**.** *Let X be an integral scheme. Then for any open subset U of X, and for any point $x \in U$, the homomorphisms $\mathscr{O}_X(U) \to \mathscr{O}_{X,x}$ and $\mathscr{O}_{X,x} \to \mathscr{O}_{X,\xi}$ are injective.*

Proof. Let $s \in \mathscr{O}_X(U)$ be a section such that $s_x = 0$. Then there exists an open neighborhood W of x such that $s|_W = 0$. For any open affine subset $V = \operatorname{Spec} A$ of U consider the restriction $s|_V$ and let $D(a)$ be any basic open set contained in $V \cap W$, which is not empty because X is irreducible (see §1.2.1). Now the restriction $A \to A_a$ is injective because A is an integral domain, hence $s|_V = 0$. This happens for any open affine subset of U, therefore $s = 0$. The injectivity of $\mathscr{O}_{X,x} \to \mathscr{O}_{X,\xi}$, results from the injectivity of the localisation of an integral domain. □

2.4.5 Field of Rational Functions The local ring $\mathscr{O}_\xi$ of the generic point ξ of an integral scheme X is a field. Indeed observe that $\mathscr{O}_\xi$ is defined to be the following direct limit

$$\mathscr{O}_\xi = \varinjlim_{U \subset X} \mathscr{O}_X(U) = \varinjlim_{U \text{ affine}} \mathscr{O}_X(U)$$

If $f_\xi \in \mathscr{O}_\xi$, then f_ξ is the equivalence class of a couple (U, f) where U is an open affine set and $f \in \mathscr{O}_X(U)$. Since $\mathscr{O}_X(U)$ is an integral domain f defines a non-empty distinguished open subset $D(f)$ of U. Now $\mathscr{O}_X\big(D(f)\big) = \mathscr{O}_X(U)_f$, hence the couple $\big(D(f), f^{-1}\big)$ represents the element f_ξ^{-1} in $\mathscr{O}_\xi$.

Lemma (Exercise II.3.6 in Hartshorne, 1977)**.** *Let X be an integral scheme with generic point ξ, and let $U = \operatorname{Spec} A$ be any open affine subset of X. Then the restriction homomorphism $\mathscr{O}_X(U) \to \mathscr{O}_{X,x}$ induces an isomorphism $\operatorname{Frac} A \cong \mathscr{O}_{X,\xi}$.*

Proof. The point ξ is also the generic point of U, and $\mathscr{O}_{X,\xi} = \mathscr{O}_{U,\xi}$. It is enough now to observe that ξ corresponds to the zero ideal. □

Proposition (II.4.18 in Liu, 2002)**.** *Let X be an integral scheme. Then by identifying $\mathscr{O}_X(U)$ and $\mathscr{O}_{X,x}$ to sub-rings of $\mathscr{O}_{X,\xi}$, we have*

$$\mathscr{O}_X(U) = \bigcap_{x \in U} \mathscr{O}_{X,x}$$

Proof. By covering U with affine open subsets, we may assume that U is affine, say $U = \operatorname{Spec} A$. Let $\gamma \in \operatorname{Frac} A$ be contained in all of the localisations $A_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$. Let I be the ideal $\{a \in A \mid a\gamma \in A\}$, then recalling the definition of localisation for every $\mathfrak{p}$ there exists $a \in I \setminus \mathfrak{p}$. But then I is not contained in any prime ideal, and therefore it is the whole of A. In particular $1 \in I$, so that $\gamma \in A$. □

Definition. Let X be an integral scheme with generic point ξ. We denote the field $\mathscr{O}_{X,\xi}$ by $K(X)$. An element of $K(X)$ is called a *rational function* on X. We call $K(X)$ the *field of rational functions* or *function field* of X.

We say that $f \in K(X)$ is *regular at* $x \in X$ if $f \in \mathscr{O}_{X,x}$. The Proposition above affirms that a rational function which is regular at every point of U is contained in $\mathscr{O}_X(U)$.

Example (Rational functions on affine n-space). If k is algebraically closed then $\mathbb{A}^n_k$, affine n-space over k, is an integral affine scheme. Its generic point corresponds to the zero ideal, its field of functions to the ring of polynomial fractions. A rational function is thus given by the quotient of two polynomials, it is regular on the whole space if and only if it is given by a single polynomial.

Example (Rational functions on Projective Space). Projective n-space $\mathbb{P}^n_k$ is an integral scheme, its generic point corresponds to the zero ideal of any of the $n+1$ distinguished affine spaces

$$D_h(x_i) = \operatorname{Spec} k\big[x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i\big]$$

The field of rational functions on $\mathbb{P}^n_k$ is given by the subalgebra of the field $\mathbb{F}\mathrm{rac}\, k[x_0, \ldots, x_n]$ consisting of fractions of polynomials of the same degree. A rational function is regular on the whole of $\mathbb{P}^n_k$ if and only if it is given by an element of k.

2.4.6 Noetherian Schemes It is well known that the polynomial ring over a field is a Noetherian ring, this is the famous Hilbert's Basis Theorem which you can find for instance in Atiyah and Macdonald (1969) as Theorem 7.5 or in Chapter 2 of Cox, Little, and O'Shea (1997). The previous two very geometric and familiar examples justify therefore the following definition.

Definition. A scheme X is *locally Noetherian* if it can be covered by open affine subsets $\operatorname{Spec} A_i$, where each A_i is a Noetherian ring. X is *Noetherian* if it is locally Noetherian and quasi-compact. Equivalently, X is Noetherian if it can be covered by a finite number of open affine subsets $\operatorname{Spec} A_i$, with each A_i a Noetherian ring.

"Note that in this definition we do not require that every open affine subset be the spectrum of a Noetherian ring. So while it is obvious from the definition that the spectrum of a Noetherian ring is a Noetherian sche-me, the converse is not obvious. It is a question of showing that the Noetherian property is a *local property*."

taken from Hartshorne (1977, §II.3)

It turns out that in fact it is, for the proof see Hartshorne (1977, Proposition II.3.2).

2.4.7 Zariski Spaces A topological space X is a *Zariski space* if it is Noetherian and every (nonempty) closed irreducible subset Z of X has a unique generic point, that is a point ζ such that $Z = \{\zeta\}^-$.

If X is a Noetherian scheme then $\mathrm{sp}(X)$ is a Zariski space, this follows by the discussion above and by §2.1.2. Any Zariski space satisfies the axiom T_0, indeed given two different points $x, y \in X$ then the closures $\{x\}^-$ and $\{y\}^-$ must be different, that is either $x \notin \{y\}^-$ or $y \notin \{x\}^-$.

Proposition (Exercise II.3.17 in Hartshorne, 1977)**.** *Any minimal nonempty closed subset of a Zariski space consists of one point. We call these* closed points*, if X is a Noetherian scheme these are precisely the closed points of X.*

Proof. It is clear that a minimal nonempty closed subset M is also irreducible, then there exists a unique generic point ξ for M. Now let $x \in M$ be any point, so that $\{x\}^- \subseteq M$. Since M is minimal this inclusion can't be strict, hence $\{x\}^- = \{\xi\}^-$ and we can conclude $x = \xi$ by the uniqueness of ξ. □

Definition. If x_0, x_1 are points of a topological space X, and if $x_0 \in \{x_1\}^-$, then we say that x_1 *specialises* to x_0, written $x_1 \rightsquigarrow x_0$. We also say x_0 is a *specialisation* of x_1, or that x_1 is a *generalisation* of x_0.

If X is a Zariski space then the minimal points, for the partial ordering determined by setting $x_1 > x_0$ if $x_1 \rightsquigarrow x_0$, are closed points, and the maximal points are the generic points of the irreducible components of X (see §1.2.3). Further a closed subset contains every specialisation of any of its points (we say closed subsets are *stable under specialisation*). Similarly, open sets are *stable under generalisation*.

Chapter 3

Attributes of Morphisms

By the end of this chapter we will have a definition of algebraic variety. As a matter of fact it took us a long time to arrive at this stage, but after all we started developing the theory from the very basic concepts. It is in fact true that algebraic varieties are schemes, but this statement is as far away from the real nature of varieties as in differential geometry is saying "manifolds are topological spaces." Starting from the main examples we have seen so far, above all affine and projective space, we will now identify other special properties that these have. The key step in here is the choice of a *base*, a fixed scheme over which to state every result. This introduces a certain level of relativity in the subject, a scheme will have a certain property *over* another and there will be to check if the same property holds when the base changes. For this we will need the powerful tool of fibered products, by means of which we will be able to construct inverse images and fibres of a morphism as well as products of varieties.

3.1 Schemes of Finite Type

3.1.1 The Category of Schemes over a Field In what follows S will be a fixed scheme that we call the *base scheme*. We formally define a *scheme over S*, or *S-scheme*, as a scheme X together with a *structure morphism* $\pi\colon X \to S$. When $S = \operatorname{Spec} A$, we also say scheme over A, or A-scheme (instead of scheme over S) and A is called base ring. A morphism of schemes over S will be a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

As usual we will write simply f with this diagram understood. In this category objects are couples (X, π) where X is a scheme and π is a morphism, in particular any scheme X can be regarded as a scheme over S in as many ways as the morphisms between X and S, and even S itself is not in general a scheme over S in a unique way. Observe however that any scheme carries a unique structure of $\mathbb{Z}$-scheme.

Let now k be a field and let V be an affine variety, that is the spectrum of a finitely generated k-algebra $A = k[x_1, \dots, x_n]/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal. Then the inclusion of k in $A = \Gamma(X, \mathscr{O}_X)$ induces a morphism of schemes $V \to \operatorname{Spec} k$. Similarly any homogeneous ideal I in $k[x_0, \dots, x_n]$ defines a projective scheme W endowed with a natural morphism $W \to \operatorname{Spec} k$ as in §2.2.4. Slightly more generally if X is a scheme over k the structure morphism of X will be induced by a homomorphism $k \to \Gamma(X, \mathscr{O}_X)$, so that we have the following characterisation.

Remark. X is a scheme over k if and only if $\mathscr{O}_X(U)$ is a k-algebra (i.e. it contains a field isomorphic to k) for any nonempty open subset $U \subseteq X$, and restriction maps are morphisms of k-algebras (i.e. they are the identity over k). Moreover the structure morphism is in this case induced by the inclusion of the field k in $\Gamma(X, \mathscr{O}_X)$.

3.1.2 Morphisms of Finite Type In the category of schemes over S properties of the structure morphism will reflect properties of the object, and viceversa the geometry of the object will help our understanding of the structure morphism. If for instance we go back to the schemes V and W above, we see that from an abstract point of view they are not merely schemes over k but their structure is richer. The affine variety V is the spectrum of a *finitely generated k-algebra* and the projective scheme W has a *finite covering* consisting of spectra of finitely generated k-algebras. To describe this situation we make a general definition concerning morphisms of schemes.

Definition. Let $f\colon X \to Y$ be a morphism of schemes. We say that an affine open subset $V = \operatorname{Spec} B$ of Y has the property (FG) if $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra. We say it has the property (FFG) if in addition $f^{-1}(V)$ can be covered by a finite number of the U_j.

The morphism f is *locally of finite type* if there exist a covering of Y by open affine subsets that have the property (FG). It is *of finite type* if the covering of Y consists of open sets which have the property (FFG).

If S is a fixed scheme, a scheme X is locally of finite type over S, or of finite type over S if it is an S-scheme and the structure morphism has the required property.

Observe that we have defined an attribute of the morphism f by means of an open affine covering of Y. This is a typical situation, we have a property $\mathscr{P}$ that in general applies whenever Y is affine but we require it to hold over an open covering only. The attribute of f will then be called *local on the base* if $\mathscr{P}$ holds for every open affine subset of Y. The word *base* here refers to Y, indeed in principle every morphism of schemes is a structure morphism, in other words given $f\colon X \to Y$ we can always regard X as a scheme over Y. In the rest of this section we are going to prove that for a morphism to be of finite type is local on the base, we begin with morphisms locally of finite type.

Proposition (Exercise II.3.1 in Hartshorne, 1977)**.** *A morphism $f\colon X \to Y$ is locally of finite type if and only if for* every *open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.*

Proof. The "if" part follows from the definition, so assuming f is locally of finite type we have to prove that every open affine subset $V = \operatorname{Spec} B$ of Y has the property (FG). First note that if $Y = \operatorname{Spec} B$ is affine and has the property (FG) then every distinguished open subset of Y has the property (FG). Indeed we can cover X by open affine subsets that are spectra of finitely generated B-algebras, and $f^{-1}\big(D(b)\big) = X_{\overline{b}}$ for any $b \in B$. Let $U = \operatorname{Spec} R$ be any of the affine open sets that cover X, then $U \cap f^{-1}\big(D(b)\big) = \operatorname{Spec} R_\beta$ where β is the restriction of $\overline{b}$ to the open set U, in other words β is the image of b under the morphism which makes R into a B-algebra. Since R was a finitely generated B-algebra we have that R_β is now a finitely generated B_b-algebra. This proves that we can cover $f^{-1}\big(D(b)\big)$ with affine open sets that are spectra of finitely generated B_b-algebras, that is $D(b)$ has the property (FG).

Going back to the general case we can say that there is a base for the topology of Y consisting of open affine sets with the property (FG). So we have reduced to proving the following statement: let $Y = \operatorname{Spec} B$ be an affine scheme, which can be covered by open affine subsets that have the property (FG); then Y has the property (FG).

Let $V = \operatorname{Spec} A$ be an open affine subset of Y with the property (FG). Then $Y_b \subseteq V$ for some $b \in B$, and $Y_b \cap V = D(a)$ where a is the restriction of b to $A = \mathscr{O}_Y(V)$; in particular $A_a \cong B_b$. By the previous discussion $D(a)$ has the property (FG), which means that we can cover $f^{-1}\big(D(a)\big) = f^{-1}(Y_b \cap V)$ with open affine subsets of X that are spectra of finitely generated A_a-algebras, i.e. B_b-algebras. Note now that B_b is a finitely generated B-algebra, it is in fact isomorphic to the quotient $B[x]/(bx-1)$, hence we can cover $f^{-1}\big(D(a)\big) = f^{-1}(Y_b \cap V)$ with open affine subsets of X that are spectra of finitely generated B-algebras. We can do this construction over an affine open covering of Y, so we can conclude that Y has the property (FG). □

Lemma (Exercise II.3.3 in Hartshorne, 1977)**.** *The morphism f is locally of finite type, if and only if for* every *open affine subset $V = \operatorname{Spec} B \subseteq Y$, and* every *open affine subset $U = \operatorname{Spec} A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra.*

Proof. Again the "if" part follows from the definition. Moreover, in view of the previous Proposition we can assume Y to be affine, say $Y = \operatorname{Spec} B$. By definition we can cover X by open affine sets that are spectra of finitely generated B-algebras.

Note that if A is a finitely generated B-algebra, then for any $s \in A$ the ring A_s is again a finitely generated B-algebra. Now we have a base for the topology of X consisting of spectra of finitely generated B-algebras.

So we have reduced to proving the following: let $X = \operatorname{Spec} A$ be an affine scheme, which can be covered by open subsets that are spectra of finitely generated B-algebras. Then A is a finitely generated B-algebra. Let $U = \operatorname{Spec} R$ be an open subset of X, with R a finitely generated B-algebra. With the same notations as in §2.1.3 we have that for some $s \in A$, $X_s \subseteq U$. Let r be the restriction of s to $R = \mathscr{O}_X(U)$, then $X_s = U \cap X_s = D(r)$. In particular $A_s = \mathscr{O}_X(X_s) \cong R_r$, hence it is a finitely generated B-algebra. So we can cover X by open subsets $X_s \cong \operatorname{Spec} A_s$ with A_s a finitely generated B-algebra. Since X is quasi-compact, a finite number will do.

Now we have reduced to a purely algebraic problem: A is a ring, $f_1, \ldots, f_r$ are a finite number of elements of A, which generate the unit ideal, and each localisation A_{f_i} is a finitely generated B-algebra. We have to show A is a finitely generated B-algebra.

Since $f_1, \ldots, f_r$ generate the unit ideal we have $1 = \sum c_i f_i$ then we can say

$$1 = \sum_{i=1}^{r} h_i f_i^{\alpha} \quad \text{for any } \alpha \in \mathbb{N}$$

where h_i is a polynomial in $c_1, \ldots, c_r, f_1, \ldots, f_r$ (one can prove it by induction). We have also that each localisation A_{f_i} is a finitely generated B-algebra, which means that for any $i = 1, \ldots, r$

$$A_{f_i} \cong B\left[(x_{i1}/f_i^{\beta_{i1}}), \ldots, (x_{is_i}/f_i^{\beta_{is_i}})\right]$$

Let $g \in A$, then for any $i = 1, \ldots, r$ in the ring A_{f_i} we have $g = g_i / f_i^{\gamma_i}$, where g_i is a polynomial in $x_{i1}, \ldots x_{is_i}$. This means that in A we have

$$f_i^{\alpha_i} g - f_i^{\delta_i} g_i = 0 \quad \text{for some integers } \alpha_i, \delta_i \in \mathbb{N}$$

Without loss of generality we can assume $\alpha_i = \alpha$, independent from i. Then we have the following relation in A.

$$\sum_{i=1}^{r} h_i \left(f_i^{\alpha} g - f_i^{\delta_i} g_i \right) = 0$$

that is

$$g = \sum_{i=1}^{r} g_i h_i f_i^{\delta_i}$$

where on the right-hand side we have a polynomial in a finite set of elements of A, and this finite set doesn't depend on g. Thus A is a finitely generated B-algebra. □

3.1.3 Quasi-compact Morphisms We are now almost ready to prove that for a morphism being of finite type is a local property. We just need a preliminary result about quasi-compactness. A morphism $f\colon X \to Y$ of schemes is *quasi-compact* if there is a covering of Y by open affine sets V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Also this attribute is local on the base.

Lemma (Exercise II.3.2 in Hartshorne, 1977)**.** *A morphism f is quasi-compact if and only if for* every *open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.*

Proof. First note that if X is quasi-compact then X_s is quasi-compact for any $s \in \Gamma(X, \mathscr{O}_X)$. Indeed we can cover X by a finite number of affine sets, so that if U is one of these the intersection $X_s \cap U$ is affine as in §2.1.3, and X_s will be therefore a finite union of affine sets.

It follows that if $Y = \operatorname{Spec} B$ is affine and X is quasi-compact then the preimage $f^{-1}\big(D(b)\big)$ is quasi compact for any $b \in B$; to check this recall that $f^{-1}\big(D(b)\big) = X_{\overline{b}}$ where $\overline{b}$ is the image of b on the global sections of $\mathscr{O}_X$. Thus we have a base for the topology in Y consisting of affine subsets V_i such that $f^{-1}(V_i)$ is quasi-compact for each i.

So we have reduced to prove the following: if Y is affine and it can be covered by affine open sets V_i such that $f^{-1}(V_i)$ is quasi-compact for each i then $f^{-1}(Y) = X$ is quasi-compact. Since Y is affine it is quasi-compact, hence we can find a finite sub-covering of Y, this means that X can be covered by a finite number of quasi-compact sets, that is X is quasi-compact. □

Proposition (Exercise II.3.3 in Hartshorne, 1977)**.** *A morphism $f\colon X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact. In particular f is of finite type if and only if* every *open affine subset $V \subseteq Y$ has the property (FFG).*

Proof. It is enough to observe that an affine open subset V of Y has the property (FFG) if and only if it has (FG) and $f^{-1}(V)$ is quasi-compact. Now the result follows from the analogue for morphisms locally of finite type (§3.1.2) and the previous Lemma. □

3.1.4 Properties of Morphisms of Finite Type A closed immersion is a morphism of finite type. To see this let $f\colon X \to Y$ be a closed immersion and assume first X and Y to be affine. In this case f is given by a surjective morphism of rings $\varphi\colon A \to A/\mathfrak{a}$ where $A/\mathfrak{a}$ is a finitely generated A-algebra, in fact a finite A-algebra. In the general case for any open affine set $U \subseteq Y$ the restriction $f^{-1}(U) \to U$ is a closed immersion, hence $f^{-1}(U)$ is also affine (this proves more, namely that a closed immersion is a *finite morphism*).

Proposition (Exercise II.3.13 in Hartshorne, 1977)**.** *Properties of morphisms of finite type.*

(a) *A quasi-compact open immersion is of finite type.*

(b) *A composition of two morphisms of finite type is of finite type.*

(c) *If $f\colon X \to Y$ is a morphism of finite type, and if Y is Noetherian, then X is Noetherian.*

Proof. Let f be a quasi compact open immersion, by the Proposition above we only need to prove that it is locally of finite type. Since it is an open immersion X is isomorphic to $(U, \mathscr{O}_Y|_U)$ where U is an open set of Y. Now if $\operatorname{Spec} B \subseteq Y$ is an open affine subset of Y we can cover $U \cap \operatorname{Spec} B$ with open basic sets, that will be $\operatorname{Spec} B_\alpha$ for some $\alpha \in B$, and $B_\alpha = B[\,x\,]/(\alpha x - 1)$ is a finitely generated B-algebra.

To prove (b) consider the composition $X \to Y \to Z$ and observe that it is enough to consider Z to be affine. Assuming $Z = \operatorname{Spec} C$ we have a finite covering of Y by open affine sets $\operatorname{Spec} B_i$ where B_i is a finitely generated C-algebra. Now for any i we have a finite covering of $f^{-1}(\operatorname{Spec} B_i)$ by open affine sets $\operatorname{Spec} A_{ij}$ where A_{ij} is a finitely generated B_i-algebra. Putting all things together we obtain a finite covering of X by open affine sets that are spectra of finitely generated C-algebras.

Statement (c) is immediate: we can cover Y by a finite number of open affine sets that are spectra of Noetherian rings, so we can cover X by a finite number of spectra of finitely generated algebras over Noetherian rings, that are Noetherian rings by Hilbert's basis Theorem. □

Lemma (Exercise II.3.13 in Hartshorne, 1977). *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes. If $g \circ f$ is of finite type and f is quasi compact then f is of finite type.*

Proof. We look at the affine case first, so let $A \to B \to C$ be a composition of morphisms of rings such that C is a finitely generated A-algebra. Then there exists a finite set of elements $x_1, \dots, x_n \in C$ such that any $c \in C$ can be written as a polynomial in $x_1, \dots, x_n$ with coefficients in A, in fact in the image of A via the composition of morphisms above, hence the same expression can be regarded as having coefficients in B that is C is a finitely generated B-algebra.

If now the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ is of finite type, we have to find a covering of Y consisting of open affine sets with the property (FFG), this is the case if we can find such a covering for $g^{-1}(U)$ for any open affine subset $U \subseteq Z$. Hence we have reduced to prove the statement assuming $Z = \operatorname{Spec} C$ affine. Since f is quasi compact, for any open affine subset $V = \operatorname{Spec} B$ of Y we can cover $f^{-1}(V)$ with a finite number of open affine subsets $\operatorname{Spec} A_i$ of X. Since the composition $g \circ f$ is of finite type, each ring A_i is a finitely generated C-algebra as in §3.1.2, and we are again in the affine case. □

If X is a scheme of finite type over k then there is a finite covering of X consisting of open affine subsets that are spectra of finitely generated k-algebras. Note that in particular X is Noetherian. In view of all the previous results we have the following characterisation.

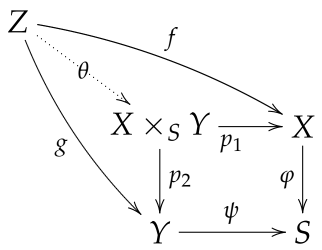
Remark. X is a scheme of finite type over k if and only if X is quasi-compact, $\mathscr{O}_X(U)$ is a finitely generated k-algebra for any nonempty open affine set U of X and restriction maps are morphisms of k-algebras. Moreover the structure morphism is in this case induced by the inclusion of the field k in $\Gamma(X, \mathscr{O}_X)$.

Let X and Y be schemes of finite type over k. Then every morphism of schemes $f\colon X \to Y$ is quasi-compact. Indeed $\operatorname{sp}(X)$ is a Noetherian topological space, so that every open subset $U \subseteq X$ is quasi compact (see §1.2.5). Therefore every morphism of schemes over k from X to Y is a morphism of finite type, by the Lemma above.

3.2 Product of Schemes

3.2.1 Fibered Product The most useful construction one can do with schemes is probably the less intuitive. The beginner will face some difficulty in accepting the idea of a product which doesn't seem to behave like a product, and indeed is not! The main Theorem we are about to describe asserts in fact that the category of schemes has *pull-backs*, the reason why we call product a pull-back is mainly because this construction allows us to define the product of varieties, which is notoriously difficult to define unambiguously. On the other hand with the same construction we can pull-back families along a morphism or give sense to the expression *base extension*. All these different aspects are discussed in the literature, in fact fibered products are to be found in every book about schemes, here we will follow Eisenbud and Harris (2000, §I.3).

Definition. Let $\varphi\colon X \to S$ and $\psi\colon Y \to S$ be morphisms of schemes. The *fibered product* of X and Y with respect to S is a scheme $X \times_S Y$ equipped with two morphisms $p_1\colon X \times_S Y \to X$ and $p_2\colon X \times_S Y \to Y$ which make a commutative diagram with the given morphisms and with the following universal property



For any scheme Z equipped with morphisms f and g which make a commutative diagram with φ and ψ, there exists a unique morphism $\theta\colon Z \to X \times_S Y$ such that the diagram above is commutative.

"There is one exceedingly important and very elementary existence theorem in the category of schemes. This asserts that arbitrary fibered products exist."

taken from Mumford (1999, §II.2)

This theorem is stated in any book about schemes, the proof is a gluing argument which is often left to the reader, but the whole of it can be found in Hartshorne (1977, Theorem II.3.3). If all the schemes involved are affine, say $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $S = \operatorname{Spec} R$, the fibered product is also affine and is given by

$$X \times_S Y = \operatorname{Spec} A \otimes_R B$$

Further in the general case the following statements are also true, they are an almost immediate consequence of the definition.

(1). Let $U \subseteq X$ and $V \subseteq Y$ be open sets then

$$U \times_S V \cong p_1^{-1}(U) \cap p_2^{-1}(V)$$

that is the open subscheme $Z \subseteq X \times_S Y$ defined as $Z = p_1^{-1}(U) \cap p_2^{-1}(V)$ is isomorphic to the product of U and V over S.

(2). Let $W \subseteq S$ and define $U = \varphi^{-1}(W)$, $V = \psi^{-1}(W)$ then

$$U \times_W V \cong p_1^{-1}\big(\varphi^{-1}(W)\big) = p_2^{-1}\big(\psi^{-1}(W)\big)$$

that is the product of U and V over W is isomorphic to the open subscheme of $X \times_S Y$ defined as $p_1^{-1}\big(\varphi^{-1}(W)\big)$.

(3). The two projection maps p_1 and p_2 are *globally monic*, which means that if Z is any scheme and $f, g\colon Z \to X \times_S Y$ is a couple of parallel morphisms such that $p_1 f = p_1 g$ and $p_2 f = p_2 g$ then $f = g$. To be precise we have the equality $(f, f^\#) = (g, g^\#)$, in particular this implies that p_1 and p_2 are globally monic as continuous maps.

3.2.2 Product of Morphisms We can use the fibered product to define an absolute product by taking S to be $\operatorname{Spec}\mathbb{Z}$, that is the terminal object in the category of schemes. This makes sense in every context, but this high level of generality loses us some geometric intuition.

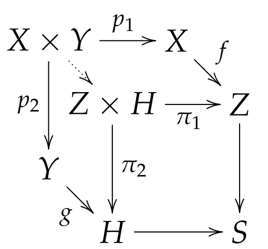
Example (In which the absolute product of nonempty sets is empty).

$$\operatorname{Spec}\mathbb{Z}/(m) \times \operatorname{Spec}\mathbb{Z}/(n)$$

This is defined to be the spectrum of $\mathbb{Z}/(m) \otimes \mathbb{Z}/(n) = \mathbb{Z}/(m,n)$, hence when $(m,n) = 1$ the absolute product is empty.

If however we work in the category of schemes over a field k these problems do not arise. Here the absolute product will be the fibered product over $\operatorname{Spec} k$, and since a k-algebra always contains a field isomorphic to k, the product of two nonempty schemes over k will always be nonempty.

More generally, if S is a fixed scheme, in the category of schemes over S we have a terminal object given by S itself and it is therefore very natural to define an absolute product as the fibered product over S. Here we are also able to construct the *product of morphisms*. Let $f\colon X \to Z$ and $g\colon Y \to H$ be morphisms of S-schemes, then we have the following commutative diagram



The morphisms $f p_1$ and $g p_2$ make a commutative diagram with $Z \to S$ and $H \to S$, hence by the universal property of $Z \times H$ there exists a unique morphism $f \times g$ such that $\pi_1 \circ f \times g = f p_1$ and $\pi_2 \circ f \times g = g p_2$.

3.2.3 Base Extensions If we have a morphism $S' \to S$ we can define a *base extension*, that is a functor from the category of schemes over S to the category of schemes over S', by pull-back. The situation is the following:

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

note that in case S and S' were spectra of fields k and k' respectively, this configuration is possible if and only if k' is a field extension of k. For any S-scheme X we define a scheme over S' as $X \times_S S'$, and for any morphism of S-schemes $f\colon X \to Z$ we define a morphism of S'-schemes as $f \times \mathrm{id}_{S'}$. It is an easy exercise (that we leave to the reader) to check that in this way we obtain in fact a functor.

Definition. Let $\mathscr{P}$ be a property of morphisms. We say that $\mathscr{P}$ is *stable under base extension* if any pull-back of any morphism satisfying $\mathscr{P}$ satisfies $\mathscr{P}$. This amounts to saying the following, let $\varphi\colon X \to S$ and $\psi\colon S' \to S$ be morphisms of schemes and construct their fibered product

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \varphi} \\ S' & \xrightarrow{\psi} & S \end{array}$$

then $\mathscr{P}$ is stable under base extension if and only if p_2 satisfies $\mathscr{P}$ whenever φ does.

We would like to prove now that closed immersions are stable under base extension, before doing so however we need a preliminary lemma.

Lemma (Exercise III.3.1 in Liu, 2002)**.** *Let $f\colon Z \to X$ be a morphism of schemes. Let $\{U_i\}_{i\in I}$ be an open covering of X. For each i consider the morphism obtained by f by restriction as follows*

$$f_i\colon f^{-1}(U_i) \longrightarrow U_i$$

then f is a closed immersion if and only if f_i is a closed immersion for all i.

Proof. First the topological statement: f is a homeomorphism onto a closed subset of X if and only if f_i is a homeomorphism onto a closed subset of U_i for all i. We only need to prove one implication, so assume that f_i is a homeomorphism onto a closed subset of U_i, then clearly f is injective. Moreover $f(Z)$ is a closed subset of X, indeed

$$X \setminus f(Z) = \bigcup_i \left(U_i \setminus \operatorname{Im} f_i\right)$$

that is $X \setminus f(Z)$ is a union of open subsets, hence it is open. The continuous function f is then a homeomorphism onto $f(Z)$ because for any open subset $U \subseteq Z$ we have $f(U) = \bigcup_i f_i(U \cap U_i)$ is open in $f(Z)$.

Next we look at the induced morphism of sheaves. We have the following equality

$$f_{i*}\left(\mathscr{O}_Z\Big|_{f^{-1}(U_i)}\right) = (f_*\mathscr{O}_Z)\,|_{U_i}$$

which proves that f_* is surjective if and only if f_{i*} is surjective for all i. □

Proposition (Exercise II.3.11 in Hartshorne, 1977)**.** *Closed immersions are stable under base extension.*

Proof. With reference to the previews diagram assume all the schemes involved are affine, so let $S = \operatorname{Spec} R$, $S' = \operatorname{Spec} R'$ and $X = \operatorname{Spec} A$. In this case that diagram is induced by the following diagram of ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{\psi} & R' \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle p_2} \\ A & \xrightarrow{p_1} & A \otimes_R R' \end{array}$$

What we have to prove is that if φ is surjective then p_2 is surjective also (follows from §1.4.6). Let therefore φ be the projection to a quotient, that is $A = R/\mathfrak{a}$ for some ideal $\mathfrak{a} \subseteq R$, then the fibered product is given by the prime spectrum of $R' \otimes_R R/\mathfrak{a} = R'/\mathfrak{a}^e$ where the extension of $\mathfrak{a}$ is made via ψ. The commutativity of the diagram now ensure us that p_2 is also the canonical projection.

Assume now only S to be affine, then if φ is a closed immersion X also is affine (we proved this in §2.1.6). We can then cover S' by open affine subsets and use the previous Lemma to check that p_2 is therefore a closed immersion. The general case, when all the three schemes are not assumed to be affine, follows by covering S by open affine sets. □

3.2.4 Inverse Image Scheme Let $f\colon X \to Y$ be a morphism of schemes and let $i\colon C \to Y$ be a closed immersion. The previous proposition allows us to define the *inverse image scheme* of C, which will be a closed subscheme of X, indeed we can pull-back the closed immersion i along f and we will have in this way a closed immersion $j\colon Z \to X$. We only need to prove the following result.

Lemma. *The underlying topological space of Z is homeomorphic to the closed subset $f^{-1}(C)$ of X.*

Proof. It is enough to consider the affine case, the general case will then follow by means of open affine coverings. The situation is the following

$$\begin{array}{ccc} X \times_Y C & \xrightarrow{p_1} & C \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{f} & Y \end{array}$$

where $X \times_Y C$ is the scheme Z and since closed immersions are stable under base extension j is a closed immersion. In the affine case this diagram is induced by the following diagram of homomorphisms of rings

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ A/\mathfrak{a} & \xrightarrow{\overline{\varphi}} & B/\mathfrak{a}^e \end{array}$$

where $\varphi\colon A \to B$ induces f. In §1.4.1 we have seen that the closed subset $f^{-1}(C)$ of X is given by $\mathcal{V}(\mathfrak{a}^e)$, hence the conclusion. □

This is perhaps the most misleading case, to call product an inverse image is indeed not intuitive at all, but nonetheless the construction is by means of what we have agreed to call fibered product. To add some confusion observe that open immersions too are stable under base extension, indeed the usual diagram

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{f|^U} & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

verifies the universal property of a fibered product, that is $f^{-1}(U) \cong X \times_Y U$.

3.2.5 Fibers of a Morphism Another very important application of fibered products is the definition of *fibers* of a morphism. If $y \in Y$ is a closed point we can of course define the fiber of f over y as above, but we want to be able to consider any point so we need the following result.

Lemma (Exercise II.2.7 in Hartshorne, 1977)**.** *Let K be any field. To give a morphism of* Spec K *to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.*

Proof. Let $f\colon \operatorname{Spec} K \to X$. Since $\operatorname{Spec} K$ is made up by a single point we see that, as a morphism of topological spaces, this is equivalent to give a point x of $\operatorname{sp}(X)$. The sheaf $f_*(\mathscr{O}_{\operatorname{Spec} K})$ is a skyscraper sheaf, and on the stalk at x the map $f^\#\colon \mathscr{O}_x \to K$ is a local homomorphism, hence it induces an inclusion map $k(x) \to K$. Conversely such an inclusion map is always induced by a local homomorphism on the stalk at x, and this defines a morphism of sheaves from $\mathscr{O}_X$ to the skyscraper sheaf whose stalk at x is K. □

Let now $y \in Y$ be any point (not necessarily closed), $k(y)$ be the residue field of y, and let $\operatorname{Spec} k(y) \to Y$ be the natural morphism of the Lemma. Then we define the *fiber* of the morphism f over the point y to be the scheme

$$X_y = X \times_Y \operatorname{Spec} k(y)$$

We want to prove that this definition coincides with the purely topological one.

Proposition (Exercise II.3.10 in Hartshorne, 1977)**.** *If $f\colon X \to Y$ is a morphism of schemes, and $y \in Y$ is a point, then $\operatorname{sp}(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.*

Proof. It is enough to consider the affine case, the general case will then follow by means of open affine coverings. The situation is the following

$$\begin{array}{ccc} A & \xrightarrow{\quad\varphi\quad} & B \\ \downarrow & & \downarrow \\ \mathbb{F}\mathrm{rac}(A/\mathfrak{p}) & \longrightarrow & B \otimes_A \mathbb{F}\mathrm{rac}(A/\mathfrak{p}) \end{array}$$

where $\varphi\colon A \to B$ induces f, and $\mathfrak{p}$ is the prime ideal of A corresponding to y. In this case we have the following canonical isomorphisms of A-modules

$$B \otimes_A \mathbb{F}\mathrm{rac}(A/\mathfrak{p}) \;\cong\; B \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \;\cong\; B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \;\cong\; S^{-1}(B/\mathfrak{p}^e)$$

Note that $B_\mathfrak{p}$ is given by $S^{-1}B$ where S is the multiplicatively closed subset $\varphi(A \setminus \mathfrak{p})$; besides the submodule $\mathfrak{p}B_\mathfrak{p}$ happen to be an ideal of the ring $B_\mathfrak{p}$. Now $f^{-1}(y)$ is given by the set of prime ideals $\mathfrak{q} \subseteq B$ such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, and this is precisely the set of prime ideals such that $\mathfrak{q} \cap S = \varnothing$ and $\mathfrak{p}^e \subseteq \mathfrak{q}$. □

3.2.6 Further Properties of Morphisms of Finite Type Fibered products are actually another general tool to construct schemes, therefore properties of schemes that are preserved by this construction will be always desirable. In here we examine the behaviour of morphisms of finite type, carry on the account in §3.1.4.

Proposition (Exercise II.3.13 in Hartshorne, 1977)**.** *The following are properties of morphisms of finite type.*

(a) *Morphisms of finite type are stable under base extension.*

(b) *The product of two morphisms of finite type is of finite type.*

(c) *If X and Y are schemes of finite type over S, then the product $X \times_S Y$ is a scheme of finite type over S.*

Proof. Let $f\colon X \to S$ be any morphism of finite type and let $g\colon S' \to S$ be any base extension, then construct the pull-back of f along g

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Let $U \subseteq S$ be an open affine set, say $U = \operatorname{Spec} R$. Then the diagram restricts to the following

$$\begin{array}{ccc} f^{-1}(U) \times_U g^{-1}(U) & \xrightarrow{p_1} & f^{-1}(U) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ g^{-1}(U) & \xrightarrow{g} & U \end{array}$$

For each $V \subseteq g^{-1}(U)$ open affine set, say $V = \operatorname{Spec} B$, we can shrink more and get the following

$$\begin{array}{ccc} f^{-1}(U) \times_U V & \xrightarrow{p_1} & f^{-1}(U) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{g} & U \end{array}$$

Now there is a finite covering of $f^{-1}(U)$ by affine open subsets of the form $\operatorname{Spec} A_i$ with A_i a finitely generated R-algebra. Hence $f^{-1}(U) \times_U V$ can be covered by open affine sets of the form $\operatorname{Spec} A_i \otimes_R B$ and each one of these rings is a finitely generated B-algebra.

Let now $f\colon X \to Z$ and $g\colon Y \to H$ be morphisms of schemes over S, in order to produce a covering of $Z \times H$ consisting of open affine sets with the property (FFG) take affine open sets $U \subseteq Z$ and $V \subseteq H$, and construct the products $U \times V$. Now $(f \times g)^{-1}(U \times V)$ will be covered by a finite number of affine open subsets, namely the products of the existing finite coverings of $f^{-1}(U)$ and $g^{-1}(V)$. Now observe the following: let A and B be two R-algebras, C be a finitely generated A-algebra and D be a finitely generated B-algebra, then $C \otimes_R D$ is a finitely generated algebra over $A \otimes_R B$.

Part (c) follows immediately by part (a) once we recall that the composition of two morphisms of finite type is of finite type. □

3.3 Separated and Proper Morphisms

3.3.1 Separated Morphisms "Many techniques of geometry yield the most complete results when applied to compact Hausdorff spaces. Although affine schemes are quasi-compact in the Zariski topology, they do not share the good properties of compact spaces in other theories because the Zariski topology is not Hausdorff."

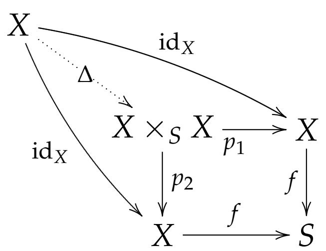
taken from Eisenbud and Harris (2000, §III.1.2)

As a result given two parallel morphisms of schemes $f, g\colon X \to Y$, the set where they are equal may not be closed. For example let Y be the affine line with a doubled origin defined in §2.3.5, and consider the two parallel morphisms given by the two inclusions $\mathbb{A}^1_k \to Y$. The set where they coincide is the open subset of $\mathbb{A}^1_k$ given by the complement of the origin, in particular it is not a closed subset (the affine line is connected).

"Such a pathology cannot happen, however, if Y is an affine scheme; nor, it turns out, can it happen when Y is a projective scheme. The desirable property that these schemes have, which is one of the most important consequences of the Hausdorff property for manifolds, is expressed by saying that Y is *separated* as a scheme over k."

taken from Eisenbud and Harris (2000, §III.1.2)

Definition. Let $f\colon X \to S$ be a morphism of schemes. The *diagonal morphism* is the unique morphism of schemes $\Delta\colon X \to X \times_S X$ whose composition with both the projection maps $p_1, p_2\colon X \times_S X \to X$ is the identity map of X:



The morphism f is *separated* if the diagonal morphism Δ is a closed immersion. In that case we also say X is separated over S.

Important results to keep in mind are the following, both of them to be found in Hartshorne (1977). The first is really an immediate consequence of the definition, and it is indeed proposed as an exercise in Eisenbud and Harris (2000), while the second is less immediate although not very difficult.

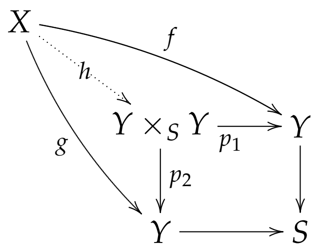
Proposition (II.4.1 in Hartshorne, 1977)**.** *If $f\colon X \to S$ is any morphism of affine schemes, then f is separated.*

Corollary (II.4.2 in Hartshorne, 1977)**.** *An arbitrary morphism $f\colon X \to S$ is separated if and only if the image of the diagonal morphism is a closed subset of $X \times_S X$*

3.3.2 Separated Schemes The first result we are going to prove is the evidence that we have indeed defined the property we were looking for. Namely we want to check that when Y is separated, given two parallel morphisms of schemes $f, g\colon X \to Y$, the set where they are equal is a closed subset of X.

Proposition (Exercise III-2 in Eisenbud and Harris, 2000)**.** *If Y is a separated scheme over S, then for any couple of parallel S-morphisms $f, g\colon X \to Y$ the set $Z \subseteq X$ consisting of $x \in X$ with $f(x) = g(x)$ is closed.*

Proof. What we are actually going to prove is a more general result, which shows that Z is always defined via the diagonal morphism. We want to see that with respect to the following diagram we have $Z = h^{-1}\big(\Delta(Y)\big)$.



"$\supseteq$" let $x \in h^{-1}\big(\Delta(Y)\big)$ then $h(x) = \Delta(y)$ for some $y \in Y$. This implies $f(x) = p_1\big(h(x)\big) = p_1\big(\Delta(y)\big) = y$ and analogously $g(x) = y$ i.e. $x \in Z$.

"$\subseteq$" let $x \in Z$ then $f(x) = g(x) = y$. This implies $p_1\big(\Delta(y)\big) = y = f(x) = p_1\big(h(x)\big)$ and analogously $p_2\big(\Delta(y)\big) = p_2\big(h(x)\big)$, being p_1 and p_2 globally monic we can conclude $\Delta(y) = h(x)$ i.e. $x \in h^{-1}\big(\Delta(Y)\big)$.

Now Y is separated, so $\Delta(Y)$ is a closed subset of $Y \times_S Y$, and h is continuous so Z is a closed subset of X. □

A *Valuative Criterion of Separation* exists (see for example Hartshorne, 1977, Theorem II.4.3), it is undoubtedly useful but in my experience at this stage is too difficult. To grasp its meaning the reader should be familiar with valuation rings, for which he will need to read Atiyah and Macdonald (1969, §V). To check separation of a scheme X over an affine scheme $S = \operatorname{Spec} B$ there is Proposition 2 in Shafarevich (1994b, §V.4.3), involving only the sheaf of structure of the scheme X. It is stated as a sufficient condition but Proposition III.3.6 in Liu (2002) is in fact the proof that it is an equivalence and besides a local property. The basic fact behind this result is the following.

Lemma (Exercise II.4.3 in Hartshorne, 1977)**.** *Let X be a separated scheme over an affine scheme S. Let U and V be open affine subsets of X. Then $U \cap V$ is also affine.*

Proof. Indeed $U \cap V = \Delta^{-1}(U \times V)$. If U and V are affine then so is $U \times V$, and if X is separated then Δ is a closed immersion, and hence $U \cap V$ is a closed subscheme of an affine scheme. This is affine, as we have proved in §2.1.6. This proof was taken from Shafarevich (1994b, §V.4.3, Proposition 3) □

3.3.3 Properties of Separated Morphisms An *algebraic variety* is an integral separated scheme of finite type over an algebraically closed field k (often $\mathbb{C}$). Just like manifolds are Hausdorff by definition varieties are separated, and this property is central in the theory. Every morphism of varieties is separated, and every construction we can do with varieties gives rise to separated schemes or morphisms, by virtue of the following result.

Proposition (Corollary II.4.6 in Hartshorne, 1977)**.** *Assume that all schemes are Noetherian in the following statements.*

(a) *Open and closed immersions are separated.*

(b) *A composition of two separated morphisms is separated.*

(c) *Separated morphisms are stable under base extension.*

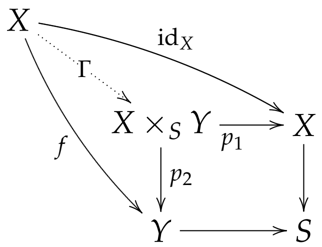
(d) *The product of two separated morphisms is separated.*

(e) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms such that $g \circ f$ is separated then f is separated.*

(f) *A morphism $f\colon X \to Y$ is separated if and only if Y can be covered by open subsets V_i such that the restriction $f^{-1}(V_i) \to V_i$ is separated for all i.*

A very similar statement is Proposition III.3.9 Liu (2002), which shows that you can prove many of these results without using the Valuative Criterion in Hartshorne (1977). It can be useful to observe that in the case of open and closed immersions the diagonal morphism is in fact an isomorphism, this is once again in contrast with the intuitive idea of product but nonetheless it is very useful when thinking about this construction as a pull-back.

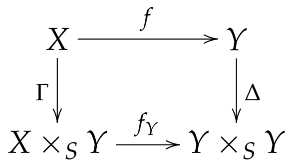
3.3.4 Graph of a Morphism If $f\colon X \to Y$ is a morphism of varieties, we can construct the *graph* of f with the fibered product, in fact this construction makes sense in the category of schemes over S as follows.



For varieties Γ has the good and desirable property to be a closed immersion. Again this is true more generally whenever Y is separated over S, and now we are going to prove it. It should be said that this result is in fact an hint to Exercise II.4.8 in Hartshorne (1977).

Proposition (Exercise II.4.8 in Hartshorne, 1977)**.** *Let $f\colon X \to Y$ be a morphism of schemes over S, and assume Y is separated. Then the graph morphism Γ defined above is a closed immersion. More precisely, it is obtained by the diagonal morphism over Y, $\Delta\colon Y \to Y \times_S Y$, by base extension.*

Proof. The proof is all about showing that the following diagram is in fact a fibered product



where f_Y is the product of f and the identity of Y. Depending on your skills in drawing commutative diagrams, this can be either obvious or a nightmare, I suggest to proceed as follows. Draw the diagrams about the construction of Δ and the construction of f_Y, being careful to call in the same way maps which are the same. Then the diagram above will be commutative because the projections of $Y \times_S Y$ are globally monic. To check the universal property, let $a\colon W \to X$ and $b\colon W \to X \times_S Y$ be two morphisms such that $\Delta \circ a = f_Y \circ b$, then $p_1 \circ b$ is the unique morphism you want to construct. □

3.3.5 Morphisms of Varieties Another good property of varieties which is a consequence of them being separated is the following: if f and g are two morphisms of varieties from X to Y which agree on an open dense subset of X, then $f = g$. We know already that $f = g$ as continuous functions (§3.3.2), but we want to prove something more, namely that the morphisms of sheaves $f^\#$ and $g^\#$ are also equal. In fact, we want to solve Exercise II.4.2 in Hartshorne (1977) which is stated for more general schemes than varieties.

Remark. Let X be a reduced scheme and let $f\colon X \to \operatorname{Spec} A$ be a morphism of schemes. Then f is induced by the ring homomorphism $\varphi\colon A \to \Gamma(X, \mathscr{O}_X)$ defined on global sections by $f^\#$ (adjunction §2.1.4). We have

$$\overline{\operatorname{Im} f} = \mathcal{V}(\ker \varphi)$$

that is, the closure of the image of f is the closed subset defined by the kernel of φ.

We denote $r_x\colon \Gamma(X, \mathscr{O}_X) \to \mathscr{O}_{X,P}$ the localisation map of the structure sheaf on X. The image of f is given by the following set

$$\operatorname{Im} f = \left\{ \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} = \varphi^{-1}\big(r_x^{-1}(\mathfrak{m}_x)\big) \text{ for some } x \in X \right\}$$

Clearly every such prime ideal contains $\ker \varphi$ so that $\overline{\operatorname{Im} f} \subseteq \mathcal{V}(\ker \varphi)$. We need to show the converse, that is for each $\mathfrak{p} \in \mathcal{V}(\ker \varphi)$ every open neighborhood of $\mathfrak{p}$ intersects $\operatorname{Im} f$. As usual we can restrict ourselves to basic open sets, so let $\mathfrak{p} \in D(a)$ that is let $a \in A$ be such that $a \notin \mathfrak{p}$. In particular $a \notin \ker \varphi$ so that $\varphi(a) \neq 0$. We claim that there is a point $x \in X$ such that $f(x) \in D(a)$, that is $r_x\big(\varphi(a)\big) \notin \mathfrak{m}_x$. Indeed if this was not the case then the open set $X_{\varphi(a)}$ was empty, and since X is reduced this implies $\varphi(a) = 0$, a contradiction. A little more of explanation is perhaps needed: for any open affine subset $\operatorname{Spec} B \subseteq X$ the intersection $X_{\varphi(a)} \cap \operatorname{Spec} B$ is given by the distinguished open subset of $\operatorname{Spec} B$ defined by $r_B\big(\varphi(a)\big)$, and this is empty if and only if $r_B\big(\varphi(a)\big) = 0$, since X is reduced.

Lemma. *Let X be a reduced scheme and let $U \subseteq X$ be an open dense subset. Then for any open set $W \subseteq X$ the restriction map $r_W\colon \mathscr{O}_X(W) \to \mathscr{O}_X(W \cap U)$ is injective.*

Proof. Assume W to be affine first, so that $i_W\colon W \cap U \to W$ is an open immersion of a reduced scheme into an affine scheme. This immersion is induced by r_W, and by the remark above we have $\overline{\operatorname{Im} i_W} = \mathcal{V}(\ker r_W)$. Since $\overline{\operatorname{Im} i_W} = W$ we have the inclusion $\ker r_W \subseteq \operatorname{Nil}\big(\mathscr{O}_X(W)\big)$ and since X is reduced this nilradical is trivial, hence $\ker r_W = 0$.
If now W is any open subset the conclusion follows immediately by covering W with open affine subsets. □

Proposition (Exercise II.4.2 in Hartshorne, 1977). *Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two S-morphisms of X to Y which agree on an open dense subset of X. Then $f = g$.*

Proof. We already know that the set $Z \subseteq X$ consisting of those $x \in X$ such that $f(x) = g(x)$ is closed (§3.3.2 above), clearly it contains U and since U is dense we can conclude that $Z = X$, that is f and g consist of the same continuous map. We have to prove that the morphisms of sheaves $f^\#\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$ and $g^\#\colon \mathscr{O}_Y \to g_*\mathscr{O}_X$ are equal, knowing that the restrictions $\left(f|_U\right)^\#$ and $\left(g|_U\right)^\#$ are equal. Observe also that the sheaves $f_*\mathscr{O}_X$ and $g_*\mathscr{O}_X$ are the same. The morphism of sheaves

$$\left(f|_U\right)^\# : \mathscr{O}_Y \longrightarrow \left(f|_U\right)_* \mathscr{O}_X$$

is defined on any open set $V \subseteq Y$ by the ring homomorphism

$$\mathscr{O}_Y(V) \longrightarrow \mathscr{O}_X\left(f^{-1}(V) \cap U\right)$$

We call this homomorphism ψ, since it is the same when induced by f or by g. Then we have the following diagram for any $V \subseteq Y$

$$\begin{array}{ccc} \mathscr{O}_Y(V) & \overset{f^\#_V}{\underset{g^\#_V}{\rightrightarrows}} & \mathscr{O}_X(f^{-1}(V)) \\ & \searrow^{\psi} & \downarrow{\scriptstyle res} \\ & & \mathscr{O}_X(f^{-1}(V) \cap U) \end{array}$$

Now X is reduced and U is a dense subset, so by the Lemma the restriction map is injective, and hence $f^\#_V = g^\#_V$ for any open subset $V \subseteq Y$, that is the two morphisms of sheaves are the same. □

3.3.6 Proper Morphisms "If $\varphi\colon X \to Y$ is a map of projective varieties, then indeed φ maps closed subvarieties of X to closed subvarieties of Y. Somewhat more generally, if we take the product of such a map with an arbitrary variety Z, to get

$$\psi = \varphi \times \mathrm{id}_Z\colon X \times Z \longrightarrow Y \times Z$$

then ψ maps closed subvarieties of $X \times Z$ to closed subvarieties of $Y \times Z$. It turns out that *this, with the separation property, is the central property of projective varieties that makes them so useful*. But it is a property satisfied by a slightly larger class of varieties than the projective ones, and it is a property that is sometimes easier to verify than projectivity, so it is of great importance to make a general definition."

taken from Eisenbud and Harris (2000, §III.1.2)

Definition. A morphism $f\colon X \to S$ is *proper* if it is separated, of finite type, and universally closed. Here we say that a morphism is *closed* if it carries closed subsets of X into closed subsets of S, it is *universally closed* if any pull-back of it along any base extension is closed. This means that for any morphism $S' \to S$ in the diagram below

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \longrightarrow & S \end{array}$$

the projection p_2 is a closed morphism. A scheme over S is proper if its structural morphism $X \to S$ is proper.

A similar result to the one in §3.3.3 about properties of separated morphisms holds for proper morphisms. In Hartshorne (1977) it is given as a corollary to the *Valuative Criterion of Properness*, which is as difficult as the Valuative Criterion of Separation. However Proposition III.3.16 in Liu (2002) is the evidence that some properties can be shown without it.

Proposition (Corollary II.4.8 in Hartshorne, 1977)**.** *Assume that all schemes are Noetherian in the following statements.*

(a) *Closed immersions are proper.*

(b) *A composition of two proper morphisms is proper.*

(c) *Proper morphisms are stable under base extension.*

(d) *The product of two proper morphisms is proper.*

(e) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms, if $g \circ f$ is proper and g is separated, then f is proper.*

(f) *A morphism $f\colon X \to Y$ is proper if and only if Y can be covered by open subsets V_i such that the restriction $f^{-1}(V_i) \to V_i$ is proper for all i.*

The first three statements all follows from properties of morphisms of finite type (see §3.1.4 and §3.2.6) and properties of separated morphisms. Each time there is only to check that the relevant map is universally closed, which is an easy exercise useful to familiarise with the definition. The rest follows by applying the more general result below.

3.3.7 The Fundamental Lemma about Attributes of Morphisms We devote this subsection to the proof of probably the most important result of this chapter. It applies to almost every construction in the category of schemes, and it allows us to derive properties of the constructed object.

Lemma (Exercise II.4.8 in Hartshorne, 1977)**.** *Let $\mathscr{P}$ be an attribute of morphisms of schemes such that*

- *a closed immersion has $\mathscr{P}$.*
- *the composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$.*
- *$\mathscr{P}$ is stable under base extension.*

Then the following holds

i) *the product of two morphisms having $\mathscr{P}$ has $\mathscr{P}$.*

ii) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms, if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$.*

iii) *If $f\colon X \to Y$ has $\mathscr{P}$, then $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ has $\mathscr{P}$. Where $\widetilde{X}$ denotes the reduced scheme associated to X.*

Proof. Let $f\colon X \to Z$ and $g\colon Y \to H$ be morphisms of schemes over S, we rewrite the diagram about the construction of their product as follows

$$\begin{array}{ccccc}
X \times Y & \xrightarrow{h_1} & X \times (Z \times H) & \xrightarrow{\tau_1} & X \\
\downarrow{\scriptstyle h_2} & \searrow{\scriptstyle f\times g} & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle f} \\
Y \times (Z \times H) & \xrightarrow[\sigma_2]{} & Z \times H & \xrightarrow[\pi_1]{} & Z \\
\downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow \\
Y & \xrightarrow{g} & H & \longrightarrow & S
\end{array}$$

where h_1 is the unique morphism such that $\tau_1 h_1$ is the first projection of $X \times Y$ and h_2 is the unique morphism such that $\sigma_1 h_2$ is the second projection. In this diagram every square is a fibered product, although you may want to check that this is true also for the top left corner one, so $f \times g$ has the property $\mathscr{P}$ by base extension and composition.

For statement *ii*) we can regard X and Y as schemes over Z, with Y separated, and construct the graph of f over Z; this is a closed immersion as in §3.3.4, hence it has the property $\mathscr{P}$. Now f has $\mathscr{P}$ by base extension and composition.

Given $f\colon X \to Y$ we construct $\widetilde{f}$ as follows. First we consider the composition of f with the closed immersion $j_X\colon \widetilde{X} \to X$, then there is a unique morphism $\widetilde{f}$ such that the following diagram is commutative (see §2.4.2)

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{j_X} & X \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \xrightarrow{j_Y} & Y \end{array}$$

Now observe that since j_X and j_Y are closed immersions, they satisfy $\mathscr{P}$ and also they are separated. So $j_Y\widetilde{f}$ satisfies $\mathscr{P}$, and since j_Y is separated $\widetilde{f}$ satisfies $\mathscr{P}$ too. □

3.4 Finite Morphisms

Definition. A morphism $f\colon X \to Y$ is called *finite* if for every point $y \in Y$ there is an open affine neighborhood $V = \operatorname{Spec} B \subseteq Y$ such that the inverse image $f^{-1}(V)$ is affine and equal to $\operatorname{Spec} A$, where A is a finite B-algebra, that is a B-algebra which is also a finitely generated B-module.

3.4.1 Finite is Local on the Base In particular a finite morphism is a morphism of finite type, but this definition is much more restrictive. For instance an open immersion need not be finite, let $Y = \mathbb{A}^2_k$ and let X be the complement of the closed point $(0,0)$. A closed immersion instead is a finite morphism as we have seen in §3.1.4. The first result we are now going to prove is that this is an attribute local on the base.

Proposition (Exercise II.3.4 in Hartshorne, 1977)**.** *A morphism $f\colon X \to Y$ is finite if and only if for* every *open affine subset $V =$* Spec *B of Y, the inverse image $f^{-1}(V)$ is affine and equal to* Spec *A, where A is a finite B-algebra.*

Proof. To simplify the argument, we say that an open affine subset $V = \operatorname{Spec} B$ of Y has the property (F) if the inverse image $f^{-1}(V)$ is affine and equal to $\operatorname{Spec} A$, where A is a finite B-algebra. We have to show that if Y can be covered by open affine subsets with the property (F) then every affine subset of Y has the property (F).

First note that if $Y = \operatorname{Spec} B$ is affine and has the property (F) then every distinguished open subset of Y has the property (F). Indeed in that case X is affine and equal to $\operatorname{Spec} A$, where A is a finite B-algebra. Then for any $b \in B$ we have $f^{-1}\big(D(b)\big) = X_{\overline{b}}$ where $X_{\overline{b}} = \operatorname{Spec} A_{\overline{b}}$, and $A_{\overline{b}}$ is a finite B_b-algebra. Back to the general case, we have a base for the topology in Y consisting of affine subsets with the property (F). So we have reduced to prove the following: if Y is affine and it can be covered by affine open sets that have the property (F) then Y has the property (F).

Let $Y = \operatorname{Spec} B$ and let V be an affine open subset with the property (F). Then for any $b \in B$ such that $Y_b \subseteq V$, Y_b has the property (F). Indeed $f^{-1}(Y_b)$ is a basic open set of $f^{-1}(V)$ and hence is a finite B_b-algebra. So we have reduced to prove the following: if Y is affine and it can be covered by basic open sets with the property (F) then Y has the property (F).

Since Y is affine it is quasi-compact, hence a finite covering will do. So let $b_1, \dots, b_r$ be a finite number of elements of B such that they generate the unit ideal and each $D(b_i)$ has the property (F). Let $A = \Gamma(X, \mathscr{O}_X)$ and consider $f^\#\colon B \to A$. We have a finite number of elements $f^\#(b_1), \dots, f^\#(b_r) \in A$ such that the open subsets $X_{f^\#(b_i)} = f^{-1}\big(D(b_i)\big)$ are affine and $f^\#(b_1), \dots, f^\#(b_r)$ generate the unit ideal in A. By the affineness criterion of §2.1.5 this is enough to say X is affine, say $X = \operatorname{Spec} A$. We have to prove A is a finite B-algebra.

This is a purely algebraic problem: A is a B-algebra, $b_1, \dots, b_r$ are a finite number of elements of B, which generate the unit ideal, and each localisation A_{b_i} is a finite B_{b_i}-algebra. We have to prove A is a finite B-algebra. Since $b_1, \dots, b_r$ generate the unit ideal we can say $1 = \sum_{i=1}^r h_i b_i^\alpha$ for any $\alpha \in \mathbb{N}$ where $h_i \in B$. For each $i = 1, \dots, r$ let the algebra A_{b_i} be generated by

$$(x_{i1}/b_i^{\beta_{i1}}), \dots, (x_{is_i}/b_i^{\beta_{is_i}})$$

Let $g \in A$, then for any $i = 1, \dots, r$ in the ring A_{b_i} we have $g = g_i/b_i^{\gamma_i}$, where g_i is a B-linear expression in $x_{i1}, \dots x_{is_i}$. This means that in A we have

$$b_i^{\alpha_i} g - b_i^{\delta_i} g_i = 0 \quad \text{for some integers } \alpha_i, \delta_i \in \mathbb{N}$$

Without loss of generality we can assume $\alpha_i = \alpha$, independent from i. Then we have the following relation in A.

$$\sum_{i=1}^r h_i \left(b_i^\alpha g - b_i^{\delta_i} g_i \right) = 0$$

that is

$$g = \sum_{i=1}^r h_i b_i^{\delta_i} g_i$$

where on the right side we have a B-linear expression in a finite set of elements of A and this finite set doesn't depend on g. Thus A is a finite B-algebra. □

We can use this result to show that the composition of two finite morphisms is finite, as follows. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be finite morphisms, then for any open affine subset $W = \operatorname{Spec} R$ of Z the inverse image $g^{-1}(W)$ is affine and equal to $\operatorname{Spec} B$, where B is a finite R-algebra, while the inverse image $f^{-1}\big(g^{-1}(W)\big)$ is affine and equal to $\operatorname{Spec} A$, where A is a finite B-algebra. Thus A is a finite R-algebra and $g \circ f$ is finite.

3.4.2 Properties of Finite Morphisms We are now going to prove that finite morphisms are stable under base extension, so that we can apply the fundamental Lemma about attributes of morphisms. This in particular means that products of finite morphisms are finite, further if $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms, with $g \circ f$ finite and g separated, then f is finite.

Proposition (Exercise III.3.17 in Liu, 2002)**.** *Finite morphisms are stable under base extension.*

Proof. Let $f\colon X \to S$ be a finite morphism and let $g\colon S' \to S$ be any morphism of schemes. Consider the product of X and S' with respect to S, that is the following commutative diagram

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

We have to show that p_2 is finite. To this purpose it is enough to assume S to be affine, in which case also X should be affine. So let $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$, and let $U = \operatorname{Spec} B$ be any open affine set of S'. Then $p_2^{-1}(U) = \operatorname{Spec} A \otimes_R B$ and we know that A is a finitely generated module over R. Now if $\{a_i\}$ is a system of generators for A over R then $\{a_i \otimes 1\}$ is a system of generators for $A \otimes_R B$ over B, and we are done. □

Let $f\colon X \to Y$ be a finite morphism of schemes. For any $y \in Y$ we look at the fiber X_y of f over y, which in this generality is just a scheme. Nevertheless without any other hypothesis we are able to prove that $\operatorname{sp}(X_y)$ is a finite set, in fact it will soon be clear also that the dimension of X_y is zero, which is a rather strong result

Corollary (Exercise II.3.5 in Hartshorne, 1977)**.** *The fibers of a finite morphism are affine schemes whose underlying topological spaces are finite and consist of closed points only.*

Proof. We start with a fibered product diagram, if $y \in Y$ is any point then the fiber X_y is given by the following

$$\begin{array}{ccc} X_y & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} k(y) & \hookrightarrow & Y \end{array}$$

Since finite morphisms are stable under base extension, X_y is an affine scheme finite over $k(y)$, which means $X_y = \operatorname{Spec} A$ where A is a finite dimensional vector space over $k(y)$. In particular by Proposition 6.10 in Atiyah and Macdonald (1969) A will be an Artin ring, therefore $\operatorname{Spec} A$ will consist of a finite number of maximal ideals only. □

3.4.3 Integral Morphisms A morphism $f\colon X \to Y$ is called *integral* if for every point $y \in Y$ there is an open neighborhood $V = \operatorname{Spec} B \subseteq Y$ such that the inverse image $f^{-1}(V)$ is affine and equal to $\operatorname{Spec} A$, where A is an integral B-algebra.

The reader may want to learn something about integral dependence in Atiyah and Macdonald (1969, Chapter 5) or Eisenbud (1995, Chapter 4). According to Liu (2002, Exercise III.3.15) this is a local property, but we are not going to describe it in this generality. In fact, the next result will clarify in which sense finite and integral morphisms are the same.

Proposition. *Let $f\colon X \to Y$ be a morphism of finite type. Then f is finite if and only if it is integral.*

Observe that we are not claiming that finite and integral morphisms are always the same, but only that they are the same *among morphisms of finite type*. The proof of the proposition reduces to a purely algebraic argument that we describe next.

Lemma (1 in Lecture 10 of Dolgachev, nd)**.** *Assume that B is a finitely generated A-algebra. Then B is integral over A if and only if B is a finitely generated module over A (i.e. B is a finite A-algebra).*

Proof. Only slight modifications have been made to the original argument in Dolgachev (nd). Assume B is integral over A. Let $x_1, \dots, x_n$ be generators of B as an A-algebra, i.e. for any element $b \in B$ there exists a polynomial $F \in A[Z_1, \dots, Z_n]$ such that $b = F(x_1, \dots, x_n)$. Since each x_i is integral over A, there exists some integer $n(i)$ such that $x_i^{n(i)}$ can be written as a linear combination of lower powers of x_i with coefficients in A. Hence every power of x_i

can be expressed as a linear combination of powers of x_i of degree less than $n(i)$. Thus there exist a number $N > 0$ such that every $b \in B$ can be written as a polynomial in $x_1, \ldots, x_n$ of degree $< N$. This shows that a finite set of monomials in $x_1, \ldots, x_n$ generate B as an A-module.

Conversely, assume that B is a finite A-algebra. Then every $b \in B$ can be written as a linear combination $b = a_1 b_1 + \ldots + a_r b_r$ where $b_1, \ldots, b_r$ is a fixed set of elements in B and $a_i \in A$. We can assume also $b_1 = 1$. Multiplying both sides by b_i and expressing each product $b_i b_j$ as a linear combination of b_i we get

$$bb_i = \sum_j a_{ij} b_j, \qquad a_{ij} \in A$$

This shows that the vector $\mathbf{b} = (b_1, \ldots, b_r)^T$ satisfies the linear equation $(bI_r - M)\mathbf{b} = 0$, where $M = (a_{ij})$. Let $D = \det(bI_r - M)$ and observe that this is a monic polynomial in b with coefficients in A. Now consider the following block matrix with coefficients in B

$$W = \begin{pmatrix} b_1 & 0 \\ \mathbf{b}' & I_{r-1} \end{pmatrix}$$

where $\mathbf{b}'$ is the vector $(b_2, \ldots, b_r)$. Since in the product matrix $(bI_r - M)W$ the first column is zero we have $\det\big((bI_r - M)W\big) = 0$, while $\det W = b_1 = 1$. Using Binet's Theorem we conclude $D = 0$ which is a monic equation for b with coefficients in A. □

Example (A morphism of affine varieties which is not finite). Project the hyperbola $xy = 1$ onto the x-axis. This map has finite fibers but is not finite. Indeed it is induced by

$$k[x] \hookrightarrow k[x,y]/(xy-1)$$

and $k[x,y]/(xy-1)$ is not a finite $k[x]$-algebra, because in it $\overline{y}$ is not the root of a monic polynomial with coefficients in $k[x]$. Observe that if we take the same projection but onto the x-axis without the origin this morphism is finite.

3.4.4 Properness of Finite Morphisms We have seen that finite morphisms have finite fibers, and in what follows we are going to prove that a finite morphism is proper. "A deep result, due to Chevalley, asserts that when Y is a separated Noetherian scheme then conversely every proper morphism $f\colon X \to Y$ with finite fibers is a finite morphism"

taken from Mumford (1999, §II.7)

Proposition (Exercise II.3.5 in Hartshorne, 1977)**.** *A finite morphism is closed.*

Proof. It is clearly enough to prove the statement in the affine case only, so let $\varphi\colon B \to A$ be a ring homomorphism inducing $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$, such that A is a finite B-algebra. Let $\mathfrak{a} \subseteq A$ be any ideal, we know from §1.4.1 that $\overline{f(\mathcal{V}(\mathfrak{a}))} = \mathcal{V}(\mathfrak{a}^c)$ so we have to show the inclusion $\mathcal{V}(\mathfrak{a}^c) \subseteq f(\mathcal{V}(\mathfrak{a}))$, which means

$$\mathcal{V}(\mathfrak{a}^c) \subseteq \Big\{\mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{q} = \varphi^{-1}(\mathfrak{p}) \text{ with } \mathfrak{p} \in \mathcal{V}(\mathfrak{a})\Big\}$$

In other words for any $\mathfrak{q} \in \mathcal{V}(\mathfrak{a}^c)$ we have to show that there exists a prime ideal $\mathfrak{p} \in \mathcal{V}(\mathfrak{a})$ such that $\mathfrak{q} = \mathfrak{p}^c$. Now it is enough to consider the quotient map $\varphi_{\mathfrak{a}}\colon B/\mathfrak{a}^c \to A/\mathfrak{a}$ and apply the following

> (Atiyah and Macdonald, 1969, Theorem 5.10) Let $B \subseteq A$ be rings, A integral over B, and let $\mathfrak{p}$ be a prime ideal of B. Then there exists a prime ideal $\mathfrak{q}$ of A such that $\mathfrak{p} = \mathfrak{q}^c$.

Note that $A/\mathfrak{a}$ is a finite algebra over $B/\mathfrak{a}^c$, hence it is integral (see §3.4.3 above). □

Corollary (Exercise II.4.1 in Hartshorne, 1977)**.** *A finite morphism is proper.*

Proof. We have already observed that a finite morphism is of finite type, moreover since finite morphisms are stable under base extension and are closed they are also universally closed. So we only need to prove that a finite morphism is separated, but this is true more generally for *affine morphisms*. □

A morphism of schemes $f\colon E \to X$ is affine if there is an open affine covering $\{U_i\}$ of X such that $f^{-1}(U_i)$ is affine for each i. In §3.4.1 we have shown among other things that this is a local property, that is the morphism $f\colon E \to X$ is affine if and only if $f^{-1}(U)$ is affine for *every* open affine subset U of X.

Proposition (Exercise II.5.17 in Hartshorne, 1977)**.** *Every affine morphism of schemes $f\colon E \to X$ is separated.*

Proof. For any open affine subset $U \subseteq X$ the restricted morphism

$$f|^U\colon f^{-1}(U) \longrightarrow U$$

is a morphism of affine schemes. Therefore it is separated, which means that the diagonal morphism

$$f^{-1}(U) \longrightarrow f^{-1}(U) \times_U f^{-1}(U)$$

is a closed immersion. Observe that this is the restriction of the diagonal $\Delta\colon E \to E \times_X E$ to a subset V_U of the product $E \times_X E$, and that sets of the form V_U cover the product as U varies over all affine subsets of X. Now use the Lemma in §3.2.3. □

Chapter 4

Basic Algebraic Geometry

For the first time in these notes we are going to see some geometry as it is meant to be. The main part of this chapter is dedicated to understand what is the dimension of a variety, it contains a lengthy discussion which will hopefully clarify how difficult a concept this is. There will be also a general discussion about Hilbert's Nullstellensatz in his most abstract formulation. We are not going to prove it, but in here the reader will find a geometric interpretation. In the process we will also look more deeply to the structure of algebraic schemes, gaining a better idea about the "shape" of the set of closed points inside a scheme. Finally thinking about morphisms in Algebraic Geometry we arrive to the definition of rational map, which is the weaker idea of a morphism characteristic of the subject. In Hartshorne (1977) this is defined without the machinery of schemes, so that by comparison the reader will be able to see how natural it is to work in the environment of schemes.

4.1 Algebraic Varieties

4.1.1 The Category of Algebraic Varieties An *Algebraic Scheme* is a separated scheme of finite type over a field k. We don't make any assumption on the base field k, assuming that whether it will be algebraically closed or not will depend on the context. An *Algebraic Variety* is an integral algebraic scheme. If a variety is proper over k, we also say it is *complete*. An *affine variety* is an affine scheme which is a variety, that is the spectrum of a finitely generated domain over k, which we usually call affine domain.

A morphism of varieties is a morphism of schemes over k, and every such morphism is of finite type and separated (this is in general a consequence of the fundamental Lemma about attributes of morphisms, see §3.3.7). Observe that an affine variety X is by definition given by the spectrum of a quotient

$k[x_1, \dots, x_n]/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal, in particular it is always endowed with a closed immersion $X \to \mathbb{A}^n_k$. Looking back at what we proved about closed immersions in §2.1.6 the converse is also true, namely if there exists a closed immersion from a variety X in affine n-space then X is affine.

Example (Exercise I.1.1 in Hartshorne, 1977). Let Y be the plane curve $y = x^2$, that is Y is the affine scheme given by the spectrum of the ring $k[x,y]/(y - x^2)$, and let Z be the plane curve $xy = 1$. Then we have the canonical isomorphisms

$$k[x,y]/(y - x^2) \cong k[x] \text{ and } k[x,y]/(xy - 1) \cong k[x, x^{-1}]$$

In particular Y and Z are not isomorphic, indeed any morphism $Y \to Z$ is induced by a ring homomorphism $\varphi\colon k[x, x^{-1}] \to k[x]$ which necessarily satisfies $\varphi(x) \in k$. Now it is a well known result that whenever F is any irreducible quadratic polynomial in $k[x,y]$, the plane curve defined by $F = 0$ is isomorphic to either Y or Z, see for example Audin (2003, §VI).

Definition. Let $\mathcal{C}$ be a category. A morphism $m\colon X \to Y$ is a *monomorphism* (or monic) in $\mathcal{C}$ when for any two parallel arrows $f, g\colon W \to X$ the equality $mf = mg$ implies $f = g$; in other words m is a monomorphism if it can always be cancelled on the left (is *left cancellable*). A morphism $\varphi\colon A \to B$ is an *epimorphism* (or epi) in $\mathcal{C}$ when for any two parallel arrows $\psi, \xi\colon B \to R$ the equality $\psi\varphi = \xi\varphi$ implies $\psi = \xi$; in other words φ is an epimorphism when it is *right cancellable*.

For instance in the category of sets epimorphisms are surjective functions while monomorphisms are injective, but in the category of algebraic varieties these properties don't seem to be fully characterised. This is partly due to the fact that they are not really well characterised in the category of rings, for instance a surjective ring homomorphism and a localisation homomorphism are both epi. Nevertheless we are able to prove that at least those morphisms that we call immersions are monic.

Proposition. *In the category of schemes open and closed immersions are monic.*

Proof. The case of open immersions is obvious, while for closed immersions we have to be slightly more careful. So let $m\colon Z \to X$ be a closed immersion, and let $a, b\colon W \to Z$ be two morphisms such that $ma = mb$. When X is affine, this situation is adjoint in the category of rings to the following

$$A \xrightarrow{\gamma} A/\mathfrak{a} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma(W, \mathscr{O}_W)$$

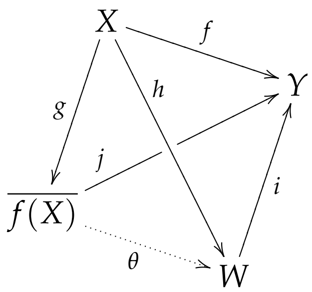
where γ is surjective and therefore epi. In the general case, for any open affine subset $U \subseteq X$ we can repeat this argument to find $a|^{U \cap Z} = b|^{U \cap Z}$, and conclude $a = b$. □

4.1.2 The Image of a Morphism If $f\colon X \to Y$ is any continuous map, we can factor it through the closed subset $\overline{f(X)}$ of Y as a dominant morphism followed by a closed immersion. The same factorisation is possible in the set-up of schemes, although not in complete generality. For instance if X and Y are affine schemes, f is induced by a ring homomorphism $\varphi\colon B \to A$ which will factor through the sub-ring of A given by the image of φ as follows

$$B \longrightarrow \operatorname{Im}\varphi \longrightarrow A$$

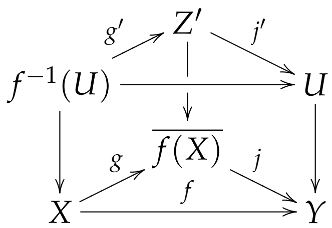
where $B \to \operatorname{Im}\varphi$ is surjective, and $\operatorname{Im}\varphi \to A$ is injective. This purely algebraic configuration corresponds precisely to the geometric one, indeed injective homomorphisms induce dominant maps and surjective homomorphisms induce closed immersions (see §1.4.6). We call $\operatorname{Spec}\operatorname{Im}\varphi$ the *scheme-theoretic image of* f, and we denote it $\overline{f(X)}$ in analogy with the topological construction.

Theorem (Exercise II.3.17 in Liu, 2002)**.** *Let $f\colon X \to Y$ be a quasi-compact morphism of schemes. Then there exists a scheme $\overline{f(X)}$, endowed with a closed immersion $j\colon \overline{f(X)} \to Y$ and a dominant morphism $g\colon X \to \overline{f(X)}$ such that $f = jg$ and with the following universal property*



for every other scheme W endowed with a closed immersion i and a morphism h such that $f = ih$ there exists a unique closed immersion θ such that $i\theta = j$.

Furthermore, for every open subset $U \subseteq Y$ we have the following commutative diagram, where $Z' = \overline{f\big(f^{-1}(U)\big)}$ and vertical arrows are open immersions



That is, the scheme-theoretic image of the restricted morphism $f|^U$ is naturally endowed with an open immersion in the scheme-theoretic image of f.

Proof. If f is a morphism of affine schemes the above argument completes the proof, the universal property being easily checked (note that W is necessarily affine). If $Y = \operatorname{Spec} A$ is affine, then f is adjoint to the homomorphism $f_Y^\#\colon A \to \Gamma(X, \mathscr{O}_X)$ and if such a scheme exists it must be affine. Since f is quasi-compact, X can be covered by a finite number of open affine subsets (see §3.1.3), say $V_i = \operatorname{Spec} B_i$; the composition $V_i \to X \to A$ is induced by the composition of ring homomorphisms $r_i \circ f_Y^\#$ where $r_i\colon \Gamma(X, \mathscr{O}_X) \to B_i$ is the restriction of the sheaf $\mathscr{O}_X$, let $\mathfrak{a}_i$ be its kernel and let $\mathfrak{a}$ be the sum of the $\mathfrak{a}_i$. Now the affine scheme $\operatorname{Spec} A/\mathfrak{a}$ comes endowed with a closed immersion and a dominant morphism as required, and the universal property is easily checked (again W is necessarily affine).

Let $U = D(\alpha)$ be an open basic subset of Y, which we still assume to be affine, then as in §2.1.3 $f^{-1}(U) = X_\alpha$ and for each i $X_\alpha \cap V_i$ is affine and given by $\operatorname{Spec} B_{i\alpha}$. This proves that, when Y is affine and U is an open basic subset of Y, there exists an open immersion $Z' \to \overline{f(X)}$ as in the statement. If now we let U to be any open affine subset of Y, we can cover U by basic open sets—which will be basic open sets for both Y and U—and construct a collection of open immersions for both $\overline{f(X)}$ and Z', indexed on the open covering of U and therefore covering Z'. Thus we construct the open immersion $Z' \to \overline{f(X)}$ by a gluing argument like the one in §2.3.2.

To complete the proof we need another gluing argument. When X and Y are not affine, we take an open affine covering $\{U_i\}$ of Y and construct the family of schemes $\{(Z_i, g_i, c_i)\}$ given by the scheme-theoretic images of the restrictions $f|^{U_i}$, where g_i is a dominant morphism and c_i is a closed immersion. In order to apply the Gluing Lemma on any of these schemes Z_i we define for each j the open subscheme $Z_{ij} \subseteq Z_i$ as $c_i^{-1}(U_i \cap U_j)$. For any open affine subset of $U_i \cap U_j$ we construct the scheme-theoretic image of the restriction of f and obtain an open immersion in both Z_{ij} and Z_{ji}, defining eventually an isomorphism. □

Since any morphism of varieties is of finite type, and any morphism of finite type is quasi-compact, this construction makes perfect sense for any morphism between algebraic varieties. Quite surprisingly however we can prove something more, namely the image of a variety is again a variety.

Corollary. *Let $f\colon X \to Y$ be a quasi-compact morphism of schemes. When X is reduced the scheme-theoretic image of f is a reduced scheme, while when X is an integral scheme it is integral.*

Proof. Let Z be the scheme-theoretic image of f as in the Theorem, and let $\widetilde{Z}$ be the reduced scheme associated to Z (see §2.4.2) with closed immersion

$r\colon \widetilde{Z} \to Z$. Since X is reduced, there exists a unique morphism $h\colon X \to \widetilde{Z}$ such that $rh = g$, and if we let $i = jr$ we obtain $ih = jrh = jg = f$. Therefore by the universal property of Z there exists a unique θ such that $i\theta = j$, which means $jr\theta = j$. Since closed immersions are monic this implies $r\theta = \mathrm{id}_Z$. Viceversa we have $jr\theta r = i\theta r = jr$ therefore $r\theta r = r$ which implies $\theta r = \mathrm{id}_{\widetilde{Z}}$. In conclusion Z is isomorphic to $\widetilde{Z}$, hence it is a reduced scheme.

Assume now X is an integral scheme, then all the restriction homomorphisms of $\mathscr{O}_X$ are injective. Therefore with reference to the proof of the Theorem when $Y = \operatorname{Spec} A$ is affine the scheme theoretic image of f is given by $\operatorname{Spec} A/\mathfrak{a}$ where $\mathfrak{a}$ is the kernel of the morphism $A \to \Gamma(X, \mathscr{O}_X)$ inducing f. The ring $\Gamma(X, \mathscr{O}_X)$ is a domain, therefore the scheme theoretic image is an integral scheme. □

Example (The Twisted Cubic Curve). This is Exercise I.1.2 in Hartshorne (1977). Consider the morphism of affine varieties given by the following morphism of rings

$$\varphi\colon k[x,y,z] \to k[\,t\,]$$

defined by $\varphi(x) = t$, $\varphi(y) = t^2$ and $\varphi(z) = t^3$. This is a surjective morphism, hence it gives rise to a closed immersion $Y \subseteq \mathbb{A}^3_k$ into the affine space. The image is an affine variety isomorphic to $\mathbb{A}^1_k$ and given by the spectrum of the quotient $k[x,y,z]/\ker\varphi$. Easy considerations bring to the conclusion that $\ker\varphi = (y - x^2, z - x^3)$. We say that Y is given by the *parametric representation* φ.

4.1.3 Product of Algebraic Varieties If X and Y are algebraic varieties, we define their product to be the fibered product $X \times_k Y$ in the category of schemes over k. But there is a tricky subtlety here, indeed from what we have seen so far we can conclude that $X \times_k Y$ is a separated scheme of finite type over k but not that it is a variety. In other words we have to show that it is an integral scheme. Given the local nature of the question it is clear that we can assume X and Y to be affine, so we can reduce to prove a purely algebraic result: the tensor product

$$k[x_1, \ldots, x_n]/\mathfrak{p} \otimes_k k[y_1, \ldots, y_m]/\mathfrak{q}$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals, is a domain. In what follows we are going to prove something more, providing a precise description of the product. In fact everything we are going to say can be found in Zariski and Samuel (1958, §III.14).

Let C be a finitely generated k-algebra. Two sub-algebras L and L' of C are said to be *linearly disjoint* over k if the following condition is satisfied: whenever $x_1, x_2, \ldots, x_n$ are elements of L which are linearly independent over k and

$x'_1, x'_2, \dots, x'_n$ are elements of L' which are linearly independent over k, then the products $x_i x'_j$ are also linearly independent over k. Let now A and B be finitely generated k-algebras, we denote by $\varphi\colon A \to A \otimes_k B$ the homomorphism $a \mapsto a \otimes 1$ and by $\psi\colon B \to A \otimes_k B$ the homomorphism $b \mapsto 1 \otimes b$.

Lemma. *Let A and B be finitely generated k-algebras as above. Then inside the tensor product, the rings $\varphi(A)$ and $\psi(B)$ are linearly disjoint over k.*

Proof. The key fact in here is that any finitely generated k-algebra admits a vector basis over k. The argument goes like this: we can assume the algebra to be the quotient $C = k[x_1, \dots, x_n]/\mathfrak{a}$, therefore the set M of all (equivalence classes of) monomials is a system of generators for C as a vector space. Assuming C to be non-trivial, thus different both from 0 and k, and n to be minimal, thus the ideal $\mathfrak{a}$ not to contain polynomials of degree one, there exists a subset $T \subseteq M$ consisting of linearly independent elements, for instance consider the set $\{1, x_1\}$. Then we can order these subsets by inclusion and conclude using Zorn's Lemma that M contains a vector basis of C over k. Let now $\{x_\alpha\}$ be a vector basis of A over k and let $\{y_\beta\}$ be a vector basis of B over k, then the set of all products $x_\alpha \otimes y_\beta$ is by definition a vector basis for $A \otimes_k B$ over k. Now the property above is immediate. □

Observe that this Lemma implies in particular that φ and ψ are injective homomorphisms. To see this choose a vector basis $\{x_i\}$ for A over k and write any element $a \in A$ as a linear combination, to get $\varphi(a) = \sum_i \lambda_i x_i \otimes 1$ where $\lambda_i \in k$. Then $\varphi(a) = 0$ if and only if $\lambda_i = 0$ for each i that is if and only if $a = 0$. Now we can proceed to study zero-divisors inside the tensor product.

Proposition. *Let A and B be finitely generated k-algebras as above. If an element of A is not a zero-divisor in A, then it is not a zero-divisor in the tensor product $A \otimes_k B$*

Proof. If we have $a\xi = 0$, where $\xi \in A \otimes_k B$ we can write ξ in the form $\xi = \sum_i a_i \otimes b_i$, where the a_i are in A and the b_i are elements of B which are linearly independent over k (and hence also over A). From $\sum_i (aa_i) \otimes b_i = 0$ follows $aa_i = 0$ and then $a_i = 0$ because a is not a zero-divisor, therefore $\xi = 0$. □

Corollary. *Let A and B be affine domains over k. Then the tensor product $A \otimes_k B$ is also an affine domain.*

Proof. Observe that $(a \otimes b)(c \otimes_k d) = 0$ if and only if $(ac \otimes_k 1)(1 \otimes_k bd) = 0$. But since A and B are domains by virtue of the previous Proposition this can only happen when $ac = 0$ or $bd = 0$, which is equivalent to having $a \otimes b = 0$ or $c \otimes d = 0$. □

Let A and B be finitely generated k-algebras as above, and let $\mathfrak{a}$ be an ideal of A and $\mathfrak{b}$ an ideal of B. We shall denote by $(\mathfrak{a}, \mathfrak{b})$ the ideal generated by $\varphi(\mathfrak{a})$ and $\psi(\mathfrak{b})$ inside the tensor product $A \otimes_k B$. In geometric terms we have two affine schemes $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ and two closed subschemes $\mathcal{V}(\mathfrak{a}) \subseteq X$ and $\mathcal{V}(\mathfrak{b}) \subseteq Y$, we are going to prove that the product $\mathcal{V}(\mathfrak{a}) \times_k \mathcal{V}(\mathfrak{b})$ is given by the ideal $(\mathfrak{a}, \mathfrak{b})$ inside $A \otimes_k B$.

Theorem. *Let $\mathfrak{a} \subseteq A$ and $\mathfrak{b} \subseteq B$ be ideals inside finitely generated k-algebras as above. Then there is a canonical isomorphism*

$$\tau\colon (A \otimes_k B)/(\mathfrak{a}, \mathfrak{b}) \xrightarrow{\simeq} A/\mathfrak{a} \otimes_k B/\mathfrak{b}$$

Proof. We define τ as follows: we take the tensor product of the two canonical projections, $A \to A/\mathfrak{a}$ and $B \to B/\mathfrak{b}$, and observe that by construction the ideal $(\mathfrak{a}, \mathfrak{b})$ is contained in the kernel. Then we consider the composition $A \to A \otimes_k B \to (A \otimes_k B)/(\mathfrak{a}, \mathfrak{b})$ and since the kernel contains $\mathfrak{a}$, we define $\alpha\colon A/\mathfrak{a} \to (A \otimes_k B)/(\mathfrak{a}, \mathfrak{b})$. Now we have a commutative diagram

$$\begin{array}{ccc} A/\mathfrak{a} & & \\ \downarrow{\scriptstyle \alpha} & \searrow & \\ (A \otimes_k B)/(\mathfrak{a}, \mathfrak{b}) & \xrightarrow{\tau} & A/\mathfrak{a} \otimes_k B/\mathfrak{b} \end{array}$$

which shows that the composition $\tau\alpha$ is injective, therefore α is injective. Similarly there exists an injective homomorphism $\beta\colon B/\mathfrak{b} \to (A \otimes_k B)/(\mathfrak{a}, \mathfrak{b})$. The diagram also proves that the composition $\tau \circ (\alpha \otimes \beta)$ is the identity, moreover the homomorphism $\alpha \otimes \beta$ is surjective, because every element in the quotient $(A \otimes_k B)/(\mathfrak{a}, \mathfrak{b})$ can be written as a linear combination of products $(\overline{a \otimes 1})(\overline{1 \otimes b})$ with $a \in A$ and $b \in B$. □

4.2 Hilbert's Nullstellensatz

Our goal will be now to understand the most general form of Hilbert's Nullstellensatz as it is formulated in Eisenbud (1995, Theorem 4.19). Since that statement is purely algebraic we need first to familiarise ourselves with the basic concepts, in particular with their geometric meaning. We focus our attention on the subset $\mathcal{U}$ of closed points of a scheme X: in general we cannot say that much about it, for instance $\mathcal{U}$ is closed in $\operatorname{Spec} k[x]_{(x)}$ where it consists of only one point, but it is not closed in $\operatorname{Spec} \mathbb{Z}$ or in $\operatorname{Spec} k[x]$, where in fact it is dense. If A is any ring the nilradical $\operatorname{Nil}(A)$ of A is the intersection of all prime ideals while the *Jacobson radical* $\operatorname{Jac}(A)$ of A is the intersection of all maximal ideals, observe that $\mathrm{Nil}(A) \subseteq \mathrm{Jac}(A)$. If $X = \operatorname{Spec} A$ then $X = \mathcal{V}\left(\mathrm{Nil}(A)\right)$ while the closure $\overline{\mathcal{U}}$ of the set of closed points of X is $\mathcal{V}\left(\mathrm{Jac}(A)\right)$. Thus $\mathcal{U}$ is dense in X if and only if $\mathrm{Nil}(A) = \mathrm{Jac}(A)$.

Proposition (Exercise V.23 in Atiyah and Macdonald, 1969)**.** *A ring A is called a* Jacobson ring *when it satisfies one of the following equivalent properties.*

i) Every prime ideal in A is an intersection of maximal ideals.

ii) In every homomorphic image of A the nilradical is equal to the Jacobson radical.

iii) Every prime ideal in A which is not maximal is equal to the intersection of the prime ideals which contain it strictly.

Proof. Observe first that condition *ii*) is equivalent to the following: for any ideal $\mathfrak{a}$ of A in the quotient $A/\mathfrak{a}$ the nilradical is equal to the Jacobson radical. Now observe that most of the implications are trivial, indeed *iii*) is a consequence of *i*) and *i*) a consequence of *ii*). To see that *i*) in fact implies *ii*) take any ideal $\mathfrak{a}$ in A and write $\mathrm{Jac}(A/\mathfrak{a})$ as the following

$$\mathrm{Jac}(A/\mathfrak{a}) = \left(\bigcap_{\mathfrak{a}\subseteq\mathfrak{m}} \mathfrak{m}\right) \Big/ \mathfrak{a} \subseteq \bigcap_{\mathfrak{a}\subseteq\mathfrak{p}} \left(\bigcap_{\mathfrak{p}\subseteq\mathfrak{m}} \mathfrak{m}\right) \Big/ \mathfrak{a} = \mathrm{Nil}(A/\mathfrak{a})$$

Now we prove *iii*) $\Rightarrow$ *i*). Suppose *i*) false, then there is a prime ideal which is not an intersection of maximal ideals. Passing to the quotient ring, we may assume that A is an integral domain whose Jacobson radical is not zero. Let f be a non-zero element of $\mathrm{Jac}(A)$. Then $A_f \neq 0$, hence A_f has a maximal ideal, whose contraction in A is a prime ideal $\mathfrak{p}$ such that $f \notin \mathfrak{p}$, and which is maximal with respect to this property. Then $\mathfrak{p}$ is not maximal and is not equal to the intersection of the prime ideals strictly containing $\mathfrak{p}$. □

Every field is a Jacobson ring. $\mathbb{Z}$ is a Jacobson ring, the unique prime ideal which is not maximal (0) being the intersection of the maximal ideals (p) of $\mathbb{Z}$, where p runs through the set of prime numbers. Let A be a Jacobson ring and let $\mathfrak{a}$ be an ideal of A. Then $A/\mathfrak{a}$ is a Jacobson ring.

Lemma (Exercise V.24 in Atiyah and Macdonald, 1969)**.** *Let A be a Jacobson ring and B an A-algebra integral over A. Then B is a Jacobson ring.*

Proof. Replacing A by its canonical image in B, we may assume that $A \subseteq B$. Let $\mathfrak{p}'$ be a prime ideal of B, and let $\mathfrak{p} = A \cap \mathfrak{p}'$. Since A is a Jacobson ring there exists a family $(\mathfrak{m}_\lambda)_{\lambda\in L}$ of maximal ideals of A whose intersection is equal to $\mathfrak{p}$. For all $\lambda \in L$ there exists a maximal ideal $\mathfrak{m}'_\lambda$ of B lying above $\mathfrak{m}_\lambda$ and containing $\mathfrak{p}'$ ("Going-up Theorem," see for example Atiyah and Macdonald, 1969, Theorem 5.11). If we write $\mathfrak{q}' = \bigcap_{\lambda \in L} \mathfrak{m}'_\lambda$ then

$$\mathfrak{q}' \cap A = \bigcap_{\lambda \in L} \mathfrak{m}_\lambda = \mathfrak{p}$$

and $\mathfrak{q}' \supseteq \mathfrak{p}'$, whence $\mathfrak{q}' = \mathfrak{p}'$ (Atiyah and Macdonald, 1969, Corollary 5.8). □

The real importance of Jacobson rings is contained in Hilbert's Nullstellensatz, as formulated in Eisenbud (1995, Theorem 4.19). It is a difficult and purely algebraic result, but to justify its name in Chapter 4 of Eisenbud (1995) the usual statement of Hilbert's Theorem is proved after this.

Theorem (Nullstellensatz - General Form)**.** *Let A be a Jacobson ring. If B is a finitely generated A-algebra, then B is a Jacobson ring. Further, if $\mathfrak{n} \subseteq B$ is a maximal ideal, then $\mathfrak{m} = \mathfrak{n} \cap A$ is a maximal ideal of A, and $B/\mathfrak{n}$ is a finite extension field of $A/\mathfrak{m}$.*

An immediate consequence of this Theorem is the following. If X is a scheme of finite type over Spec A, where A is a Jacobson ring, then the subset $\mathcal{U}$ of closed points is dense in X. However this is not completely satisfactory, as the general formulation of the Nullstellensatz remains in this way in the realm of abstract algebra. The following result will reconcile geometry with it.

Theorem (Nullstellensatz - Geometric Form)**.** *Let $f\colon X \to Y$ be a morphism of schemes of finite type over a Jacobson ring R. Then for any closed point $x \in X$ the image $f(x)$ is a closed point in Y, and the local homomorphism $f^\#_x\colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ induces a finite field extension $k(x) : k\big(f(x)\big)$.*

Proof. This is nothing but a reformulation of Theorem 4.19 in Eisenbud (1995) in the language of geometry. Since it is a local statement we can assume everything to be affine, so f will be induced by a homomorphism of finitely generated R-algebras $\varphi\colon A \to B$. It is easy to see that the morphism f is of finite type (apply the Proposition in §1.2.5 and the Lemma in §3.1.4), hence B will be a finitely generated algebra over $A/\ker(\varphi)$, which is a Jacobson ring. □

4.3 Rational Maps

Definition. Let X and Y be algebraic varieties. A *rational map from X to Y*, denoted by $X \dashrightarrow Y$, is an equivalence class of morphisms of varieties from a non-empty open subscheme of X to Y. Two such morphisms $U \to Y$, $V \to Y$ are called equivalent if they coincide on the intersection $U \cap V$.

It is an exercise to show that in the definition we have in fact described an equivalence relation, only one has to bear in mind that two morphisms of varieties which agree on a nonempty open subset are equal (see §3.3.5).

Proposition (Exercise III.3.13 in Liu, 2002)**.** *In every equivalence class, there exists a unique element $f\colon U \to Y$ such that U is maximal for the inclusion relation, and that every element $g\colon V \to Y$ of the class verifies $g = f|_V$.*

This result follows at once by a gluing argument on morphisms as in §2.3.2. We call U the *domain of definition* of $X \dashrightarrow Y$. We then denote the rational map by $f\colon X \dashrightarrow Y$. Observe that the generic point of X is always contained in the domain of f.

Let $Y = \operatorname{Spec} A$ be affine, and let $V = \operatorname{Spec} B$ be any open affine subset of X. Let $\beta \in B$ be any element such that the distinguished open subset $D(\beta)$ is contained in the domain of the rational map f. Hence $f|_{D(\beta)}$ is a morphisms of affine varieties and as such it is induced by a ring homomorphism $\varphi\colon A \to B_\beta$. The rings A and B are actually affine domains, so in particular we have a morphism

$$\varphi\colon k[x_1,\dots,x_n]/\mathfrak{p} \longrightarrow k[y_1,\dots,y_m]_F/\mathfrak{q}$$

where F is a polynomial, $\mathfrak{p}$ and $\mathfrak{q}$ prime ideals, and φ is the identity on k. In accordance with the definition given for instance in Shafarevich (1994a), such a morphism is uniquely determined by the image of the indeterminates $\varphi(x_i)$, which is an m-tuple of rational functions over Y.

A rational map $f\colon X \dashrightarrow Y$ is *dominant* if the corresponding morphism $U \to Y$ is dominant, where U is the domain of definition of f. If $g\colon Y \dashrightarrow Z$ is another rational map with domain $V \subseteq Y$, then $f^{-1}(V)$ is a nonempty open subset of U and we can shrink to define the morphism

$$f^{-1}(V) \xrightarrow{f|^V} V \xrightarrow{g} Z$$

which defines a rational map from X to Z that we call *the composition* $g \circ f$. This proves that we can consider the category of varieties and dominant rational maps. An isomorphism in this category is called a birational map.

Definition. A *birational map* $f\colon X \dashrightarrow Y$ is a rational map which admits an inverse, namely a rational map $g\colon Y \dashrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ as rational maps. If there is a birational map from X to Y, we say that X and Y are *birationally equivalent* or simply *birational*.

The main result of this section is that the category of varieties and dominant rational maps is equivalent to the category of finitely generated field extensions of k, with arrows reversed. But first we need a preliminary result about morphisms of varieties.

Lemma. *Let X and Y be two varieties with generic point η and ζ respectively, and let $f\colon X \to Y$ be a morphism of varieties. Then f is dominant if and only if the morphism of sheaves $f^\#$ is injective, further in this case $f(\eta) = \zeta$.*

Proof. For any $V = \operatorname{Spec} A$ open affine subset of Y, let $U = \operatorname{Spec} B$ be any open affine subset of X contained in $f^{-1}(V)$. Then the restriction $f|_U^V\colon U \to V$ is induced by the composition of homomorphisms

$$\mathcal{O}_Y(V) \xrightarrow{f_V^\#} \mathcal{O}_X\big(f^{-1}(V)\big) \xrightarrow{res} \mathcal{O}_X(U)$$

The morphism f is dominant if and only if so is this restriction for any V and U. Indeed the restriction is given by the composition $f|^V \circ i_U$ where i_U is the inclusion of U in $f^{-1}(V)$ and is therefore dominant. Maybe it is easier to understand the argument from the purely topological point of view: we have two morphisms $\varphi\colon S_1 \to S_2$ and $\psi\colon S_2 \to S_3$ such that φ is dominant, hence we have

$$\overline{\psi\big(\varphi(S_1)\big)} = \overline{\psi\big(\overline{\varphi(S_1)}\big)} = \overline{\psi(S_2)}$$

therefore $\psi \circ \varphi$ is dominant if and only if ψ is.

Now since $A = \mathcal{O}_Y(V)$ is an integral domain, the morphism of affine schemes $f|_U^V$ is dominant if and only if the homomorphism above is injective (see §1.4.1). Recall now that in §2.4.4 we have seen that the restrictions of $\mathcal{O}_X$ are injective and the proof is complete. Observe in particular that η corresponds to the zero ideal in $B = \mathcal{O}_X(U)$, while ζ to the zero ideal in A therefore when f is dominant we have also $f(\eta) = \zeta$. □

Let X and Y be two varieties with generic point η and ζ respectively, and let $f\colon X \dashrightarrow Y$ be a dominant rational map. Then any morphism $U \to Y$ representing f is dominant, and applying the Lemma induces an injective homomorphism of k-algebras from $K(Y) = \mathcal{O}_{Y,\zeta}$ to $K(X) = \mathcal{O}_{X,\eta}$. Observe however that this field extension depends only on f and not on the choice of the morphism representing it.

Theorem (I.4.4 in Hartshorne, 1977)**.** *For any two varieties X and Y, the above construction gives a bijection between*

i) the set of dominant rational maps from X to Y, and

ii) the set of k-algebra homomorphisms from $K(Y)$ to $K(X)$.

Furthermore, this correspondence gives an arrow-reversing equivalence of categories.

Proof. The construction described above associates a homomorphism of k-algebras to any dominant rational map, here we will describe its inverse. Let $\theta\colon K(Y) \to K(X)$ be a homomorphism of k-algebras, we want to define a rational map from X to Y. By a gluing argument we can assume $Y = \operatorname{Spec} A$ to be affine, then A will be an affine domain say generated by $y_1, \dots, y_n$. Let U be the maximal open subset of X on which the n rational functions $\theta(y_1), \dots, \theta(y_n)$ are regular, observe incidentally that for any open affine subset $W = \operatorname{Spec} B$ of X the intersection $U \cap W$ is given by the open basic subset of $\operatorname{Spec} B$ defined by the denominators of the rational functions. Thus θ restricts to an injective k-algebra homomorphism $A \to \mathscr{O}_X(U)$, which by adjunction (§2.1.4) defines a morphism of varieties $U \to Y$, such that the corresponding morphism of sheaves is injective hence by the previous Lemma it is a dominant morphism of varieties. □

Corollary (I.4.5 in Hartshorne, 1977)**.** *For any two varieties X and Y, the following conditions are equivalent*

i) *X and Y are birationally equivalent;*

ii) *there are open subsets $U \subseteq X$ and $V \subseteq Y$ with U isomorphic to V;*

iii) *the fields of rational functions $K(X)$ and $K(Y)$ are isomorphic k-algebras.*

Proof. Observe first that *iii*) follows from *ii*) by the definition of function field, while *i*) follows from *iii*) by the Theorem, therefore there is only to prove that *i*) implies *ii*). Let $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow X$ be rational maps which are inverse to each other. Let U be the domain of f and V be the domain of g and take $f^{-1}\big(g^{-1}(U)\big)$ as the open subset of X and $g^{-1}\big(f^{-1}(V)\big)$ as the open subset of Y. □

"In Algebraic Geometry we work with two different equivalence relations between varieties, isomorphism and birational equivalence. Birational equivalence is clearly a coarser equivalence relation than isomorphism; in other words, two varieties can be birational without being isomorphic. Thus it often turns out that the classification of varieties up to birational equivalence is simpler and more transparent than the classification up to isomorphism. Since it is defined at every point, isomorphism is closer to geometric notions such as homeomorphism and diffeomorphism, and so more convenient. Understanding the relation between these two equivalence relations is an important problem; the question is to understand how much coarser birational equivalence is compared to isomorphism, or in other words, how many varieties are distinct from the point of view of isomorphism but the same from that of birational equivalence."

taken from Shafarevich (1994a, §I.3.3)

4.4 Dimension

4.4.1 Topological Definition If X is a topological space, we define the *dimension* of X (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain $Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$ of distinct irreducible closed subsets of X. Observe that a Hausdorff space is zero-dimensional, indeed we have already observed that points are the only irreducible subsets of a Hausdorff space.

Proposition (Exercise I.1.10 in Hartshorne, 1977)**.**

(*a*) *If Y is any subset of a topological space X, then* $\dim Y \leq \dim X$.

(*b*) *If X is a topological space which is covered by a family of open subsets $\{U_i\}$, then* $\dim X = \sup \dim U_i$.

(*c*) *If Y is a closed subset of an irreducible finite-dimensional topological space X, and if* $\dim Y = \dim X$, *then* $Y = X$.

Proof. Part (a) follows immediately from the following remark: if $Y_0 \subsetneq Y_1 \subsetneq Y_2 \subsetneq \ldots$ is an ascending chain of distinct irreducible closed subsets of Y then $\overline{Y_0} \subsetneq \overline{Y_1} \subsetneq \overline{Y_2} \subsetneq \ldots$ is an ascending chain of distinct irreducible closed subsets of X. Where $\overline{Y_i}$ denotes closure in X.

Let now $\{U_i\}$ be an open covering of X and observe that by part (a) we have $\dim X \geq \dim U_i$ for all i, hence $\dim X \geq \sup \dim U_i$. Conversely let $Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \ldots$ be an ascending chain of distinct irreducible closed subsets of X and let U_{i_0} be such that $U_{i_0} \cap Z_0 \neq \varnothing$, hence the chain $(Z_0 \cap U_{i_0}) \subseteq (Z_1 \cap U_{i_0}) \subseteq (Z_2 \cap U_{i_0}) \subseteq \ldots$ is an ascending chain of irreducible closed subsets of U_{i_0}. Note that this chain is strictly increasing in fact if $(Z_k \cap U_{i_0}) = (Z_{k+1} \cap U_{i_0})$ then

$$Z_{k+1} = (Z_{k+1} \cap U_{i_0}) \cup \big(Z_{k+1} \cap (X \setminus U_{i_0})\big) = Z_k \cup \big(Z_{k+1} \cap (X \setminus U_{i_0})\big)$$

since Z_{k+1} is irreducible and $Z_k \subsetneq Z_{k+1}$ we can conclude that Z_{k+1} doesn't meet U_{i_0}, but this is impossible since $U_{i_0} \cap Z_0 \subseteq Z_{k+1}$. In this way we have proved that for any chain of irreducible closed subsets of X, of any length, there is an open subset in the family $\{U_i\}$ of dimension greater or equal than the length of the chain. By definition of dimension we can now say $\dim X \leq \sup \dim U_i$ and conclude.

To prove (c) let $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n$ be a maximal chain of irreducible closed subsets of Y. Since Y is closed, this is also a chain of irreducible closed subsets of X, and hence is maximal in X also for $\dim Y = \dim X$. If we assume now $Y \subsetneq X$ we can add to the chain X itself since it is irreducible and since $Y_n \subseteq Y \subsetneq X$, contradicting the maximality of the chain. □

Remark (Dimension and coverings). In part (b) of the Proposition above we proved that $\dim X = \sup \dim U_i$ for any *open* covering $\{U_i\}$ of a topological space X. If we drop the hypothesis of the covering being open this is no longer true, indeed a quadric cone is a union of lines each one being of dimension 1 while the quadric cone is clearly of dimension 2. However if we cover X by closed subsets $\{Z_i\}$ a similar result holds provided the covering is *locally finite*, that is every point x admits an open neighborhood U which meets only a finite number of Z_i.

4.4.2 Local Definition Let X be a topological space. Let $x \in X$. We define the *dimension of X at x* to be

$$\dim_x X = \inf \left\{ \dim U \,\middle|\, U \text{ open neighborhood of } x \right\}$$

Proposition (II.5.5 in Liu, 2002)**.** *Let X be a topological space. Then we have the equality* $\dim X = \sup\{\dim_x X \mid x \in X\}$

Proof. Statement (b) of the previous Proposition is in fact a corollary of this result, it is not surprising therefore that the proof is almost the same. We only need to prove the inequality $\dim X \leq \sup\{\dim_x X \mid x \in X\}$, to this purpose take any chain of distinct irreducible closed subsets $Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \ldots$ and let $x \in Z_0$. Let U be any open neighborhood of x and repeat the argument in the Proposition above. □

Corollary (Exercise II.5.1 in Liu, 2002)**.** *Let X be a topological space. Let Y_i be a locally finite covering of X by closed subsets. Then* $\dim X = \sup \dim Y_i$.

Proof. First we prove the following. Let U be a topological space which is the union of a finite number of closed subsets $U = Y_1 \cup \cdots \cup Y_n$ then $\dim U = \max\{\dim Y_1, \ldots, \dim Y_n\}$. This follows immediately once we observe that if Z is an irreducible subset of U then it is contained in one of the Y_i. To convince yourself that this is true write Z as follows

$$Z = (Z \cap Y_1) \cup \left(Z \cap \left(\bigcup_{j=2}^{n} Y_j \right) \right)$$

since Z is irreducible is either contained in Y_1 or in the union of all the others, now conclude inductively. This implies that any chain $Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$ of distinct irreducible closed subsets of U, is contained in some of the Y_i and eventually that $\dim U = \max\{\dim Y_1, \ldots, \dim Y_n\}$.

Now we have an open covering $\mathscr{U}$ of X made by open sets U that meet only a finite number of the given closed subsets Y_i, hence

$$\begin{aligned}\dim X &= \sup\{\dim U \mid U \in \mathscr{U}\} \\ &= \sup\left\{ \max\left\{ \dim(Y_{U,1} \cap U), \ldots, \dim(Y_{U,N(U)} \cap U)\right\} \,\Big|\, U \in \mathscr{U}\right\} \\ &\leq \sup\left\{ \max\{\dim Y_{U,1}, \ldots, \dim Y_{U,N(U)}\} \mid U \in \mathscr{U}\right\}\end{aligned}$$

therefore $\dim X \leq \sup \dim Y_i$ □

4.4.3 The Dimension of a Scheme To help the reader gain confidence with the abstract notion of dimension, but at the same time to warn him from being too much confident, we collect in this subsection some pathological examples.

Example (In which there is a dense open subset of strictly smaller dimension). Take the real line and identify points in the open interval $(0, 1)$. Call ε the open point in $X = \mathbb{R}/\sim$ that is the quotient of the interval. The closure $\overline{\varepsilon}$, of the set $\{\varepsilon\}$ is the irreducible closed subset $\{0, \varepsilon, 1\}$. So in X there is the chain $\{0\} \subsetneq \overline{\varepsilon}$, and $\dim X = 1$. The set $X \setminus \{0, 1\}$ is open and dense, moreover zero dimensional, since it is Hausdorff.

Example (In which there is a Noetherian space of infinite dimension). As a set $X = \mathbb{N} \setminus \{0\}$, we define a topology on X setting $C_n = \{1, 2, \ldots, n\}$ to be closed for all n, with $C_0 = \varnothing$ and $C_\infty = X$. It is clear that C_n is irreducible for all $n \neq 0$, so that $C_1 \subsetneq C_2 \subsetneq \ldots$ is an infinite ascending chain of irreducible closed subsets of X, and hence X is infinite dimensional. On the other hand every descending chain of closed subsets is of the form $C_{i_0} \supseteq C_{i_1} \supseteq C_{i_2} \supseteq \ldots$, where $I = \{i_0, i_1, i_2 \ldots\} \subseteq \mathbb{N}$ and $i_0 \geq i_1 \geq i_2 \geq \ldots$. Then there exists $\alpha = \inf I$ and $N \in \mathbb{N}$ such that for each $j \geq N$ we have $i_j = \alpha$, i.e. the chain is stationary and X is Noetherian.

Definition. The *dimension* of a scheme X, denoted $\dim X$, is its dimension as a topological space. That is the supremum of all integers n such that there exists a chain $Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$ of distinct irreducible closed subsets of X.

Recall that for a ring A the *Krull dimension* is defined to be the supremum of all integers n such that there exists a chain of prime ideals $\mathfrak{p}_n \supsetneq \mathfrak{p}_{n-1} \supsetneq \cdots \supsetneq \mathfrak{p}_0$ of length n. It is obvious from the definition that the dimension of an affine scheme coincides with the Krull dimension of the ring.

"Of course, not every topological space has finite dimension. This is false in general for Spec A even if A is Noetherian. Nevertheless there is a series of important types of rings for which the dimension of Spec A is finite."

taken from Shafarevich (1994b, §V.1.4)

Example (In which there is a Noetherian ring of infinite dimension). Let R be the algebra $\mathbb{C}[\{x_{ij} \mid i \leq j\}]$ and let S be the multiplicatively closed subset

$$S = \bigcap_{j \geq 0} \left(R \setminus (x_{0j}, \ldots, x_{jj})\right) = R \setminus \bigcup_{j \geq 0} (x_{0j}, \ldots, x_{jj})$$

Each polynomial $F \in R$ such that $F \notin S$ is contained in $(x_{0j}, \ldots, x_{jj})$ for finitely many j only. More precisely for $(x_{0j}, \ldots, x_{jj})$ to contain F it is necessary and sufficient that it contains all of the monomials in F. We consider the ring $S^{-1}R$, it is clear that for any $n \in \mathbb{N}$ there exists a chain of prime ideals of length n, namely $(x_{0n}) \subsetneq (x_{1n}) \subsetneq \ldots \subsetneq (x_{0n}, \ldots, x_{nn})$, so that $S^{-1}R$ has infinite dimension. The difficult part is to prove that $S^{-1}R$ is Noetherian.

Let $S^{-1}\mathfrak{a}$ be an ideal of $S^{-1}R$, where $\mathfrak{a}$ is an ideal of R such that $\mathfrak{a} \cap S$ is empty. Then $\mathfrak{a} \subseteq (x_{0j}, \ldots, x_{jj})$ for some j, indeed assume this is not the case, let $F \in \mathfrak{a}$ be any polynomial and let $j_1, \ldots, j_q$ be the finite set of indices above, then $\mathfrak{a}$ is not contained in the union of the ideals $(x_{0j}, \ldots, x_{jj})$ for $j \in \{j_1, \ldots, j_q\}$ (otherwise $\mathfrak{a}$ would be contained in one of them, see Atiyah and Macdonald, 1969, Proposition 1.11) and there exists $G \in \mathfrak{a}$ that avoids this union. Then the sum $F + G$ is not contained in the ideal $(x_{0j}, \ldots, x_{jj})$, for $j \in \{j_1, \ldots, j_q\}$ because otherwise G would be contained in it, for any other j because there is some monomial of F which is not contained in it. Therefore $F + G \in S$ which is a contradiction.

Since any ideal in $S^{-1}R$ is of the form $S^{-1}\mathfrak{a}$ for some $\mathfrak{a}$ as above, we conclude that the set of maximal ideals in $S^{-1}R$ is given by ideals of the form $\mathfrak{m}_j = S^{-1}(x_{0j}, \ldots, x_{jj})$. If we localise $S^{-1}R$ in one of its maximal ideals we obtain

$$[S^{-1}R]_{\mathfrak{m}_h} = R_{(x_{0h}, \ldots, x_{hh})} \cong \mathbb{F}[x_{0h}, \ldots, x_{hh}]_{(x_{0h}, \ldots, x_{hh})}$$

Where $\mathbb{F} = \mathbb{F}\mathrm{rac}\left(\mathbb{C}\left[\{x_{ij} \mid i \leq j, j \neq h\}\right]\right)$. Hence for any maximal ideal $\mathfrak{m}_j$ the ring $[S^{-1}R]_{\mathfrak{m}_j}$ is a localisation of a Noetherian ring and therefore it is Noetherian (see Atiyah and Macdonald, 1969, Proposition 7.3). Now we can conclude that $S^{-1}R$ is Noetherian by the following Lemma.

Lemma (9.4 in Eisenbud, 1995)**.** *Let A be a ring such that for every maximal ideal $\mathfrak{m} \subset A$ the local ring $A_{\mathfrak{m}}$ is Noetherian. If for every nonzero element $s \in A$ there are only finitely many maximal ideals containing s, then A is Noetherian.*

Proof. Let $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots$ be any ascending chain of ideals of A. For any maximal ideal $\mathfrak{m}$ there exists an integer $i(\mathfrak{m})$ such that for all $j > i \geq i(\mathfrak{m})$ we have $\mathfrak{a}_j^e = \mathfrak{a}_i^e$, where the extension is made in the localisation $R_{\mathfrak{m}}$.
Since the sequence is increasing we have the following inclusion between finite sets

$$\{\mathfrak{m} \subseteq A \mid \mathfrak{a}_i \subseteq \mathfrak{m}\} \subseteq \{\mathfrak{m} \subseteq A \mid \mathfrak{a}_0 \subseteq \mathfrak{m}\} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_q\}$$

Let now $i_0 = \max\{i(\mathfrak{m}_1), \dots, i(\mathfrak{m}_q)\}$ and $j > i \geq i_0$. Take $F \in \mathfrak{a}_j$, we will prove that $F \in \mathfrak{a}_i$. For all $k = 1, \dots, q$ we have $\mathfrak{a}_j^e = \mathfrak{a}_i^e$ in $R_{\mathfrak{m}_k}$, in particular $F \in \mathfrak{a}_i^e$ i.e.

$$\frac{F}{1} = \frac{G_k}{H_k}$$

where $G_k \in \mathfrak{a}_i$ and $H_k \notin \mathfrak{m}_k$, this means that there exists $T_k \notin \mathfrak{m}_k$ such that

$$T_k H_k F = T_k G_k$$

Now the ideal generated by $\mathfrak{a}_i$ and $T_1 H_1, \dots, T_q H_q$ is the unit ideal, indeed it is not contained in any maximal ideal, so there is an expression of the form $1 = a + \sum_{k=1}^{q} \alpha_k T_k H_k$, where $a \in \mathfrak{a}_i$. Hence

$$F = Fa + \sum_{k=1}^{q} \alpha_k T_k H_k F = Fa + \sum_{k=1}^{q} \alpha_k T_k G_k$$

where the right-hand side is an element of $\mathfrak{a}_i$. □

4.4.4 Codimension Let X be a topological space. If Z is an irreducible closed subset of X, we define the *codimension* of Z in X, denoted $\operatorname{codim}(Z, X)$ to be the supremum of integers n such that there exists a chain $Z \subsetneq Z_1 \subsetneq \dots \subsetneq Z_n$ of distinct irreducible closed subsets of X, beginning with Z.

Proposition (Exercise II.5.2 in Liu, 2002)**.** *Let X be a scheme and $x \in X$ be any point. Then we have the equality* $\dim \mathscr{O}_{X,x} = \operatorname{codim}\left(\{x\}^-, X\right)$.

Proof. Take any chain $\{x\}^- \subsetneq Z_1 \subsetneq \dots \subsetneq Z_n$ of distinct irreducible closed subsets of X, beginning with $\{x\}^-$, then for any open neighborhood U of x the generic points $\xi_1, \dots, \xi_n$ of the Z_i are contained in U. Thus if $U = \operatorname{Spec} A$ is any open affine neighborhood of x these generic points correspond to prime ideals $\mathfrak{p}_x \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n$ of A. □

4.4.5 Zero Dimensional Schemes In our intuition a geometric space of dimension zero is the same as a finite set, and we would be very surprised to call *Geometry* any theory where this is not the case. The next result prevents Algebraic Geometry from upsetting us, but further comments will be required.

Proposition (Lemma I.1.10 in Iitaka, 1982)**.** *Let X be a Noetherian T_0 topological space. Then* $\dim X = 0$ *if and only if X is a finite set consisting of closed points.*

If $X = \operatorname{Spec} A$ is the spectrum of a Noetherian ring this proposition is a well known result, it is Theorem 8.5 Atiyah and Macdonald (1969) which asserts that a ring is Noetherian of dimension zero if and only if it is Artin. Observe that we require X to be not just a finite set but consisting of closed points, indeed it is easy to find a Noetherian ring of dimension 1 with only a finite set of prime ideals, for instance $k[\,x\,]$ localised in the maximal ideal $(\,x\,)$ which consists of two points only one of which is closed.

The Noetherian hypothesis is necessary, indeed if X is any scheme then the dimension of X is zero if and only if $\dim U = 0$ for any open affine subset U of X (it follows from §4.4.1). Therefore a zero-dimensional scheme will always consist of closed points, for in a ring A with Krull dimension zero every prime ideal is maximal. But this set can be infinite, already in the affine case.

Example (Zero dimensional schemes with an infinite number of points). A ring A is *Boolean* if $x^2 = x$ for any $x \in A$. According to Atiyah and Macdonald (1969, Exercise I.11) the dimension of a Boolean Ring is always zero, and by Exercise I.24 in there we can give a structure of Boolean ring to the set of all subsets of a set Σ, which we denote with $R(\Sigma)$. The product in $R(\Sigma)$ will be given by intersection in Σ. Now let $e \in \Sigma$ be any element, let E be the subset $\Sigma \setminus \{e\}$ and let $\mathscr{M}$ be the ideal generated by E in $R(\Sigma)$. Unravelling the definitions we have

$$A \in \mathscr{M} \iff e \notin A$$

From which it follows immediately that $\mathscr{M}$ is prime. Therefore the ring $R(\Sigma)$ contains a set of prime ideals in one-to-one correspondence with elements of Σ, and if Σ is an infinite set $R(\Sigma)$ contains infinitely many prime ideals.

4.4.6 The Dimension of a Variety "Be careful in applying the concepts of dimension and codimension to arbitrary schemes. Our intuition is derived from working with schemes of finite type over a field, where these notions are well-behaved. [...] But in arbitrary (even Noetherian) schemes, funny things can happen."

taken from Hartshorne (1977, Caution II.3.2.8)

There are essentially two key algebraic results that we need, although both of them can be derived from the celebrated *Noether Normalisation Theorem*. In Chapter 13 of Eisenbud (1995) everything is explained in details, so we can just state what we need, remember that by affine ring we mean finitely generated k-algebra.

Theorem. *If R is an affine domain over a field k (not necessarily algebraically closed), then*

$$\dim R = \operatorname{tr.deg}_k R,$$

and this number is the length of every maximal chain of primes in R.

By $\operatorname{tr.deg}_k R$ here we mean the transcendence degree of the field $\mathbb{F}\mathrm{rac}\, R$ over k, while the dimension of a ring is always the Krull dimension. If $\mathfrak{p}$ is a prime ideal then the *codimension*, or *height* of $\mathfrak{p}$, is by definition the dimension of the local ring $R_{\mathfrak{p}}$. Equivalently, it is the supremum of lengths of chains of primes descending from $\mathfrak{p}$. If $\mathfrak{a}$ is any ideal then $\operatorname{codim}\mathfrak{a}$ is the minimum of the codimensions of primes containing $\mathfrak{a}$.

Corollary. *If R is an affine domain over a field k (not necessarily algebraically closed), and $\mathfrak{a} \subseteq R$ is an ideal, then*

$$\dim R/\mathfrak{a} + \operatorname{codim}\mathfrak{a} = \dim R$$

Definition. Let X be a topological space. If Z is an irreducible closed subset of X, we define the *codimension* of Z in X, denoted $\operatorname{codim}(Z, X)$ to be the supremum of integers n such that there exists a chain $Z \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$ of distinct irreducible closed subsets of X, beginning with Z. If Y is any closed subset of X, we define

$$\operatorname{codim}(Y, X) = \inf_{Z \subseteq Y} \operatorname{codim}(Z, X)$$

where the infimum is taken over all closed irreducible subsets of Y. Equivalently we can take the infimum over the irreducible components of Y (defined in §1.2.3).

Proposition (Exercise II.3.20 in Hartshorne, 1977)**.** *Let X be an integral sche-me of finite type over a field k (not necessarily algebraically closed). Then for any nonempty open subset $U \subseteq X$*

$$\dim U = \dim X = \operatorname{tr.deg}_k K(X)$$

Further $\dim X = \dim \mathscr{O}_{X,x}$ *for any closed point $x \in X$, and if Y is a closed subset of X then*

$$\operatorname{codim}(Y, X) = \inf\{\dim \mathscr{O}_{X,x} \mid x \in Y\}$$
$$\dim Y + \operatorname{codim}(Y, X) = \dim X$$

Proof. The dimension of X is given by the supremum of the dimensions of open affine subsets of X as in §4.4.1, but by §2.4.5 (Lemma) combined with the theorem above for any open affine subset $U = \operatorname{Spec} A$ of X the dimension of U is given by $\operatorname{tr.deg}_k K(X)$.

Further for any closed point x in X the dimension of the local ring $\mathscr{O}_{X,x}$ is the dimension of $A_{\mathfrak{m}}$ where $U = \operatorname{Spec} A$ is any open affine neighborhood of x and $\mathfrak{m}$ is a maximal ideal of A. By the theorem above, $\dim A$ is the length of every maximal chain of primes in A, in particular it is the length of those maximal chains contained in $\mathfrak{m}$ which is $\dim A_{\mathfrak{m}}$.

From §4.4.4 it follows that the dimension of the local rings $\mathscr{O}_{X,x}$, when x varies in a closed subset Y of X, is the same as the codimension of the irreducible subsets $\{x\}^-$, which are all the closed irreducible subsets of Y.

Now let Z be any irreducible closed subset of X. For any open subset U of X such that $U \cap Z \neq \varnothing$ we then have the following

$$\dim Z = \dim Z \cap U$$
$$\operatorname{codim}(Z, X) = \operatorname{codim}(Z \cap U, U)$$

Once we observe that the reduced induced closed subscheme structure on Z makes it into an integral scheme of finite type over k, the first of these equalities follows by the previous discussion while the second by §4.4.4. Now take U to be any open affine subset of X and apply the corollary above to conclude that the last formula in the statement holds for any irreducible closed subset of X.

Since $\operatorname{sp}(X)$ is Noetherian Y has only a finite number of irreducible components $Z_1, \ldots, Z_r$, therefore using the Corollary in §4.4.2 we can conclude that there exists i such that $\dim Z_i = \dim Y$. Moreover the codimension of Y is given by definition by the least codimension of its irreducible components, that is there exists j such that $\operatorname{codim}(Z_j, X) = \operatorname{codim}(Y, X)$. Now we can use the formula on irreducible closed subsets to conclude that $i = j$ and eventually that the same formula holds for any closed subset. □

Corollary. *Let X and Y be integral schemes of finite type over k. Then the product $X \times_k Y$ is an integral scheme of finite type over k and*

$$\dim X \times_k Y = \dim X + \dim Y$$

Proof. We have proved already in §4.1.3 that the product is an integral scheme of finite type over k. The dimension of $X \times_k Y$ is given by the dimension of any open subset, in particular by the dimension of $\operatorname{Spec} A \otimes_k B$ where $U = \operatorname{Spec} A$ is a subset of X and $V = \operatorname{Spec} B$ a subset of Y, and this dimension is given by the transcendence degree of $A \otimes_k B$ over k, which is given by the sum $\operatorname{tr.deg}_k A + \operatorname{tr.deg}_k B$. You may wish to solve Exercise 13.13 Eisenbud (1995). □

Chapter 5

Sheaves over Schemes

In order to deepen our understanding of varieties we need to define coherent and quasi-coherent sheaves and describe how these are related with vector bundles, divisors, differential forms and so on. We will see later that tangent and normal bundles, morphisms to projective space, and even subvarieties can be defined in terms of sheaves, preparing the ground for the introduction of Homological Algebra (Gelfand and Manin, 2003). Historically the material was covered in the long article by Serre (1955), but the beginner may want to read more recent accounts such as Liu (2002) or Ueno (2001). Not surprisingly some of the concepts involved are better understood when we refer to Commutative Algebra, especially to Eisenbud (1995) whose complete title, "with a view toward Algebraic Geometry," appears to be particularly appropriate in this context. But the subject is so broad that it also concerns other disciplines, sheaves of modules for instance are covered in standard books of sheaf theory such as Tennison (1975).

5.1 Sheaves of Modules

5.1.1 Abstract Definitions Let X be a ringed space with structure sheaf $\mathscr{O}_X$. A *sheaf of $\mathscr{O}_X$-modules* (or simply an $\mathscr{O}_X$-module) is a sheaf $\mathscr{F}$ on X, such that

i) for each open set $U \subseteq X$, the group $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module, and

ii) for each inclusion of open sets $V \subseteq U$, the restriction homomorphism $\mathscr{F}(U) \to \mathscr{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$.

A *morphism* $\mathscr{F} \to \mathscr{G}$ of sheaves of $\mathscr{O}_X$-modules is a morphism of sheaves, such that for each open set $U \subseteq X$, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is a homomorphism of $\mathscr{O}_X(U)$-modules.

Note that to satisfy properties $i)$ and $ii)$ it is not necessary for $\mathscr{F}$ to be a sheaf. For some purposes it is better to bear in mind that an $\mathscr{O}_X$-module is a *sheaf*, but dealing with presheaves can make life easier, especially in the moment you have to verify certain properties. The following proposition is the missing link.

Proposition. *Let $\mathscr{F}$ be a presheaf on a ringed space X which satisfies the requirements to be an $\mathscr{O}_X$-module, that is properties $i)$ and $ii)$ above. Then the sheaf $\mathscr{F}^+$ associated to $\mathscr{F}$ carries a natural structure of $\mathscr{O}_X$-module.*

Proof. To see this recall the definition of $\mathscr{F}^+$, we will recall here the one given in Hartshorne (1977), but the reader can refer to his favorite construction. For any open set U, $\mathscr{F}^+(U)$ is the set of functions s from U to the union $\bigcup_{x\in U}\mathscr{F}_x$ of the stalks of $\mathscr{F}$ over points of U, such that

(1) for each $x \in U$, $s(x) \in \mathscr{F}_x$, and

(2) for each $x \in U$, there is a neighborhood V of x, contained in U, and an element $t \in \mathscr{F}(V)$, such that for all $x' \in V$, the germ $t_{x'}$ of t at x' is equal to $s(x')$.

One sees immediately from this definition that $\mathscr{F}^+(U)$ is an $\mathscr{O}_X(U)$-module in the same way it is a group, with operations defined pointwise and this structure is clearly compatible with restriction homomorphisms. □

In view of the previous Proposition it is clear that the kernel, cokernel, and image of a morphism of $\mathscr{O}_X$-modules are again $\mathscr{O}_X$-modules. If $\mathscr{F}'$ is a subsheaf of $\mathscr{O}_X$-modules of an $\mathscr{O}_X$-module $\mathscr{F}$, then the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is an $\mathscr{O}_X$-module. Any direct sum, direct product, direct limit, or inverse limit of $\mathscr{O}_X$-modules is an $\mathscr{O}_X$-module.

A sequence of $\mathscr{O}_X$-modules and morphisms is *exact* if it is exact as a sequence of sheaves of abelian groups. A *sheaf of ideals* on X is a sheaf of modules $\mathscr{I}$ which is a subsheaf of $\mathscr{O}_X$. In other words, for every open set U, $\mathscr{I}(U)$ is an ideal of $\mathscr{O}_X(U)$.

5.1.2 Hom and Tensor Product Given two sheaves of $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$ we can define the *tensor product* as the sheaf associated to the presheaf

$$U \longmapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)\,.$$

It is clear that this presheaf satisfies properties $i)$ and $ii)$ above, hence it defines an $\mathscr{O}_X$-module which is denoted $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ or simply $\mathscr{F} \otimes \mathscr{G}$, with $\mathscr{O}_X$ understood.

Proposition (§IV.4.9 in Tennison, 1975). *Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of $\mathscr{O}_X$-modules and let $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ be their tensor product. Then on the stalks we have a canonical identification*

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x$$

"Tensor product of $\mathscr{O}_X$-modules inherits many of the properties of tensor product of modules; for instance, tensor product with a fixed $\mathscr{O}_X$-module $\mathscr{F}$ gives a right exact covariant functor

$$\mathscr{F} \otimes - \colon \ (\mathit{Modules}/\mathscr{O}_X) \longrightarrow (\mathit{Modules}/\mathscr{O}_X)$$

where with $(\mathit{Modules}/\mathscr{O}_X)$ we denote the category of $\mathscr{O}_X$-modules."

taken from Tennison (1975, §IV.4.10)

The definition of the sheaf $\mathscr{H}om$ requires more details. Again $\mathscr{F}$ and $\mathscr{G}$ are sheaves of $\mathscr{O}_X$-modules, and for any open set $U \subseteq X$ we define

$$U \longmapsto \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)\ ,$$

that is we consider the set of morphisms of sheaves of $\mathscr{O}_X|_U$-modules from $\mathscr{F}|_U$ to $\mathscr{G}|_U$. Now we want to define a structure of $\mathscr{O}_X(U)$-module on the set $\mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$, and since we are considering morphisms of restricted sheaves it is clear that there is no loss of generality in treating the global sections case only. So we want to define a structure of $\Gamma(X, \mathscr{O}_X)$-module on the set $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$. First of all we need it to be an abelian group and we proceed as follows: define the sum of two morphisms of sheaves φ and ψ as $\varphi_U + \psi_U$ on any open set U, that is set

$$(\varphi + \psi)_U := \varphi_U + \psi_U.$$

This is a collection of group homomorphisms, one for each open set $U \subseteq X$. Now let $s \in \mathscr{F}(U)$ and look at the following diagram chase, where vertical arrows are restrictions

$$\begin{array}{ccc} s & & \\ \downarrow & & \\ s|_V & \longrightarrow & \varphi_V(s|_V) + \psi_V(s|_V) \end{array} \qquad \begin{array}{ccc} s & \longrightarrow & \varphi_U(s) + \psi_U(s) \\ & & \downarrow \\ & & (\varphi_U(s) + \psi_U(s))|_V \end{array}$$

The two paths coincide because of the following equalities, where we use that restriction maps of $\mathscr{G}$ are group homomorphisms, and φ and ψ are morphisms of sheaves

$$\big(\varphi_U(s) + \psi_U(s)\big)|_V = \varphi_U(s)|_V + \psi_U(s)|_V = \varphi_V(s|_V) + \psi_V(s|_V)$$

Note that, when φ and ψ are morphisms of $\mathscr{O}_X$-modules, the sum $\varphi + \psi$ is again a morphism of $\mathscr{O}_X$-modules. Now we have to define the product of a morphism $\varphi \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ with a section $\alpha \in \Gamma(X, \mathscr{O}_X)$, and for any open set $U \subseteq X$ we put

$$(\alpha\varphi)_U := res_U^X(\alpha)\varphi_U$$

This defines a morphism of $\mathscr{O}_X(U)$-modules for each U, and again we take a section $s \in \mathscr{F}(U)$ and look at a diagram chase

$$\begin{array}{ccc} s & & \\ \downarrow & & \\ s|_V & \longrightarrow & res_V^X(\alpha)\varphi_V(s|_V) \end{array} \qquad \begin{array}{ccc} s & \longrightarrow & res_U^X(\alpha)\varphi_U(s) \\ & & \downarrow \\ & & \big(res_U^X(\alpha)\varphi_U(s)\big)|_V \end{array}$$

The two paths coincide because of the following equalities, where we use that $\mathscr{G}$ is an $\mathscr{O}_X$-module and in particular it satisfies property *ii*), and φ is a morphism of sheaves

$$\big(res_U^X(\alpha)\varphi_U(s)\big)|_V = res_U^X(\alpha)|_V\varphi_U(s)|_V = res_V^X(\alpha)\varphi_V(s|_V)$$

Setting $U \mapsto \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ we have defined a presheaf, which satisfies properties *i*) and *ii*) above. We claim that this presheaf is already a sheaf.

> Indeed let $\{U_i\}$ be an open covering of X and let φ_i be a section in $\mathrm{Hom}_{\mathscr{O}_X|_{U_i}}(\mathscr{F}|_{U_i}, \mathscr{G}|_{U_i})$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. Define the section $\varphi \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ as the unique morphism of sheaves such that $\varphi_V|_{V\cap U_i} = \varphi_i|_{V \cap U_i}$ for every open set $V \subseteq X$.

This is called the *sheaf of local morphisms* of $\mathscr{F}$ into $\mathscr{G}$, and usually denoted $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ or simply $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ if the structure sheaf is understood.

Proposition (Exercise II.5.1 in Hartshorne, 1977)**.** *Let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Then for any $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$ we have the following isomorphism of modules over $\Gamma(X, \mathscr{O}_X)$*

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathscr{O}_X}\big(\mathscr{F}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{G})\big)$$

that is the functor $\mathscr{E} \otimes -$ is left adjoint *to the functor $\mathscr{H}om(\mathscr{E}, -)$.*

Proof. In analogy with the case of modules over a ring, given a morphism of sheaves of $\mathscr{O}_X$-modules $\varphi\colon \mathscr{E} \otimes \mathscr{F} \to \mathscr{G}$, for any open set $U \subseteq X$ we define

$$\begin{array}{rcl} \gamma(\varphi)_U\colon \mathscr{F}(U) & \longrightarrow & \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{E}|_U, \mathscr{G}|_U) \\ s & \mapsto & \big(\alpha^{(s)}\colon \mathscr{E}|_U \to \mathscr{G}|_U\big) \end{array}$$

where $\alpha^{(s)}$ is a morphism of sheaves, and for any $V \subseteq U$ open set in X

$$\begin{aligned} \alpha^{(s)}_V \colon \mathscr{E}(V) &\longrightarrow \mathscr{G}(V) \\ t &\longmapsto \varphi_V(t \otimes s|_V) \end{aligned}$$

the reader can check that γ is a homomorphism of $\Gamma(X, \mathscr{O}_X)$-modules. To see that this is in fact an isomorphism we should check injectivity and surjectivity, but in this case it is far quicker to construct the inverse. Again we are led by the analogy with the case of modules over a ring, and for any open set $U \subseteq X$ we define

$$\begin{aligned} \delta(\psi)_U \colon \mathscr{E}(U) \otimes \mathscr{F}(U) &\longrightarrow \mathscr{G}(U) \\ t \otimes s &\longmapsto [\psi_U(s)]_U\,(t) \end{aligned}$$

where $\psi_U(s) \colon \mathscr{E}|_U \to \mathscr{G}|_U$ is a morphism of sheaves. In the same way as γ before, δ is a homomorphism of $\Gamma(X, \mathscr{O}_X)$-modules, and it is immediate to check that $\gamma\big(\delta(\psi)\big) = \psi$ for any ψ and $\delta\big(\gamma(\varphi)\big) = \varphi$ for any φ. □

5.1.3 Pull-backs – Adjunction Let $f \colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. If $\mathscr{F}$ is an $\mathscr{O}_X$-module then the direct image sheaf $f_*\mathscr{F}$ is an $f_*\mathscr{O}_X$-module, but since we have the morphism $f^\# \colon \mathscr{O}_Y \to f_*\mathscr{O}_X$ of sheaves of rings on Y, this gives $f_*\mathscr{F}$ a natural structure of $\mathscr{O}_Y$-module also. In this way we have defined the push-forward of $\mathscr{O}_X$-modules, while to define the right operation of pull-back we need a little bit more of explanation.

Definition. Let $f \colon X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathscr{G}$ on Y, we define the *inverse image* sheaf $f^{-1}\mathscr{G}$ on X to be the sheaf associated to the presheaf which sends an open set $U \subseteq X$ to

$$f^{-1}\mathscr{G}(U) = \varinjlim_{V \supset f(U)} \mathscr{G}(V),$$

where the limit is taken over all open sets V of Y containing $f(U)$.

"Note that f_* is a functor from the category $(Sheaves/X)$ of sheaves on X to the category $(Sheaves/Y)$ of sheaves on Y. Similarly, f^{-1} is a functor from $(Sheaves/Y)$ to $(Sheaves/X)$."

taken from Hartshorne (1977, §II.1)

Lemma (Exercise II.1.18 in Hartshorne, 1977)**.** *The functor f^{-1} is left adjoint to the functor f_*, that is for any sheaf $\mathscr{F}$ on X and any sheaf $\mathscr{G}$ on Y there is a natural bijection of sets*

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \longleftrightarrow \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

Proof. In Smith (2002) the reader will find a good explanation of all the concepts involved: "there are natural maps $\mathscr{G} \to f_* f^{-1}\mathscr{G}$ of sheaves on Y and $f^{-1} f_* \mathscr{F} \to \mathscr{F}$ of sheaves on X. These maps are little more than glorified restriction maps, and defining them is just a matter of unravelling the definitions." □

If $\mathscr{G}$ is an $\mathscr{O}_Y$-module it is not true that $f^{-1}\mathscr{G}$ is an $\mathscr{O}_X$-module, so we cannot use the inverse image operation alone. Anyway observe that $f^{-1}\mathscr{G}$ is an $f^{-1}\mathscr{O}_Y$-module. Because of the adjointness between f^{-1} and f_* the morphism $f^\#\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$ naturally defines a morphism $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ of sheaves of rings on X. Therefore also the sheaf of structure $\mathscr{O}_X$ is an $f^{-1}\mathscr{O}_Y$-module. We define the pull-back of $\mathscr{G}$ as the sheaf of $\mathscr{O}_X$-modules

$$f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$$

Observe that $f^*\mathscr{O}_Y = \mathscr{O}_X$. For any $x \in X$ it is obvious that the stalk on X of the inverse image $f^{-1}\mathscr{G}$ over x is given by the stalk on Y of $\mathscr{G}$ over $f(x)$, that is $f^{-1}\mathscr{G}_x = \mathscr{G}_{f(x)}$, so we have also the following equality

$$f^*\mathscr{G}_x = \mathscr{G}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x}$$

Proposition (Exercise 6 in Smith, 2002)**.** *The functor f^* is left adjoint to f_*, that is for any $\mathscr{O}_X$-module $\mathscr{F}$ and any $\mathscr{O}_Y$-module $\mathscr{G}$ there is a natural bijection of sets*

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \longleftrightarrow \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F})$$

Proof. We start observing that the adjunction of the previous Lemma respects the module structure in that it defines the following natural bijection between homomorphisms of $\mathscr{O}_Y$-modules and homomorphisms of $f^{-1}\mathscr{O}_Y$-modules

$$\mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{G}, \mathscr{F}) \longleftrightarrow \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F})$$

Tensoring out with $\mathscr{O}_X$ over $f^{-1}\mathscr{O}_Y$ we can also define a bijection

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X) \longleftrightarrow \mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{G}, \mathscr{F})$$

Now observe that $\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$ is canonically isomorphic to $\mathscr{F}$. □

5.2 Locally Free Sheaves and the Picard Group

5.2.1 Locally Free Sheaves Let X be a ringed space with structure sheaf $\mathscr{O}_X$. A sheaf of $\mathscr{O}_X$-modules $\mathscr{E}$ is *free* if it is isomorphic to a direct sum of copies of

$\mathscr{O}_X$. It is *locally free* if there exists a *trivialising covering* of X for $\mathscr{E}$, that is if X can be covered by open sets U_i for which $\mathscr{E}|_{U_i}$ is a free $\mathscr{O}_X|_{U_i}$-module. It is clear that any refinement of the trivialising covering $\{U_i\}$ is again a trivialising covering, moreover we can also assume $\{U_i\}$ to be a *base for the topology* on X, indeed given a base $\mathscr{B}$ for the topology on X we can consider the refinement of the given covering $\{U_i\}$ made by every open set $V \in \mathscr{B}$ that is contained in some U_i. Clearly every open set in X is a union of subsets of this kind.

The *rank* of $\mathscr{E}$ on any open set of the trivialising covering is the number of copies of the structure sheaf needed (finite or infinite). If X is connected, the rank of a locally free sheaf is the same everywhere, however it is often convenient to let X be possibly disconnected but work anyway with *locally free sheaves of finite rank*, meaning that we require the rank to be at least everywhere finite, if we want it to be also constant we will specify *of rank n*.

Given a locally free $\mathscr{O}_X$-module of finite rank $\mathscr{E}$, we define the *dual* of $\mathscr{E}$, denoted $\mathscr{E}^\vee$, to be the sheaf $\mathscr{H}om(\mathscr{E}, \mathscr{O}_X)$. Observe that $\mathscr{E}^\vee$ is again a locally free $\mathscr{O}_X$-module, of the same rank as $\mathscr{E}$. Indeed for all members U of a sufficiently fine open covering of X we have

$$\begin{aligned} \mathscr{E}^\vee|_U = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X)|_U &= \mathscr{H}om_{\mathscr{O}_X|_U}(\mathscr{E}|_U, \mathscr{O}_X|_U) \\ &\cong \mathscr{H}om_{\mathscr{O}_X|_U}(\mathscr{O}_X^n|_U, \mathscr{O}_X|_U) \end{aligned}$$

Now it is enough to apply the following result.

Lemma. *Let $(X, \mathscr{O}_X)$ be a ringed space, then there is a natural isomorphism*

$$\varphi \colon \mathscr{O}_X^n \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{O}_X)$$

Proof. Recall first that the direct sum of two sheaves $\mathscr{F}$ and $\mathscr{G}$ is defined simply by assigning $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ as in §B.2.2. Now it is enough to observe that the global sections $e_1, \dots, e_n$ in $\Gamma(X, \mathscr{O}_X)^n$ define by restriction a base for any module $\mathscr{O}_X^n(U) = \mathscr{O}_X(U)^n$, so for any sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ every morphism $\psi \colon \mathscr{O}_X^n \to \mathscr{F}$ is uniquely determined by the n-tuple $\psi_X(e_1), \dots, \psi_X(e_n)$ of global sections of $\mathscr{F}$. □

5.2.2 Two Canonical Isomorphisms We present here two important canonical isomorphisms, the reader will recognise the statements as they are very well known results in commutative algebra about modules over a ring. What makes them special in this case is that they still hold for sheaves.

Proposition (Exercise II.5.1 in Hartshorne, 1977)**.** *Let $\mathscr{E}$ be a locally free sheaf of $\mathscr{O}_X$-modules of finite rank. Then*

$$\mathscr{E} \cong \left(\mathscr{E}^\vee\right)^\vee.$$

Proof. We begin recalling some commutative algebra: if M is any module over a ring R we have a homomorphism $M \to \operatorname{Hom}_R\big(\operatorname{Hom}_R(M,R),R\big)$ defined by the action of the elements of M on $\operatorname{Hom}_R(M,R)$. If M is free of finite rank, after choosing a base on M one shows that this homomorphism is in fact an isomorphism with easy arguments of linear algebra. Going back to the set up on sheaves if $\mathscr{E}$ is any $\mathscr{O}_X$-module we define a morphism

$$\Phi\colon \mathscr{E} \longrightarrow \mathscr{H}om\big(\mathscr{H}om(\mathscr{E},\mathscr{O}_X),\mathscr{O}_X\big)$$

as follows

$$\begin{array}{rcl}\Phi_U\colon \mathscr{E}(U) & \longrightarrow & \operatorname{Hom}_{\mathscr{O}_X|_U}\big(\mathscr{H}om_{\mathscr{O}_X|_U}(\mathscr{E}|_U,\mathscr{O}_X|_U),\mathscr{O}_X|_U\big)\\ s & \mapsto & \big(\psi^{(s)}\colon \mathscr{H}om_{\mathscr{O}_X|_U}(\mathscr{E}|_U,\mathscr{O}_X|_U)\to\mathscr{O}_X|_U\big)\end{array}$$

where $\psi^{(s)}$ is a morphism of sheaves defined for any open set $V \subseteq U$ in X as

$$\begin{array}{rcl}\psi^{(s)}_V\colon \operatorname{Hom}_{\mathscr{O}_X|_V}(\mathscr{E}|_V,\mathscr{O}_X|_V) & \longrightarrow & \mathscr{O}_X(V)\\ \big(\alpha\colon \mathscr{E}|_V\to\mathscr{O}_X|_V\big) & \mapsto & \alpha_V(s|_V)\end{array}$$

When $\mathscr{E}$ is locally free of finite rank we can assume U to be a trivialising open set, so s is an n-tuple $(s_1,\dots,s_n)$ in $\mathscr{O}_X(U)^n$ and the morphism α is uniquely determined by an n-tuple of global sections. Therefore with easy considerations of linear algebra we can conclude that Φ_U is an isomorphism. □

Proposition (Exercise II.5.1 in Hartshorne, 1977)**.** *Let $\mathscr{E}$ be a locally free sheaf of $\mathscr{O}_X$-modules of finite rank. Then for any $\mathscr{O}_X$-module $\mathscr{F}$ we have*

$$\mathscr{E}^\vee\otimes\mathscr{F}\cong\mathscr{H}om(\mathscr{E},\mathscr{F})$$

Proof. We begin recalling some commutative algebra: if M and N are modules over a ring R we have a morphism $\operatorname{Hom}(M,R)\otimes N \to \operatorname{Hom}(M,N)$ defined by $\varphi\otimes n\mapsto\varphi(\,\cdot\,)n$. When M is free of finite rank every element of $\operatorname{Hom}(M,R)\otimes N$ can be written uniquely as $\varphi_1\otimes n_1+\cdots+\varphi_k\otimes n_k$ where k is the rank of M and φ_i is the i-th coordinate map from M to R and with easy arguments of linear algebra one shows that this is an isomorphism. Going back to the set up on sheaves we define a morphism

$$\Phi\colon \mathscr{H}om(\mathscr{E},\mathscr{O}_X)\otimes\mathscr{F}\longrightarrow\mathscr{H}om(\mathscr{E},\mathscr{F})$$

for any open set $U\subseteq X$ as follows

$$\begin{array}{rcl}\Phi_U\colon \operatorname{Hom}_{\mathscr{O}_X|_U}(\mathscr{E}|_U,\mathscr{O}_X|_U)\otimes\mathscr{F}(U) & \longrightarrow & \operatorname{Hom}_{\mathscr{O}_X|_U}(\mathscr{E}|_U,\mathscr{F}|_U)\\ \varphi\otimes t & \mapsto & \Big(\varphi^{(t)}\colon \mathscr{E}|_U\to\mathscr{F}|_U\Big)\end{array}$$

where φ and $\varphi^{(t)}$ are morphisms of sheaves, and for any inclusion of open sets $V \subseteq U$ in X $\varphi^{(t)}$ is defined by

$$\begin{aligned} \varphi_V^{(t)} : \mathscr{E}(V) &\longrightarrow \mathscr{F}(V) \\ s &\longmapsto \varphi_V(s)(t|_V) \end{aligned}$$

With these definitions when $\mathscr{E}$ is locally free of finite rank we can assume U to be a trivialising open set, and with easy considerations of linear algebra we can conclude that Φ_U is an isomorphism. □

5.2.3 Projection Formula We will prove here the Projection Formula, a useful formula dealing with pull-backs and push-forwards. But first we need a preliminary result.

Lemma (Exercise 7 in Smith, 2002)**.** *Let $f\colon X \to Y$ be a morphism of ringed spaces. If $\mathscr{E}$ is a locally free $\mathscr{O}_Y$-module of finite rank then $f^*\mathscr{E}$ is a locally free $\mathscr{O}_X$-module of the same rank.*

Proof. Observe that for any open subset U of X we have the following equality on the restricted sheaves

$$f^*\mathscr{E}|_U = f^{-1}\mathscr{E}|_U \otimes_{f^{-1}\mathscr{O}_Y|_U} \mathscr{O}_X|_U$$

Therefore it is enough to prove that f^{-1} commutes with direct sums, and this follows easily from the more general result that direct limit commutes with direct sums. □

Proposition (Exercise II.5.1 in Hartshorne, 1977)**.** *Let $f\colon X \to Y$ be a morphism of ringed spaces. If $\mathscr{F}$ is an $\mathscr{O}_X$-module and if $\mathscr{E}$ is a locally free $\mathscr{O}_Y$-module of finite rank, then there is a natural isomorphism of $\mathscr{O}_Y$-modules*

$$f_*\left(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E}\right) \cong f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{E}$$

Proof. For any sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ and any sheaf of $\mathscr{O}_Y$-modules $\mathscr{E}$ we have the canonical isomorphism

$$\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E} = \mathscr{F} \otimes_{\mathscr{O}_X} \left(f^{-1}\mathscr{E} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X\right) \cong \mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{E}$$

Therefore by tensoring the identity with the natural map $\mathscr{E} \to f_*f^{-1}\mathscr{E}$ we can define a morphism of $\mathscr{O}_Y$-modules

$$f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{E} \longrightarrow f_*\left(\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{E}\right)$$

that, when $\mathscr{E}$ is locally free of rank n, on any open subset of a trivialising covering is just the identity of $f_*\mathscr{F}^n$. □

5.2.4 The Picard Group The locally free $\mathscr{O}_X$-modules of rank one are called *invertible modules* or invertible sheaves, the reason for this name is that the set of isomorphism classes of invertible sheaves over X is a group under the operation of tensor product. This is a well known result and can be found for instance in Hartshorne (1977) or in Tennison (1975), nevertheless we are now going to prove it.

If $\mathscr{L}$ and $\mathscr{M}$ are invertible sheaves, then for all members U of a sufficiently fine open covering of X we have

$$(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{M})|_U \cong \mathscr{L}|_U \otimes_{\mathscr{O}_X|_U} \mathscr{M}|_U \cong \mathscr{O}_X|_U \otimes_{\mathscr{O}_X|_U} \mathscr{O}_X|_U \cong \mathscr{O}_X|_U$$

so that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{M}$ is also invertible. Observe also that the structure sheaf is a unit element for this operation, indeed $\mathscr{L} \otimes \mathscr{O}_X \cong \mathscr{L}$ for any $\mathscr{L}$.

Lemma. *Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{L}$ be an invertible sheaf on X. Then there is a natural isomorphism*

$$\varphi\colon \mathscr{O}_X \longrightarrow \mathscr{H}om(\mathscr{L}, \mathscr{L})$$

Proof. For every open subset $U \subseteq X$ we define φ_U to be the morphism sending the section $s \in \mathscr{O}_X(U)$ to the morphism of sheaves $\mathscr{L}|_U \to \mathscr{L}|_U$ defined on any open set $V \subseteq U$ by multiplication for $s|_V$. This morphism of sheaves coincides with the isomorphism we have seen in §5.2.1 above on any open subset U such that $\mathscr{L}|_U \cong \mathscr{O}_X|_U$, and therefore it is an isomorphism. □

Proposition. *Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{L}$ be an invertible sheaf on X. Then there exists an invertible sheaf $\mathscr{L}^{-1}$ on X such that $\mathscr{L} \otimes \mathscr{L}^{-1} \cong \mathscr{O}_X$.*

Proof. Take $\mathscr{L}^{-1}$ to be the dual sheaf $\mathscr{L}^{\vee}$. We know from §5.2.1 that this is again an invertible sheaf, and in view of §5.2.2 there is a natural isomorphism $\mathscr{L}^{\vee} \otimes \mathscr{L} \cong \mathscr{H}om(\mathscr{L}, \mathscr{L})$. To conclude apply the previous Lemma. □

Definition. For any ringed space X, we define the *Picard group* of X, $\operatorname{Pic} X$, to be the group of isomorphism classes of invertible sheaves on X, under the operation $\bigotimes$. In view of the previous results $\operatorname{Pic} X$ is in fact a group.

5.3 Coherent Sheaves

5.3.1 The Functor Shf We will apply now all the machinery of $\mathscr{O}_X$-modules to schemes. First consider the affine case, so let A be a ring and $X = \operatorname{Spec} A$ be its spectrum. To any A-module M we are now going to associate a sheaf of $\mathscr{O}_X$-modules, denoted $\operatorname{Shf} M$; the construction is identical to the one of the structure sheaf, in fact the ring A is a particular A-module and we should have dealt with modules all the time. Just like in §1.3 we have two possible ways of defining Shf$M$, by means of functions $s\colon U \to \coprod_{\mathfrak{p}\in U} M_{\mathfrak{p}}$ or by specifying a $\mathscr{B}$-sheaf where $\mathscr{B}$ is the base for the Zariski topology given by open sets $D(\alpha)$ for any $\alpha \in A$. Either way the main properties that this sheaf will have are the following

i) for any $\mathfrak{p} \in X$, the stalk $(\mathrm{Shf}M)_{\mathfrak{p}}$ is isomorphic to $M_{\mathfrak{p}}$;

ii) for any $\alpha \in A$, the A_α-module $\mathrm{Shf}M\big(D(\alpha)\big)$ is isomorphic to M_α;

iii) in particular, $\Gamma(\operatorname{Spec} A, \mathrm{Shf}M) \cong M$.

In Hartshorne (1977, §II.5) the first approach is explained in details, while I would invite the reader to work out the $\mathscr{B}$-sheaf business on its own. Observe that in this case property *ii*) alone is enough to define Shf$M$.

Of course for this association to be useful we need it to be functorial, even better if the functor will be fully faithful. If $\varphi\colon M \to N$ is a homomorphism of A-modules, in view of the properties above we can define the morphism of $\mathscr{O}_X$-modules Shfφ on any distinguished open subset $D(\alpha)$ of X as the localised map $\varphi_\alpha\colon M_\alpha \to N_\alpha$. Again there is a strong analogy, this time with the morphism of schemes induced by a homomorphism of rings, and functoriality shows up without any surprise. Also standard constructions on A-modules are respected by this functor, as shown by the following result.

Proposition (II.5.2 in Hartshorne, 1977)**.** *Let A be a ring and let $X = \operatorname{Spec} A$. Also let $A \to B$ be a ring homomorphism, and let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the corresponding morphism of spectra. Then:*

(*a*) *the map $M \mapsto \mathrm{Shf}M$ gives an exact, fully faithful functor from the category of A-modules to the category of $\mathscr{O}_X$-modules;*

(*b*) *if M and N are two A-modules, then $\mathrm{Shf}(M \otimes_A N) \cong \mathrm{Shf}M \otimes_{\mathscr{O}_X} \mathrm{Shf}N$;*

(*c*) *if $\{M_i\}$ is any family of A-modules, then $\mathrm{Shf}(\bigoplus M_i) \cong \bigoplus \mathrm{Shf}M_i$;*

(*d*) *for any B-module N we have $f_*(\mathrm{Shf}N) \cong \mathrm{Shf}({}_AN)$, where ${}_AN$ means N considered as an A-module;*

(*e*) *for any A-module M we have $f^*(\mathrm{Shf}M) \cong \mathrm{Shf}(M \otimes_A B)$.*

5.3.2 Adjoint Property of Shf In analogy with the adjunction we have seen in §2.1.4 between Spec and the global section functor we have the following adjunction between Shf and the global section functor.

Proposition (Exercise II.5.3 in Hartshorne, 1977). *Let $X = \operatorname{Spec} A$ be an affine scheme. The functors* Shf *and* Γ *are adjoint in the following sense: for any A-module M and for any sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$, there is a natural isomorphism of A-modules*

$$\operatorname{Hom}_{\mathscr{O}_X}(\operatorname{Shf}M, \mathscr{F}) \longrightarrow \operatorname{Hom}_A\big(M, \Gamma(X, \mathscr{F})\big)$$

Proof. For any morphism of sheaves $\xi\colon \operatorname{Shf}M \to \mathscr{F}$, taking global sections gives a morphism $\xi_X\colon M \to \Gamma(X, \mathscr{F})$ and this association defines a morphism of A-modules. Conversely let $\varphi\colon M \to \Gamma(X, \mathscr{F})$, for any $\alpha \in A$ observe that $\mathscr{F}\big(D(\alpha)\big)$ is an A_α-module and define a morphism $M_\alpha \to \mathscr{F}\big(D(\alpha)\big)$ as the composition

$$M_\alpha \xrightarrow{\varphi_\alpha} \Gamma(X, \mathscr{F})_\alpha \xrightarrow{res_\alpha} \mathscr{F}\big(D(\alpha)\big)$$

It is now routine to check that these maps are inverse to each other. □

5.3.3 Quasi-coherent Sheaves As usual once we have a well understood construction on affine schemes we generalise it to every scheme by means of an open affine covering. In this particular case we are going to define what is a quasi-coherent sheaf, the result that follows is that quasi-coherence is a local property.

Definition. Let $(X, \mathscr{O}_X)$ be a scheme. A sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is *quasi-coherent* if X can be covered by open affine subsets $\{U_i = \operatorname{Spec} A_i\}$, such that for each i there is an A_i-module M_i with $\mathscr{F}|_{U_i} \cong \operatorname{Shf}M_i$.

Proposition (II.5.4 in Hartshorne, 1977). *Let X be a scheme. Then an $\mathscr{O}_X$-module $\mathscr{F}$ is quasi-coherent if and only if for every open affine subset $U = \operatorname{Spec} A$ of X, there is an A-module M with $\mathscr{F}|_U \cong \operatorname{Shf}M$.*

Observe that a locally free $\mathscr{O}_X$-module $\mathscr{E}$ of rank n is a quasi-coherent sheaf. Indeed X can be covered by open sets U such that $\mathscr{E}|_U \cong (\mathscr{O}_X|_U)^n$ and by refining the existing covering if necessary we can assume U to be affine. Now we have

$$(\mathscr{O}_X|_U)^n = \big(\operatorname{Shf}\mathscr{O}_X(U)\big)^n \cong \operatorname{Shf}\big(\mathscr{O}_X(U)^n\big)$$

But of course quasi-coherent sheaves are more general than locally free sheaves. The kernel, cokernel, and image of any morphism of quasi-coherent sheaves are quasi-coherent. Any extension of quasi-coherent sheaves is quasi-coherent; this refers to short exact sequences of $\mathscr{O}_X$-modules, more precisely given $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, if $\mathscr{F}'$ and $\mathscr{F}''$ are quasi-coherent then so is $\mathscr{F}$ (see Hartshorne, 1977, for a proof). It is possible, and sometimes useful, to give an equivalent definition of quasi-coherence. We will state it as a Theorem, but first we need another result.

Proposition (II.5.6 in Hartshorne, 1977)**.** *Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of $\mathscr{O}_X$-modules, where X is an affine scheme, and assume that $\mathscr{F}'$ is quasi-coherent. Then the following sequence is exact*

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0$$

Theorem (Exercise II.5.4 in Hartshorne, 1977)**.** *Let X be a scheme and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. The following are equivalent*

i) *$\mathscr{F}$ is quasi-coherent;*

ii) *for every $x \in X$, there exists an open neighborhood U of x and an exact sequence of $\mathscr{O}_X$-modules*

$$\mathscr{O}_X^{(J)}|_U \longrightarrow \mathscr{O}_X^{(I)}|_U \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

where I and J are sets and by $\mathscr{O}_X^{(I)}$ we mean the free $\mathscr{O}_X$-module over I.

Proof. In Chapter 5 of Liu (2002) statement *ii*) is given as the definition of quasi-coherence, and this Theorem is stated from the opposite point of view as Theorem 1.7, but the proof in there needs X to be Noetherian or separated and quasi-compact.

Observe that if $\mathscr{F}$ is quasi-coherent then statement *ii*) holds trivially. For the converse let U be an open affine set such that there exists an exact sequence as above. Since the sheaf $\mathscr{O}_X^{(J)}|_U$ is quasi-coherent the corresponding sequence on global sections is exact, then we can apply the functor Shf to obtain another exact sequence of sheaves. In this way we can construct the following commutative diagram

$$\begin{array}{ccccccc}
\text{Shf}\left(\mathscr{O}_X(U)^{(J)}\right) & \longrightarrow & \text{Shf}\left(\mathscr{O}_X(U)^{(I)}\right) & \longrightarrow & \text{Shf}\,\mathscr{F}(U) & \longrightarrow & 0 \\
\| & & \| & & \downarrow \psi & & \\
\mathscr{O}_X^{(J)}|_U & \longrightarrow & \mathscr{O}_X^{(I)}|_U & \longrightarrow & \mathscr{F}|_U & \longrightarrow & 0
\end{array}$$

where vertical arrows are induced by adjunction as in §5.3.2. For any point $x \in U$ we have a diagram of $\mathscr{O}_{X,x}$-modules. An easy diagram chase will now prove that ψ is an isomorphism. □

5.3.4 Coherent Sheaves over Noetherian Schemes Let X be a Noetherian scheme. We say that a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is *coherent* if it can be covered by open affine subsets $\{U_i = \operatorname{Spec} A_i\}$, such that for each i there is a *finitely generated* A_i-module M_i with $\mathscr{F}|_{U_i} \cong \operatorname{Shf} M_i$. In particular $\mathscr{F}$ is quasi-coherent.

Just like quasi-coherence also coherence is a local property, again the kernel, cokernel, and image of any morphism of coherent sheaves are coherent and any extension of coherent sheaves is coherent. The equivalent definition we have seen above holds also for coherent sheaves over a Noetherian scheme with the extra condition that the sets I and J be finite.

Proposition. *Let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves over a Noetherian scheme X. Then we have an isomorphism on the stalks of the sheaf $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$*

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})_x \longrightarrow \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)$$

Proof. Observe that we can always define such a morphism, even when $\mathscr{F}$ and $\mathscr{G}$ are just sheaves of $\mathscr{O}_X$-modules. To be precise any element of the stalk $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})_x$ is given by the equivalence class of a couple (U,φ) where U is an open subset of X and $\varphi\colon \mathscr{F}|_U \to \mathscr{G}|_U$ is a morphism of sheaves, and φ induces a morphism of $\mathscr{O}_{X,x}$-modules that doesn't depend on the particular choice inside the equivalence class. The reader can check that this association defines in fact a morphism of $\mathscr{O}_{X,x}$-modules, the issue here is to prove that this is in fact an isomorphism. The statement is local, so it is enough to assume the scheme X to be affine, say $X = \operatorname{Spec} A$. Then by adjunction (§5.3.2) for any $\alpha \in A$ we have the following isomorphism

$$\operatorname{Hom}_{\mathscr{O}_X|_{D(\alpha)}}\left(\operatorname{Shf} M|_{D(\alpha)}, \operatorname{Shf} N|_{D(\alpha)}\right) \cong \operatorname{Hom}_{A_\alpha}\left(M_\alpha, N_\alpha\right)$$

where M is assumed to be finitely generated, say by $m_1, \dots, m_r$.

Injectivity. Let $f\colon M \to N$ be a morphism of A-modules such that the localisation $f_\mathfrak{p}\colon M_\mathfrak{p} \to N_\mathfrak{p}$ is the zero map. Then for any $i = 1,\dots,r$ we have $f_\mathfrak{p}(m_i) = 0$, this is equivalent to say that for any $i = 1,\dots,r$ there exists $\alpha_i \notin \mathfrak{p}$ such that $\alpha_i m_i \in \ker f$. If we define α to be the product of the α_i's the localised morphism $f_\alpha\colon M_\alpha \to N_\alpha$ is the zero map.

Surjectivity. Let $\psi\colon M_\mathfrak{p} \to N_\mathfrak{p}$ be a morphism of $A_\mathfrak{p}$-modules. Then ψ is uniquely determined by $\psi(m_1), \dots, \psi(m_r)$, because $m_1, \dots, m_r$ is a system of generators for $M_\mathfrak{p}$, and these will be fractions $n_1/\beta, \dots, n_r/\beta$ where β can be taken to be independent from i. Now the same set of elements defines a unique morphism $M_\beta \to N_\beta$ whose localisation in $\mathfrak{p}$ is ψ. □

5.3.5 Tensor Operations For any sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$, we define the *tensor algebra*, *symmetric algebra*, and *exterior algebra* of $\mathscr{F}$ by taking the sheaves associated to the presheaves which to each open set U assign the corresponding tensor operation applied to $\mathscr{F}(U)$ as an $\mathscr{O}_X(U)$-module. The results are $\mathscr{O}_X$-algebras, and their components in each degree are $\mathscr{O}_X$-modules. From the construction it is also clear that if $\mathscr{F}$ is quasi-coherent (respectively coherent) then all of these are quasi-coherent (respectively coherent).

To fix notations we recall the definitions of these tensor operations on a module. As usual we are forced to assume the reader familiar with the basic concepts, referring him to the appendices in Eisenbud (1995) or Fulton and Harris (1991) or to Northcott (1984) for a more complete exposition.

Let A be a ring and let M be an A-module. Let $T^n(M)$ be the tensor product $M \otimes \cdots \otimes M$ of M with itself n times, for $n \geq 1$. For $n = 0$ we put $T^0(M) = A$. Then

$$T(M) = \bigoplus_{n \geq 0} T^n(M)$$

is a (noncommutative) A-algebra, which we call the *tensor algebra* of M.

We define the *symmetric algebra* of M

$$\operatorname{Sym}(M) = \bigoplus_{n \geq 0} \operatorname{Sym}^n(M)$$

to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes y - y \otimes x$, for all $x, y \in M$. Then $\operatorname{Sym}(M)$ is a commutative A-algebra. Its component $\operatorname{Sym}^n(M)$ in degree n is called the *n-th symmetric product* of M. We denote the image of $x \otimes y$ in $\operatorname{Sym}(M)$ by xy, for any $x, y \in M$. If M is a free A-module of rank r, then the choice of a basis for M determines an isomorphism $\operatorname{Sym}(M) \to A[x_1, \dots, x_n]$.

We define the *exterior algebra* of M

$$\bigwedge(M) = \bigoplus_{n \geq 0} \bigwedge^n(M)$$

to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes x$, for $x \in M$. Note that this ideal contains all expressions of the form $x \otimes y + y \otimes x$, so that $\bigwedge(M)$ is a *skew commutative* graded A-algebra. This means that if $u \in \bigwedge^r(M)$ and $v \in \bigwedge^s(M)$, then

$$u \wedge v = (-1)^{rs} v \wedge u$$

where we denote by $\wedge$ the multiplication in this algebra; so the image of $x \otimes y$ in $\bigwedge^2(M)$ is denoted by $x \wedge y$. The n-th component $\bigwedge^n(M)$ is called the *n-th exterior power* of M.

5.3.6 Coherence for Affine Schemes – Projective Modules Let $X = \operatorname{Spec} A$ be an affine scheme and let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of rank n. Since quasi-coherence is a local property, we know that there exists an A-module M such that $\mathscr{E} = \operatorname{Shf} M$. What can we say about the module M? We can reasonably expect to characterise modules that give rise to locally free sheaves, if not to prove that they are all free. To begin with observe that the localisation $M_{\mathfrak{p}}$ of M on any prime ideal $\mathfrak{p}$ is a free $A_{\mathfrak{p}}$-module, a module with this property is called *locally free*.

Proposition. *Let $X = \operatorname{Spec} A$ be a Noetherian affine scheme and let $\mathscr{E} = \operatorname{Shf} M$ be a coherent sheaf. Then $\mathscr{E}$ is a locally free sheaf if and only if M is a projective module.*

Proof. Let $\mathscr{E}$ be a locally free sheaf, then we can assume the trivialising covering of X for $\mathscr{E}$ to consist of basic open sets, and since any affine scheme is quasi-compact we can also assume it to be finite. Now we are in the following situation: M is an A-module and there is a finite set of elements $\alpha_1, \dots, \alpha_r$, that generate the unit ideal of A, such that M_i is free over A_i for each i, where M_i and A_i denote localisation on the element α_i. Conversely if we are in this situation the sheaf $\operatorname{Shf} M$ is by definition locally free. Now we have the following characterisation, which can be found in Eisenbud (1995) as Theorem 19.2 or Theorem A3.2. □

Theorem (19.2 in Eisenbud, 1995)**.** *Let M be a finitely generated module over a Noetherian ring A. The following statements are equivalent:*

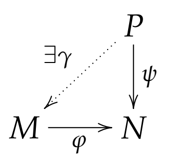
(*a*) *M is a projective module;*

(*b*) *$M_{\mathfrak{p}}$ is a free module for every prime ideal $\mathfrak{p}$ of A;*

(*c*) *There is a finite set of elements $\alpha_1, \dots, \alpha_r$ that generate the unit ideal of A such that M_i is free over A_i for each i.*

In particular, every projective module over a local ring is free. Every graded projective module over a positively graded ring A with A_0 a field is a graded free module.

Projective modules appear naturally in Homological Algebra, where together with the dual concept of *injective modules* they represent the building block for the construction of derived functors. A module P is *projective* if for every epimorphism of modules $\varphi\colon M \to N$ and every map $\psi\colon P \to N$, there exists a map $\gamma\colon P \to M$ such that $\psi = \varphi \circ \gamma$, as in the following diagram



An introduction to these concepts can be found in Appendix 3 of Eisenbud (1995), or in Berrick and Keating (2000) from a more categorical point of view. The characterisation of projectives as locally free modules is also true in general, without the Noetherian and finitely generated hypotheses, the proof is in Kaplansky (1958).

Not all projective modules are free, but in the mid-1950s, Jean-Pierre Serre conjectured that every projective module over a polynomial ring over a field must be free. This was open until 1976, when a proof was given simultaneously, and independently, by Daniel Quillen (Fields Medal in 1978) in Cambridge, Massachusetts and Andrei Suslin in Moscow. As a result, the statement is often referred to as the *Quillen-Suslin Theorem*. It can be found in Mandal (1997), together with other interesting results about projective modules and Algebraic Geometry. It clearly implies that the Picard Group of affine n-space over a field k is trivial.

5.3.7 Pull-backs The next step in the study of quasi-coherent sheaves is to describe their behaviour under pull-backs. Given a morphism of schemes $f\colon X \to Y$ we consider quasi-coherent sheaves $\mathscr{F}$ over X and $\mathscr{G}$ over Y. It turns out that $f^*\mathscr{G}$ is always quasi-coherent, and even more importantly it is coherent whenever $\mathscr{F}$ is coherent. The sheaf $f_*\mathscr{F}$ instead is much more problematic. The main result is the following.

Proposition (II.5.8 in Hartshorne, 1977)**.** *Let $f\colon X \to Y$ be a morphism of schemes.*

(a) *If $\mathscr{G}$ is a quasi-coherent sheaf of $\mathscr{O}_Y$-modules, then $f^*\mathscr{G}$ is a quasi-coherent sheaf of $\mathscr{O}_X$-modules.*

(b) *If X and Y are Noetherian, and if $\mathscr{G}$ is coherent, then $f^*\mathscr{G}$ is coherent.*

(c) *Assume that either X is Noetherian, or f is quasi-compact and separated. Then if $\mathscr{F}$ is a quasi-coherent sheaf of $\mathscr{O}_X$-modules, $f_*\mathscr{F}$ is a quasi-coherent sheaf of $\mathscr{O}_Y$-modules.*

Looking at statement (c) observe that a closed immersion is a quasi-compact and separated morphism, other examples include finite morphisms (§3.4) and proper morphisms (§3.3.6). But we should keep in mind that when X is Noetherian quasi-coherence is not an issue.

The real issue is coherence. Indeed when X and Y are Noetherian and $\mathscr{F}$ is a coherent sheaf it is not true that $f_*\mathscr{F}$ is a coherent sheaf. Not even when f is quasi-compact and separated, in fact not even when f is a morphism of affine schemes.

Example (A coherent sheaf $\mathscr{F}$ such that $f_*\mathscr{F}$ is not coherent). Assume f to be a morphism of affine schemes, induced by the homomorphism of rings $\varphi\colon A \to B$. Then a coherent sheaf of $\mathscr{O}_X$-modules is defined to be Shf M for some finitely generated B-module M, and $f_*\mathscr{F}$ is given by $\mathrm{Shf}({}_AM)$ where ${}_AM$ means M considered as an A-module. In this situation as soon as B is not finitely generated as an A-module we have the easiest counterexample of all, $\mathscr{F} = \mathscr{O}_Y = \mathrm{Shf}\, B$. This is the case for instance when $\varphi\colon \mathbb{C}[x] \to \mathbb{C}[x]_{\mathfrak{m}}$ is the localisation in some maximal ideal $\mathfrak{m} = (x - \lambda)$.

Inspired by this example we are now able to spot a sufficient condition for $f_*\mathscr{F}$ to be coherent. The following is the first and easiest result in this direction, in fact so easy to be almost useless.

Lemma (Exercise II.5.5 in Hartshorne, 1977)**.** *Let X and Y be Noetherian schemes, and let $f\colon X \to Y$ be a finite morphism. If $\mathscr{F}$ is a coherent sheaf of $\mathscr{O}_X$-modules, then $f_*\mathscr{F}$ is a coherent sheaf of $\mathscr{O}_Y$-modules.*

Proof. It is clearly enough to consider Y to be affine, in which case, since the morphism is finite, X also will be affine. So the morphism f will be induced by a homomorphism of rings $\varphi\colon A \to B$, and the sheaf $\mathscr{F}$ will be in fact given by Shf M for some finitely generated B-module M. But since f is finite B is a finitely generated A-module, so that M viewed as an A-module is also finitely generated. This completes the proof because $f_*\mathscr{F}$ is precisely given by $\mathrm{Shf}({}_AM)$. □

The real importance of this result is that it holds more generally for proper morphisms. The proof is in Grothendieck's *Éleménts de géométrie algébrique* (Théorème 3.2.1 in EGA III), but for most purposes it is enough to keep in mind that this holds for projective morphisms (§6.1.1), which is the content of Theorem III.8.8 in Hartshorne (1977). However let us remind the interested reader that Grothendieck's original work is available from <http://www.numdam.org/>.

5.3.8 Global Spec Let $\mathscr{A}$ be a quasi-coherent sheaf of $\mathscr{O}_X$-algebras over a scheme X, that is a sheaf of rings which is also quasi-coherent. Equivalently we can think about $\mathscr{A}$ as a sheaf of rings endowed with a structure morphism $\mathscr{O}_X \to \mathscr{A}$ which makes it into a quasi-coherent sheaf of $\mathscr{O}_X$-modules. We are going to define the *spectrum of* $\mathscr{A}$, this will be a scheme E, which we usually denote by $\mathbf{Spec}\,\mathscr{A}$, endowed with a morphism $\pi\colon \mathbf{Spec}\,\mathscr{A} \to X$ such that $\pi_*(\mathscr{O}_E) = \mathscr{A}$. In the affine case, when X is the spectrum of a ring R, there exists an R-algebra A such that $\mathscr{A} = \mathrm{Shf} A$, thus it is natural to define $\mathbf{Spec}\,\mathscr{A} = \mathrm{Spec}\, A$: indeed, by the adjunction in §5.3.2, the structure morphism of $\mathscr{A}$ will be induced by the structure morphism $R \to A$ of A, which will also define a morphism of affine schemes $\pi\colon \operatorname{Spec} A \to X$ as required.

Proposition (Exercise II.5.17 in Hartshorne, 1977)**.** *Let X be a scheme. Then there exists a fully faithful functor* **Spec** *from the category of quasi-coherent sheaves of $\mathscr{O}_X$-algebras to the category of schemes over X. When $X = \operatorname{Spec} R$ is an affine scheme the functor* **Spec** *coincides with* Spec *from the category of R-algebras to the category of schemes over R.*

Proof. By the previous discussion we know how to define **Spec** when X is affine, and also that in this case it coincides with Spec. Now we want to extend this definition by means of a gluing argument. For any open affine subset U of X consider the affine scheme $\operatorname{Spec}\mathscr{A}(U)$ and observe that whenever $W \subseteq U$ is an inclusion of open affine subsets we have a natural open immersion $\operatorname{Spec}\mathscr{A}(W) \to \operatorname{Spec}\mathscr{A}(U)$ induced by the restriction morphism. It is an exercise to check that this family of schemes glue together and this association is functorial (you may want to use the Gluing Lemma as in §2.3.3). Note that for any scheme X we have $\mathbf{Spec}\, \mathscr{O}_X = X$. □

Observe that in the statement above we could be more precise, indeed the functor Spec from the category of R-algebras to the category of schemes over R coincides with the composition $\mathbf{Spec} \circ \operatorname{Shf}$. But the analogy with the affine case is so strong that no confusion can arise. In fact **Spec** satisfies an adjunction property in the same way as Spec.

Lemma. *Let $\mathscr{A}$ be a quasi-coherent sheaf of $\mathscr{O}_X$-algebras and let S be a scheme over X, with structure morphism $p\colon S \to X$. If we denote by $\mathfrak{Alg}$ the category of quasi-coherent $\mathscr{O}_X$-algebras, there is a natural bijection*

$$\operatorname{Hom}_{\mathfrak{Sch}/X}(S, \mathbf{Spec}\,\mathscr{A}) \longleftrightarrow \operatorname{Hom}_{\mathfrak{Alg}}\left(\mathscr{A}, p_*\mathscr{O}_S\right)$$

given by the adjunction in §2.1.4.

Proof. Let $E = \mathbf{Spec}\,\mathscr{A}$ and $\pi\colon E \to X$ its structure morphism. Then the statement above is very easy to check if we observe that for any open affine subset $U \subseteq X$ we have $V = \pi^{-1}(U) = \operatorname{Spec}\mathscr{A}(U)$ and the restriction $f|^V$ of any morphism of schemes $f\colon S \to E$ is given by a morphism of algebras $\mathscr{A}(U) \to \mathscr{O}_S(p^{-1}(U))$. □

A couple of remarks about this construction are needed. First observe that if $\mathscr{A}$ is a quasi-coherent sheaf of $\mathscr{O}_X$-algebras, the projection morphism $\pi\colon \mathbf{Spec}\,\mathscr{A} \to X$ is obviously an affine morphism (§3.3.6). Viceversa for any affine morphism $f\colon Z \to X$ the direct image sheaf $f_*\mathscr{O}_Z$ is a quasi-coherent sheaf of $\mathscr{O}_X$-algebras and $Z = \mathbf{Spec}\, f_*\mathscr{O}_Z$. Indeed an affine morphism is quasi-compact and separated, so we can apply the results in §5.3.7 above, in particular Proposition II.5.8 in Hartshorne (1977). Observe that a closed immersion is an affine morphism, therefore any closed subscheme of X arises in this way. Observe also the obvious equality $X = \mathbf{Spec}\, \mathscr{O}_X$.

5.4 Vector Bundles on Schemes

"Locally free modules are the most convenient algebraic form of the concept of vector bundles familiar in Topology and Differential Geometry. And invertible sheaves are the algebraic analogs of line bundles."

taken from Mumford (1999, §III.2)

A vector bundle over a manifold X is, roughly speaking, another manifold E obtained from X by gluing an n-dimensional vector space over each point. In his book Mumford explains the statement above by a gluing argument, mimicking the definition of vector bundle as it is for instance in Griffiths and Harris (1994). We will adopt here a slightly different point of view, in that we will start with a purely algebraic definition before actually see that this is the right analogy with Differential Geometry. We begin with the basic construction of *affine n-space* $\mathbb{A}^n_X$ over a scheme X, this is defined to be simply the product

$$\mathbb{A}^n_X = \mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} X$$

where $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$. Observe that if X is a scheme over a field k we can replace $\mathbb{Z}$ with k. When $X = \operatorname{Spec} R$ is affine $\mathbb{A}^n_X$ is affine too and is given by $\operatorname{Spec} R[x_1, \dots, x_n]$, in this case we also denote affine n-space over X by $\mathbb{A}^n_R$. The R-algebra $R[x_1, \dots, x_n]$ is naturally isomorphic to the symmetric algebra $\operatorname{Sym} R^n$, therefore we have also a natural identification

$$\mathbb{A}^n_X = \mathbf{Spec}\operatorname{Sym} \mathscr{O}^n_X$$

We call the product $\mathbb{A}^n_X$ between X and affine space the *trivial bundle*. The projection p onto X is an affine morphism, such that the fiber over each point x of X is given by $\mathbb{A}^n_{k(x)}$.

Definition. A (*geometric*) *vector bundle* of rank n over a scheme X is another scheme E, endowed with a morphism $\pi\colon E \to X$, which is isomorphic to $\mathbf{Spec}\operatorname{Sym} \mathscr{E}$ (as schemes over X), for some locally free sheaf $\mathscr{E}$ of rank n. A *line bundle* is a vector bundle of rank one. A *morphism of vector bundles* is a morphism of schemes compatible with this structure, namely it's a morphism of schemes from $E = \mathbf{Spec}\operatorname{Sym} \mathscr{E}$ to $M = \mathbf{Spec}\operatorname{Sym} \mathscr{M}$ induced by a morphism of $\mathscr{O}_X$-modules from $\mathscr{M}$ to $\mathscr{E}$.

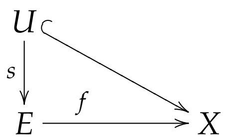
This definition surely fits with our intuition of a vector bundle as a scheme *locally isomorphic* to the trivial bundle, but a few comments are needed. Observe that, for any open affine subset $U = \operatorname{Spec} R$ of X over which the sheaf $\mathscr{E}$ is trivial there exists an isomorphism $\psi_U \colon \pi^{-1}(U) \to \mathbb{A}^n_U$, in other words $\pi^{-1}(U)$ is given by the spectrum of $R[x_1, \ldots, x_n]$, and this family of affine schemes is glued together by means of *linear* morphisms. By this we mean that for any $V = \operatorname{Spec} B$ and for any open affine subset $W \subseteq U \cap V$, $W = \operatorname{Spec} A$, the automorphism

$$\psi = (\psi_U|^{\bullet}) \circ (\psi_V|^{\bullet})^{-1} \colon \mathbb{A}^n_W \longrightarrow \mathbb{A}^n_W$$

is given by a *linear* automorphism θ, that is $\theta(x_i) = \sum a_{ij} x_j$ for suitable elements $a_{ij} \in A$. Similarly, over each open affine subset $U = \operatorname{Spec} R$ of X over which the sheaves $\mathscr{E}$ and $\mathscr{M}$ are trivial, a morphism of vector bundles is given by a linear homomorphism $\mathbb{A}^n_U \to \mathbb{A}^m_U$.

It is a well known result that for any scheme X vector bundles over X and locally free sheaves on X are equivalent concepts, the equivalence being given by a natural one-to-one correspondence up to isomorphism. Therefore, although conceptually different, the expressions "vector bundle over X" and "locally free sheaf on X" are often used interchangeably, and one finds himself dealing almost exclusively with his favourite point of view regardless of which language is being used. We are now going to prove the equivalence, which is left as an exercise in Hartshorne (1977) and discussed very briefly in Mumford (1999).

Definition. For any morphism of schemes $f \colon E \to X$, a *section* of f over an open set $U \subseteq X$ is a morphism $s \colon U \to E$ such that $f \circ s$ is the open immersion of U in X.



It is clear how to restrict sections to smaller open subsets, or how to glue them together, so we see that the presheaf $U \mapsto \{\text{sections of } f \text{ over } U\}$ is a sheaf of sets on X, which we denote by $\mathscr{S}_{E/X}$.

Proposition (Exercise II.5.18 in Hartshorne, 1977). *Let $E = \mathbf{Spec}\, \mathscr{E}$ be a vector bundle of rank n over a scheme X. Then the sheaf $\mathscr{S}_{E/X}$ of sections of the projection morphism $\pi \colon E \to X$ over X is naturally identified with the locally free sheaf of $\mathscr{O}_X$-modules $\mathscr{E}^{\vee}$.*

Proof. As we have seen in §5.3.8 above, sections of π over the open subset U are in one-to-one correspondence with morphisms of quasi-coherent algebras $\operatorname{Sym} \mathscr{E}|_U \to \mathscr{O}_X|_U$, moreover if the open set U is also affine then by adjunction (§5.3.2) morphisms like these correspond to $\mathscr{O}_X(U)$-algebras homomorphisms $\operatorname{Sym} \mathscr{E}(U) \to \mathscr{O}_X(U)$. Now by the very definition of symmetric algebra we have another bijection with homomorphisms of $\mathscr{O}_X(U)$-modules between $\mathscr{E}(U)$ and $\mathscr{O}_X(U)$. □

Theorem (Exercise II.5.18 in Hartshorne, 1977)**.** *Let X be a scheme. The two functors $E \mapsto \mathscr{S}_{E/X}$ and $\mathscr{E} \mapsto \mathbf{Spec}\operatorname{Sym} \mathscr{E}^\vee$ define an equivalence of categories between vector bundles of rank n over X and locally free sheaves of rank n on X.*

The proof of the Theorem is immediate from what we have seen so far, observe by the way that the two functors are covariant. The statement involves a process of duality, which is harmless but nevertheless can lead to confusion. If $\mathscr{E}$ is a locally free sheaf of rank n on X, then $\mathscr{E}$ is the sheaf of sections of a uniquely defined vector bundle over X and this vector bundle is the one usually taken to be in direct correspondence with $\mathscr{E}$. Observe therefore that the actual one-to-one correspondence is straightforward only in one direction, namely from vector bundles to locally free sheaves.

Chapter 6

Projective Geometry

This chapter is essentially divided in two parts. The first is rather technical and contains lots of abstract results concerning schemes and graded rings while in the second we are going to see some modern geometry. The two are unfortunately indivisible, because we need to build a pretty strong bridge between algebra and geometry to understand projective schemes. Indeed Proj doesn't satisfy all the good functorial properties of Spec and this, although really frustrating at the beginning, is the main reason why it is so much more interesting. We are going to study projective and quasi-projective morphisms, analyse the connection between graded modules and quasi-coherent sheaves, and define very ample invertible sheaves. As a climax we will describe the very classic construction of blow-up, starting from the easy example of the blow-up of the plane at the origin to reach in increasing generality the construction of global Proj and the blow-up of a Noetherian scheme along an arbitrary closed subscheme.

6.1 Projective Schemes

6.1.1 Projective Morphisms The reader with some experience in basic Algebraic Geometry may think about a *projective variety* as a closed irreducible subset of projective n-space, defined by some homogeneous ideal I inside the polynomial ring $k[x_0, \dots, x_n]$, in other words as $\operatorname{Proj} k[x_0, \dots, x_n]/I$. But this point of view, although very useful in most cases, hides a subtle problem: let X be an algebraic variety and assume that there exists a closed immersion $X \to \mathbb{P}^n_k$, is this a projective variety? Surely we would be very disappointed if it was not, but truth is that we cannot answer this question without developing more theory. The abstract approach that we are about to describe gives the word *projective* a relative meaning, effectively adopting the idea that a projective variety should be a variety endowed with a closed immersion in projective space.

Definition. If Y is any scheme, we define *projective n-space* over Y, denoted $\mathbb{P}^n_Y$, to be the fibered product $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$. A morphism $f\colon X \to Y$ of schemes is *projective* if it factors into a closed immersion $i\colon X \to \mathbb{P}^n_Y$ for some n, followed by the projection $\mathbb{P}^n_Y \to Y$. A morphism $f\colon X \to Y$ is *quasi-projective* if it factors into an open immersion $j\colon X \to X'$ followed by a projective morphism $g\colon X' \to Y$. A scheme over Y is projective or quasi-projective if its structure morphism is.

When we are working inside the category of schemes over k we can equivalently say that $\mathbb{P}^n_Y$ is the fibered product $\mathbb{P}^n_k \times_k Y$. In particular any algebraic scheme X is projective if and only if it comes endowed with a closed immersion into projective space $\mathbb{P}^n_k$.

Theorem (II.4.9 in Hartshorne, 1977)**.** *A projective morphism of Noetherian schemes is proper. A quasi-projective morphism of Noetherian schemes is of finite type and separated.*

This Theorem is highly non-trivial, but the proof reduces quickly to the case of projective space itself. In other words the hardest work is to show that the structure morphism of projective space $\mathbb{P}^n_k \to \operatorname{Spec} k$ is proper, which means $\mathbb{P}^n_k$ is a complete algebraic variety. The result itself is quite restrictive, namely any scheme projective over k is already an algebraic scheme (recall §3.3.6), in particular any integral scheme projective over k is already a complete algebraic variety.

6.1.2 Properties of Projective Morphisms Projective morphisms satisfy all the good properties of morphisms between schemes, for instance they are stable under base extension. This result has furthermore some very interesting consequences about projective varieties and projective algebraic schemes in general, but before we can see them we need to convince ourselves that for any scheme S the product $\mathbb{P}^n_S \times_S \mathbb{P}^m_S$ is a projective scheme over S. This is a very classic result and the reader may have seen it before, it is obtained by a gluing argument which is explained in details in Liu (2002, Lemma III.3.31).

Segre Embedding. *Let S be any scheme. Then there exists a closed immersion*

$$\mathbb{P}^n_S \times_S \mathbb{P}^m_S \longrightarrow \mathbb{P}^{nm+n+m}_S$$

Proof. Let R and T be two graded rings with $R_0 = T_0 = A$, and let P be the graded ring $\bigoplus_{d\geq 0}(R_d \otimes_A T_d)$. If we denote $X = \operatorname{Proj} R$ and $Y = \operatorname{Proj} T$

then $\operatorname{Proj} P \cong X \times_A Y$. To see this observe that for any homogeneous decomposable element $s \otimes t$ in P_+ there is an isomorphism $P_{(s\otimes t)} \cong S_{(s)} \otimes_A T_{(t)}$. When the base scheme is affine, say $S = \operatorname{Spec} A$, we can consider the rings $R = A[x_0, \ldots, x_n]$ and $T = A[y_0, \ldots, y_m]$, so that $\mathbb{P}^n_S = \operatorname{Proj} R$ and $\mathbb{P}^m_S = \operatorname{Proj} T$. In this case the obvious surjective homomorphism $A[z_{00}, \ldots, z_{ij}, \ldots, z_{nm}] \to P$ defines the desired closed immersion. In the general case we can repeat this construction over any open affine subset of S. □

Proposition (Corollary III.3.32 in Liu, 2002)**.** *Projective morphisms satisfy the following properties.*

(a) *A closed immersion is projective;*

(b) *The composition of two projective morphisms is projective;*

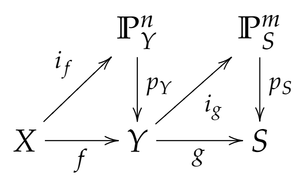
(c) *Projective morphisms are stable under base extension;*

(d) *The product of two projective morphisms is projective;*

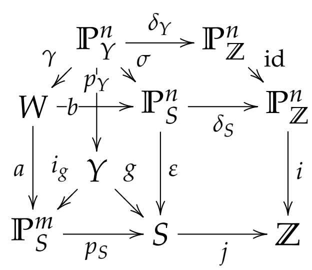
(e) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms, if $g \circ f$ is projective and g is separated, then f is projective.*

Proof. It is enough to prove the first three statements, the rest will follow by the fundamental Lemma about attributes of morphisms (§3.3.7). Observe that $\mathbb{P}^0_Y = Y$ for any scheme Y, so a closed immersion $f\colon X \to Y$ is projective because it factors through a zero dimensional projective space over Y.

If now $f\colon X \to Y$ and $g\colon Y \to S$ are two projective morphisms, there are two closed immersions i_f and i_g as in the following diagram

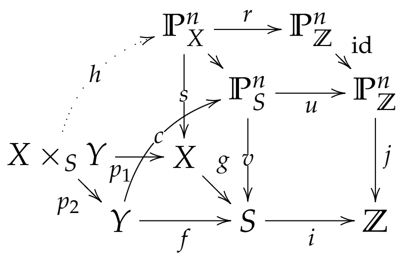


By the Segre embedding, to prove statement (b) it is enough to prove that there exists a closed immersion $\gamma\colon \mathbb{P}^n_Y \to \mathbb{P}^n_S \times_S \mathbb{P}^m_S$. For this purpose, we refer to the following diagram, where $W = \mathbb{P}^n_S \times_S \mathbb{P}^m_S$ and σ is the product morphism $g \times \mathrm{id}$.



The existence of γ is guaranteed because the square $p_S a = \varepsilon b$ is a fibered product, indeed σ and $i_g p_Y$ make a commutative diagram with ε and p_S. It remains to prove that γ is a closed immersion, and this follows by checking that the diagram $i_g p_Y = a\gamma$ is a fibered product.

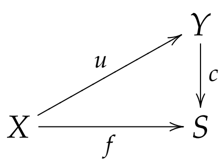
To prove (c) let $f\colon Y \to S$ be a projective morphism, and let $g\colon X \to S$ be a base extension. We refer to the following diagram, where the cube is the diagram of definition of the product morphism $\beta\colon \mathbb{P}^n_X \to \mathbb{P}^n_S$ and on the left there is the diagram of the fibered product of f and g:



The two morphisms p_1 and $ucp_2\colon X \times_S Y \to \mathbb{P}^n_{\mathbb{Z}}$ make a commutative diagram with $j\mathrm{id}$ and ig, therefore there exists a unique h as above. It remains to prove that h is a closed immersion, and this follows by checking that the diagram $\beta h = cp_2$ is a fibered product. □

The statement of this Proposition may appear to be quite abstract, but if you try and translate it into English you will discover the following: products of projective varieties are projective varieties, a variety which is projective over $\mathbb{R}$ will be also projective when considered over $\mathbb{C}$, a morphism between projective varieties is always projective (in particular proper). The same result, but under more restrictive hypotheses, holds for quasi-projective morphisms, however we need a little bit of extra work to be able to see it.

6.1.3 Immersions A morphism of schemes $f\colon X \to S$ is an *immersion* if it factors into an open immersion followed by a closed immersion.



In other words if there exists a scheme Y endowed with a closed immersion $c\colon Y \to S$ and an open immersion $u\colon X \to Y$ such that $f = cu$.

According to the very definition a morphism of schemes $f\colon X \to Y$ is quasi-projective if it factors into an immersion $i\colon X \to \mathbb{P}^n_Y$ for some n, followed by the projection $\mathbb{P}^n_Y \to Y$, and it is projective when the immersion i is closed. In particular immersions are quasi-projective morphisms.

Lemma. *Immersions are stable under base extension.*

Proof. Let $f\colon X \to S$ be an immersion, and let $f = cu$ its decomposition as above. Let also $g\colon W \to S$ be any morphism of schemes. We refer to the following diagram, where $W' = W \times_S Y$ and $W'' = (W \times_S Y) \times_Y X$.

$$\begin{array}{ccccc} W'' & \xrightarrow{r_1} & W' & \xrightarrow{p_1} & W \\ \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{u} & Y & \xrightarrow{c} & S \end{array}$$

The reader can convince himself that the commutative square $cur_2 = gp_1r_1$ is the fibered product diagram for $W \times_S X$. Now p_1r_1 is an immersion because closed and open immersions are stable under base extension. □

Unfortunately in this generality it is not true that the composition of two immersions is an immersion, we need more hypotheses as in the following result.

Proposition (Exercise III.2.3 in Liu, 2002)**.**

i) *Let $f\colon X \to S$ be an immersion. Then f factors into a closed immersion followed by an open immersion.*

ii) *Let $q\colon X \to S$ be a closed immersion and $v\colon S \to Y$ be an open immersion, and assume that the composition $f = vq$ is quasi-compact* (*see* §3.1.3)*. Then f is an immersion, that is it factors into an open immersion followed by a closed immersion.*

iii) *Let $f\colon X \to Y$ and $g\colon Y \to S$ be two immersions, with g quasi-compact. Then the composition gf is an immersion.*

Proof. The first statement is little more than a trivial remark, indeed we can factor f as cu where c is a closed immersion and u is an open immersion. Therefore $\mathrm{sp}(X)$ is an open subset of some scheme Y, and since c is a homeomorphism there exists an open subset $U \subseteq S$ such that $c^{-1}(U) = \mathrm{sp}(X)$. Now $f = v \circ c|^U$, where v is the inclusion of U.

Statement *ii*) is a consequence of the Theorem in §4.1.2, indeed we can factor f through its scheme-theoretic image Z, and the dominant morphism $g\colon X \to Z$ will be an open immersion. To see this, with reference to the second diagram in the statement of the Theorem, just take $U = S$ and observe that in this case $Z' = X$.

To prove part *iii*) observe first that we can factor $f = c_1u_1$ and $g = c_2u_2$, where c_i is a closed immersion and u_i is an open immersion, therefore we have

$gf = c_2(u_2c_1)u_1$. If we assume u_2 is quasi-compact we can use statement $ii)$ to conclude that u_2c_1 is an immersion, that is it factors as c_3u_3, and we obtain the factorisation $gf = c_2c_3u_3u_1$ which proves gf is an immersion. The reader can check that quasi-compact morphisms are closed under composition, and that if the composition cu of an open immersion followed by a closed immersion is quasi-compact, then u is quasi-compact. □

6.1.4 Quasi-Projective Morphisms We are now ready to prove that, when we restrict ourselves to well-behaved categories, quasi-projective morphisms satisfy all the good properties we expect.

Proposition (Exercise III.3.20 in Liu, 2002)**.** *Let S be any scheme. In the category of separated schemes of finite type over S, quasi-projective morphisms satisfy the following properties.*

(a) *An immersion is a quasi-projective morphism;*

(b) *The composition of two quasi-projective morphisms is quasi-projective;*

(c) *Quasi-projective morphisms are stable under base extension;*

(d) *The product of two quasi-projective morphisms is quasi-projective;*

(e) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two quasi-projective morphisms, if $g \circ f$ is quasi-projective and g is separated, then f is quasi-projective.*

Proof. Observe that in the category of separated schemes of finite type over S every morphism is quasi-compact. Now the proof goes on precisely as the one in §6.1.2, only we will have to replace "projective" with "quasi-projective" and "closed immersion" with "immersion." □

Lemma. *Let k be a field and let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} R$ be affine algebraic schemes. Then any morphism $f\colon X \to Y$ as schemes over k is quasi-projective.*

Proof. The morphism f is of finite type, therefore it is induced by a ring homomorphism $R \to A$ which makes A into a finitely generated algebra over R. Thus f factors into an immersion $i\colon X \to \mathbb{A}^n_R$ for some n, followed by the projection $\mathbb{A}^n_R \to \operatorname{Spec} R$, and we have reduced to prove that the latter is quasi-projective, which is obvious. □

Once again these statements may appear to be quite abstract, but in fact they show that products of quasi-projective varieties are quasi-projective varieties, and morphisms between quasi-projective varieties are always quasi-projective. The Lemma in particular implies that every affine variety is quasi-projective.

6.2 Projective Varieties

6.2.1 Sheaves of Ideals By definition an algebraic variety is projective if it comes endowed with a closed immersion into projective space $\mathbb{P}^n_k$, we are therefore interested at first in the following set up: X is a scheme and $c\colon Z \to X$ is a closed immersion. The machinery of quasi-coherent sheaves allows us to study the geometry of Z in terms of the geometry of X, for instance when X is affine Z is given by the spectrum of the quotient ring $A/\mathfrak{a}$ and we can read all the geometry of Z out of this A-algebra. More generally we will be able to identify Z with a sheaf of $\mathscr{O}_X$-algebras, but first we need to introduce the ideal sheaf of a closed immersion.

Definition. Let Z be a closed subscheme of a scheme X, and let $c\colon Z \to X$ be the corresponding closed immersion. We define the *ideal sheaf* of Z, denoted $\mathscr{I}_Z$, to be the kernel of the morphism $c^\#\colon \mathscr{O}_X \to c_*\mathscr{O}_Z$.

Proposition (II.5.9 in Hartshorne, 1977)**.** *Let X be a scheme. For any closed subscheme Z of X, the corresponding ideal sheaf $\mathscr{I}_Z$ is a quasi-coherent sheaf of ideals on X. If X is Noetherian, it is coherent. Conversely, any quasi-coherent sheaf of ideals on X is the ideal sheaf of a uniquely determined closed subscheme Z.*

Proof. Since a closed immersion is quasi-compact and separated (it is indeed finite), we can conclude using Proposition II.5.8 in Hartshorne (1977) that $c_*\mathscr{O}_Z$ is a quasi-coherent sheaf. Therefore $\mathscr{I}_Z$, being the kernel of a morphism of quasi-coherent sheaves, is quasi-coherent. If X is Noetherian then it is coherent, since for any open affine subscheme $U = \operatorname{Spec} A$ of X the ring A is Noetherian, and therefore the ideal $\mathscr{I}_Z(U)$ is finitely generated.

Conversely, given any quasi-coherent sheaf of ideals $\mathscr{I}$ on X, we need to define a scheme $(Z, \mathscr{O}_Z)$ together with a closed immersion $c\colon Z \to X$ such that the kernel of the surjective morphism $c^\#\colon \mathscr{O}_X \to c_*\mathscr{O}_Z$ be $\mathscr{I}$. Observe that the quotient sheaf $\mathscr{O}_X/\mathscr{I}$ is a quasi-coherent sheaf of $\mathscr{O}_X$-algebras and define Z to be $\mathbf{Spec}\, \mathscr{O}_X/\mathscr{I}$ (see §5.3.8). □

An immediate consequence of the Proposition above is the following: for any closed immersion $c\colon Z \to X$ we can identify the closed subscheme Z with the quasi-coherent sheaf of $\mathscr{O}_X$-algebras $\mathscr{O}_X/\mathscr{I}_Z$. Incidentally observe that from this point of view we can study the relative position of "closed subspaces" inside the "ambient space" X as follows. We say that the closed subscheme Z *contains* the closed subscheme Y if Y is in turn a closed subscheme of Z, that is if $\mathscr{I}_Y \supseteq \mathscr{I}_Z$. The *union* $Y \cup Z$ is defined as $\mathbf{Spec}\, \mathscr{O}_X/(\mathscr{I}_Z \cap \mathscr{I}_Y)$, and their *intersection* $Y \cap Z$ as $\mathbf{Spec}\, \mathscr{O}_X/(\mathscr{I}_Z + \mathscr{I}_Y)$. It is important to note that the notions of containment, intersection, and union do *not* satisfy all the usual properties of their set-theoretical counterparts, although they are consistent with them.

In particular when Z is any closed subscheme of X we may want to study the geometry of Z relatively to the given immersion $c\colon Z \to X$. The following result clarifies how to do it, characterising quasi-coherent sheaves over Z in terms of quasi-coherent sheaves over X.

Lemma. *Let X be a scheme and let $c\colon Z \to X$ be a closed immersion. Then the pull-back functor c^* and the push-forward c_* give an isomorphism of categories between quasi-coherent sheaves of $\mathscr{O}_Z$-modules over Z and quasi-coherent sheaves of $\mathscr{O}_X/\mathscr{I}_Z$-modules over X. If X is Noetherian the same is true with coherent sheaves.*

Proof. We can apply Proposition II.5.8 in Hartshorne (1977) and conclude that c^* and c_* actually are functors between quasi-coherent sheaves. Then, since c is an affine morphism we can reduce to prove the statement in the affine case only, and the conclusion is then given by Proposition II.5.2 in Hartshorne (1977). □

Definition. Under the identification of the previous Lemma the pull-back of any quasi-coherent sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ will be given by $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathscr{I}_Z$ which will be rewritten as $\mathscr{F} \otimes \mathscr{O}_Z$ and called the *restriction* of $\mathscr{F}$ to Z.

6.2.2 Graded Modules and Quasi-coherent Sheaves Let $I \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal and let Z be the projective scheme $\operatorname{Proj} k[x_0, \dots, x_n]/I$. Then Z is a closed subscheme of projective space $\mathbb{P}^n_k$ and as we have seen above it defines a unique quasi-coherent sheaf of ideals $\mathscr{I}_Z$. In what follows we describe how to construct the sheaf $\mathscr{I}_Z$ starting from the ideal I.

Let S be a graded ring. A *graded S-module* is an S-module M, together with a decomposition $M = \bigoplus_{d\in\mathbb{Z}} M_d$ of M into a direct sum of abelian groups M_d, such that for every $d, e \in \mathbb{Z}$, with $d \geq 0$, $S_d M_e \subseteq M_{d+e}$. In analogy with the affine case a graded S-module M defines a quasi-coherent sheaf $\mathrm{Shf}_h M$ over the scheme $X = \operatorname{Proj} S$.

Proposition (V.1.17 in Liu, 2002)**.** *With the notations above, there exists a unique quasi-coherent $\mathscr{O}_X$-module $\mathrm{Shf}_h M$ such that*

i) for any homogeneous $\alpha \in S_+$, the restriction $\mathrm{Shf}_h M|_{D_h(\alpha)}$ is the quasi-coherent sheaf $\mathrm{Shf} M_{(\alpha)}$ on $D_h(\alpha) = \operatorname{Spec} S_{(\alpha)}$;

ii) for any $\mathfrak{p} \in \operatorname{Proj} S$, the stalk $(\mathrm{Shf}_h M)_{\mathfrak{p}}$ is isomorphic to $M_{(\mathfrak{p})}$.

Proof. There are many different ways of constructing $\mathrm{Shf}_h M$, all essentially equivalent. Observe for instance that property i) above characterises a sheaf as in §1.3, provided that the gluing condition is satisfied. This is the direct approach in Liu (2002) where another one is also shown: we can pull-back the sheaf $\mathrm{Shf} M$ over the canonical injection $\mathrm{Proj}\, S \to \mathrm{Spec}\, S$. In Hartshorne (1977) we find instead the usual very geometric way of visualising sheaves, by means of functions $s\colon U \to \coprod_{\mathfrak{p}\in U} M_{(\mathfrak{p})}$. □

Corollary. *The association $M \mapsto \mathrm{Shf}_h M$ defines an exact functor from the category of graded S-modules, with morphisms preserving degrees, to the category of quasi-coherent sheaves over* $\mathrm{Proj}\, S$.

Proof. We already know that $M \mapsto \mathrm{Shf}\, M$ defines an exact functor. The pull-back operation is right-exact because it is a left adjoint functor between abelian categories (see Mac Lane, 1998, Theorem X.1.2), so we can conclude that Shf_h is a right exact functor. It remains to prove that it preserves injective morphisms, but if $\varphi\colon M \to N$ is injective then the homogeneous localisation $\varphi_{(\alpha)}$ is just the restriction of the localisation $\varphi_\alpha\colon M_\alpha \to N_\alpha$, and therefore is injective. □

Despite the analogy with the affine case when M is a graded S-module many good results no longer apply. For instance it is obvious that any homogeneous element of degree zero $m \in M_0$ defines a global section of the sheaf $\mathrm{Shf}_h M$ (observe indeed that it is contained in every localisation $M_{(\alpha)}$) but the same is not true for every element of M. In fact the sheaf $\mathrm{Shf}_h M$ doesn't determine the module M: for instance assume that S is generated by S_1 as an S_0-algebra, in this situation $\mathrm{Proj}\, S$ is covered by the open affine subsets $D_h(\alpha)$ where α runs through the elements of S of degree one, and the two localisations $M_{(\alpha)}$ and $N_{(\alpha)}$ are equal. Note the analogy with §2.2.5.

Lemma (Exercise II.3.12 in Liu, 2002)**.** *Let S be a Noetherian graded ring. Then* $\mathrm{Proj}\, S$ *is a Noetherian scheme and every finitely generated graded module over S defines a coherent sheaf.*

Proof. If S is Noetherian then the maximal ideal S_+ is finitely generated, say by $\alpha_1, \dots, \alpha_r$. Therefore applying the results in §2.2.3 we conclude that $\mathrm{Proj}\, S = \bigcup_{i=1}^r D_h(\alpha_i)$. To complete the proof it is enough to show that $S_{(\alpha)}$ is a Noetherian ring for any homogeneous $\alpha \in S_+$.

Let I be any ideal inside $S_{(\alpha)}$ and consider I^e in S_α. The ring S_α is a Noetherian ring because S is, and the set of all the elements of I is a system of generators for I^e. We want to prove that I^e is actually finitely generated.

So we have reduced to prove the following: A is a Noetherian ring, $\mathfrak{a}$ is an ideal, and $S \subseteq \mathfrak{a}$ is a system of generators for $\mathfrak{a}$, then there exists a finite part of S that generates $\mathfrak{a}$. Clearly if S is finite there is nothing to prove. Otherwise consider the collection of ideals in A generated by a finite part of S, which must have a maximal element $\mathfrak{a}'$ because A is Noetherian. Then $\mathfrak{a}'$ is finitely generated by a subset of S and is therefore contained in $\mathfrak{a}$. If $S \subseteq \mathfrak{a}'$ then equality holds otherwise we reach a contradiction. □

6.2.3 Twisted Sheaves Let S be a graded ring and let $X = \operatorname{Proj} S$. For any graded S-module M, and for any $n \in \mathbb{Z}$, we define the *twisted module* $M(n)$ by shifting the degree in M as follows: $M(n)_d = M_{n+d}$. We define the sheaf $\mathscr{O}_X(n)$ to be the quasi-coherent sheaf $\mathrm{Shf}_h S(n)$. We call $\mathscr{O}_X(1)$ the *twisting sheaf* of Serre. For any sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ we denote by $\mathscr{F}(n)$ the *twisted sheaf* $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$.

Example. Let S be the polynomial ring $A[x_0, \dots, x_n]$ and let X be the projective space $\mathbb{P}^n_A$. Then the sheaf $\mathscr{O}_X(1)$ is a locally free sheaf of rank one, indeed for any $i = 0, \dots, n$ the restriction $\mathscr{O}_X(1)|_{D_h(x_i)}$ is given by the module of the homogeneous elements of degree one in S_{x_i}, and multiplication by $1/x_i$ defines an isomorphism between this module and $S_{(x_i)}$. On the intersection $D_h(x_i x_j)$ the gluing condition is therefore given by multiplication with x_i/x_j.

Proposition (II.5.12 in Hartshorne, 1977)**.** *Let S be a graded ring and let $X =$* $\operatorname{Proj} S$*. Assume that S is generated by S_1 as an S_0-algebra. Then*

(a) *The sheaf $\mathscr{O}_X(n)$ is an invertible sheaf on X;*

(b) *For graded S-modules M and N, we have* $\mathrm{Shf}_h M \otimes_S N = \mathrm{Shf}_h M \otimes_{\mathscr{O}_X} \mathrm{Shf}_h N$.

(c) *For any graded S-module M, we have* $(\mathrm{Shf}_h M)(n) = \mathrm{Shf}_h\big(M(n)\big)$*, in particular we have the identity* $\mathscr{O}_X(n) \otimes \mathscr{O}_X(m) = \mathscr{O}_X(n+m)$.

Proof. The same argument as in the example above proves (a). Indeed for any homogeneous element α of degree one multiplication by $1/\alpha^n$ gives an isomorphism from $\mathscr{O}_X(n)|_{D_h(\alpha)}$ and $S_{(\alpha)}$. For statement (b) we need to observe the following: the tensor product of graded modules $M \otimes_S N$ is a graded S-module, where for homogeneous $s \in M_i$ and $t \in N_j$ we set the degree of $s \otimes t$ to $d = i + j$. Since the degree of α is one, we can write

$$\frac{s \otimes t}{\alpha^d} = \frac{s}{\alpha^i} \otimes \frac{t}{\alpha^j}$$

defining in this way an isomorphism $(M \otimes_S N)_{(\alpha)} = M_{(\alpha)} \otimes_{S_{(\alpha)}} N_{(\alpha)}$. Statement (c) follows by the previous ones. □

Observe that according to this Proposition the *twisting functor*, defined over the category of $\mathscr{O}_X$-modules by $\mathscr{F} \mapsto \mathscr{F} \otimes \mathscr{O}_X(n)$, is exact. Indeed on stalks this corresponds to taking tensor product with a free module, which is in particular flat.

6.2.4 Projective Varieties Going back for a moment to the beginning of our discussion, let X be a projective variety, that is a variety endowed with a closed immersion $X \to \mathbb{P}^n_k$. We have seen that X is uniquely defined by a quasi-coherent sheaf of ideals $\mathscr{I}_X$ over $\mathbb{P}^n_k$. We are now going to prove that there exists a homogeneous ideal $I \subseteq k[x_0, \dots, x_n]$ such that not only $\mathscr{I}_X = \operatorname{Shf}_h I$ but also $X = \operatorname{Proj} k[x_0, \dots, x_n]/I$.

First we want to prove that any quasi-coherent sheaf over $\mathbb{P}^n_k$ arises as the sheaf associated to some graded module over $k[x_0, \dots, x_n]$, more generally we investigate the situation for an arbitrary graded ring S. We need a functor going in the opposite direction, that is associating a graded S-module to any quasi-coherent sheaf.

Definition. Let S be a graded ring and let $X = \operatorname{Proj} S$. Assume that S is generated by S_1 as an S_0-algebra, and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We define the *graded S-module associated to $\mathscr{F}$* as follows. We consider the group

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma\big(X, \mathscr{F}(n)\big)$$

Any homogeneous element $\alpha \in S_d$ determines a global section of the sheaf $\mathscr{O}_X(d)$ and we have seen above that $\mathscr{F}(n) \otimes \mathscr{O}_X(d) \cong \mathscr{F}(n+d)$. We define the product $\alpha \cdot t$ of a global section t with an homogeneous $\alpha \in S_d$ to be the tensor product $\alpha \otimes t$.

Example. Let A be a ring, S be the polynomial ring $A[x_0, \dots, x_n]$ and X be the n-dimensional projective space $\mathbb{P}^n_A$. Any homogeneous $\alpha \in S_d$ is an element of degree zero in $S(d)$, therefore it defines a global section of $\mathscr{O}_X(d)$. If we now take $\alpha \in S_d$ and $\beta \in S_r$ the product we have defined over $\Gamma_*(\mathscr{O}_X)$ is just the usual product $\alpha\beta$ over S. In fact much more is true, according to Hartshorne (1977, Proposition II.5.13) there is an isomorphism $\Gamma_*(\mathscr{O}_X) = S$.

The next Theorem is the cornerstone of Projective Geometry, it allows us to investigate properties of quasi-coherent sheaves over $\operatorname{Proj} S$ by studying graded S-modules. Namely any quasi-coherent sheaf over $\operatorname{Proj} S$ is the sheaf associated to some graded S-module.

Theorem (Proposition II.5.15 in Hartshorne, 1977)**.** *Let S be a graded ring, which is finitely generated by S_1 as an S_0-algebra. Let $X = \operatorname{Proj} S$, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then there is a natural isomorphism $\beta\colon \operatorname{Shf}_h \Gamma_*(\mathscr{F}) \to \mathscr{F}$.*

Let A be a ring. With this result understood we can now characterise projective schemes over A, namely a scheme over A is projective if and only if it is the projective spectrum of some graded ring S with $S_0 = A$.

Proposition (V.1.30 in Liu, 2002)**.** *Let A be a ring. A scheme X over* $\operatorname{Spec} A$ *is projective if and only if it is isomorphic to* $\operatorname{Proj} S/I$*, where* $S = A[x_0, \dots, x_n]$ *and I is a homogeneous ideal of S contained in S_+.*

Proof. This is just a corollary of the previous Theorem, the complete argument can be found also in Hartshorne (1977, Corollary II.5.16). The point is to prove that $\Gamma_*(\mathscr{I}_X)$ is a homogeneous ideal of S, and this follows once we realise that Γ_* is a left-exact functor. □

6.2.5 Saturated Ideals Let A be a ring. We describe next how to decide when two different homogeneous ideals in $A[x_0, \dots, x_n]$ give rise to the same closed subscheme. Quite surprisingly this construction turns out to be also computational, however we are not going to give any details about algorithms referring the interested reader to Cox, Little, and O'Shea (1997).

Definition. Let A be a ring, let $S = A[x_0, \dots, x_n]$ and let α be a homogeneous element in S. For any homogeneous ideal I in S, we define the *saturation* $I : \alpha^\infty$ of I with respect to α to be the homogeneous ideal

$$I : \alpha^\infty = \left\{ s \in S \,\middle|\, \alpha^m s \in I \text{ for some } m > 0 \right\}$$

We define the *saturation* $\overline{I}$ of I to be the intersection $\overline{I} = \bigcap_{i=0}^n (I : x_i^\infty)$. We say that I is *saturated* if $I = \overline{I}$.

Proposition (Exercise II.5.10 in Hartshorne, 1977)**.** *Let A be a ring and let S be the polynomial ring $A[x_0, \dots, x_n]$.*

(*a*) *Two homogeneous ideals I and J of S define the same subscheme of $\mathbb{P}^n_A$ if and only if they have the same saturation;*

(*b*) *If Z is any closed subscheme of $\mathbb{P}^n_A$, then the ideal $\Gamma_*(\mathscr{I}_Z)$ is saturated;*

(*c*) *Saturated ideals of S are in one-to-one correspondence with closed subschemes of projective space $\mathbb{P}^n_A$.*

Proof. Let $I \subseteq S$ be homogeneous. Observe first that for any $i = 0, \dots, n$ we have $I_{(x_i)} = \overline{I}_{(x_i)}$; indeed since $I \subseteq \overline{I}$ one inclusion is obvious while the other follows immediately from the definition. Hence $\mathrm{Shf}_h I = \mathrm{Shf}_h \overline{I}$ for any homogeneous ideal I of S. Two homogeneous ideals I and J define the same closed subscheme if and only if they define the same sheaf of ideals (see §6.2.1 above), thus we have proved the "if" part of (*a*). Now let $\mathrm{Shf}_h I = \mathrm{Shf}_h J$. For any homogeneous element $\lambda \in \overline{I}$ of degree d, the fraction λ / x_i^d is an element of $\overline{J}_{(x_i)}$, hence there exists some homogeneous $\varepsilon \in \overline{J}$ of degree t such that

$\lambda/x_i^d = \varepsilon/x_i^t$. This is equivalent to the relation $x_i^{d+q}\varepsilon = x_i^{t+q}\lambda$ in S, which implies $\lambda \in \overline{J}$. Viceversa for every ε there exists λ and a similar relation, which implies $\varepsilon \in \overline{I}$.

To prove (b) it is enough to show that $\Gamma_*(\mathrm{Shf}_h I) = \overline{I}$, furthermore this will prove (c) also. Now the result will follow easily if we realise that a global section of $(\mathrm{Shf}_h I)(d) = \mathrm{Shf}_h\big(I(d)\big)$ is given by an homogeneous element $\lambda \in S_d$ such that the fraction λ/x_i^d is an element of $I_{(x_i)}$ for every $i = 0, \ldots, n$. $\square$

6.3 Very Ample Invertible Sheaves

6.3.1 Sheaves Generated by Global Sections Let $(X, \mathscr{O}_X)$ be a ringed space. We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is *generated by global sections* if there is a family of global sections $\{s_i\}_{i\in I} \subseteq \Gamma(X, \mathscr{F})$, such that for each $x \in X$, the images of s_i in the stalk $\mathscr{F}_x$ generate that stalk as an $\mathscr{O}_{X,x}$-module.

The obvious example of a sheaf generated by global sections is the free $\mathscr{O}_X$-module over a set I. But more generally we have the following characterisation.

Proposition. *An $\mathscr{O}_X$-module $\mathscr{F}$ is generated by global sections if and only if it can be written as a quotient of a free sheaf.*

Proof. If $\varphi\colon \mathscr{O}_X^{(I)} \to \mathscr{F}$ is a surjective morphism of sheaves, then $\mathscr{F}$ is generated by the global sections $\big\{\varphi_X(e_i)\big\}$ where $\{e_i\}$ is the canonical basis for $\Gamma(X, \mathscr{O}_X)^{(I)}$. Viceversa if $\mathscr{F}$ is generated by the global sections $\{s_i\}$ the association $\varphi_X(e_i) = s_i$ extends to a unique morphism of sheaves. $\square$

Examples. Any quasi-coherent sheaf on an affine scheme is generated by global sections. Indeed, if $\mathscr{F} = \mathrm{Shf}\, M$ on $\mathrm{Spec}\, A$, any set of generators for M as an A-module will do.

Let $S = A[x_0, \ldots, x_n]$ and let X be the projective space $\mathbb{P}^n_A$. Then $x_0, \ldots, x_n$ give global sections of $\mathscr{O}_X(1)$ which generate it. Indeed for each $i = 0, \ldots, n$ the restriction of $\mathscr{O}_X(1)$ to the open affine subsets $D_h(x_i)$ is (by definition) the sheaf associated to $S(1)_{(x_i)}$ which is generated by x_i over the localised ring $S_{(x_i)} = A[x_0/x_i, \ldots, x_n/x_i]$.

Example. Here is a less trivial example, showing that it actually happens to have a sheaf generated by a set of global sections which is not a system of generators for the module of global sections. Let X be the conic in projective plane defined by the equation $z^2 = xy$, that is

$$X = \mathrm{Proj}\, k[x, y, z]/(xy - z^2)$$

Then the sheaf $\mathscr{O}_X(1)$ is generated by the global sections $\overline{x}, \overline{y}$, while $\overline{z}$ defines a different global section not contained in the span of $\overline{x}$ and $\overline{y}$ over $\Gamma(X, \mathscr{O}_X)$. To see this, we first investigate the local structure of the conic by means of the distinguished open affine covering $X = D_h(\overline{x}) \cup D_h(\overline{y}) \cup D_h(\overline{z})$, in the following way

$$\begin{aligned} D_h(\overline{z}) &= \operatorname{Spec} k\left[\frac{x}{z}, \frac{y}{z}\right] \Big/ \left(1 - \frac{xy}{z^2}\right) = \operatorname{Spec} k[\, t, t^{-1}\,] \\ D_h(\overline{x}) &= \operatorname{Spec} k\left[\frac{y}{x}, \frac{z}{x}\right] \Big/ \left(\frac{z^2}{x^2} - \frac{y}{x}\right) = \operatorname{Spec} k[\, t^{-1}\,] \\ D_h(\overline{y}) &= \operatorname{Spec} k\left[\frac{x}{y}, \frac{z}{y}\right] \Big/ \left(\frac{z^2}{y^2} - \frac{x}{y}\right) = \operatorname{Spec} k[\, t\,] \end{aligned}$$

It is immediate to observe that $D_h(\overline{z}) \subseteq D_h(\overline{x}) \cap D_h(\overline{y})$, and that X is isomorphic to $\mathbb{P}^1_k$. Besides we know from §2.3.5 that $\Gamma(X, \mathscr{O}_X) = k$. In the same fashion as above the restriction of $\mathscr{O}_X(1)$ to $D_h(\overline{x})$ is generated by $\overline{x}$ and the restriction over $D_h(\overline{y})$ is generated by $\overline{y}$. This is enough to conclude that $\overline{x}$ and $\overline{y}$ generate the sheaf. If now we assume $\overline{z}$ to be contained in the span of $\overline{x}$ and $\overline{y}$ we reach the contradiction $\overline{z} = \alpha\overline{x} + \beta\overline{y}$ with $\alpha, \beta \in k$.

6.3.2 Morphisms to Projective Space We are now interested in studying morphisms of a given scheme to projective space. More precisely let A be a fixed ring and let X be any scheme over A. We want to characterise in intrinsic terms all the morphisms from X to the projective space $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$. We will use the machinery of quasi-coherent sheaves, in particular the twisting sheaf $\mathscr{O}(1)$ over $\mathbb{P}^n_A$ will play a central role. First observe that if $f\colon X \to \mathbb{P}^n_A$ is a morphism of A-schemes, then $\mathscr{L} = f^*\mathscr{O}(1)$ is an invertible sheaf on X (see §5.2.3). We also have that $\mathscr{O}(1)$ is generated by the global sections $x_0, \dots, x_n$, and this implies that $\mathscr{L}$ is generated by $n+1$ global sections $s_0, \dots, s_n$. If this is not completely obvious the following result will prove it.

Lemma. *Let $f\colon X \to Y$ be a morphism of ringed spaces. Let $\mathscr{G}$ be a sheaf of $\mathscr{O}_Y$-modules generated by a finite number of global sections. Then $f^*\mathscr{G}$ is generated by the same number of global sections.*

Proof. According to Mac Lane (1998, Theorem X.1.2) a left adjoint functor between abelian categories is right exact, therefore f^* is right exact. Now we have an exact sequence $\mathscr{O}_Y^n \to \mathscr{G} \to 0$ from which we obtain $\mathscr{O}_X^n \to f^*\mathscr{G} \to 0$, recalling that $f^*\mathscr{O}_Y^n = \mathscr{O}_X^n$. □

If $x_0, \dots, x_n$ are the global sections of $\mathscr{G}$ that generate it, we denote $f^*(x_i)$ the image of e_i under the homomorphism $\Gamma(X, \mathscr{O}_X)^n \to \Gamma(X, f^*\mathscr{G})$ determined by f^*, where $e_0, \dots, e_n$ is the canonical basis for $\Gamma(X, \mathscr{O}_X)^n$. With these notations the sheaf $\mathscr{L}$ above is generated by the global sections $s_i = f^*(x_i)$. This associates an invertible sheaf on X generated by a finite number of global sections to any morphism of A-schemes from X to $\mathbb{P}^n_A$. The converse is also true and we can establish a one-to-one correspondence.

Theorem (II.7.1 in Hartshorne, 1977)**.** *Let A be a ring and let X be an A-scheme.*

- (a) *If $f\colon X \to \mathbb{P}^n_A$ is an A-morphism, then $f^*\mathscr{O}(1)$ is an invertible sheaf on X, which is generated by the global sections $s_i = f^*(x_i)$, $i = 0, \dots, n$.*
- (b) *Conversely, if $\mathscr{L}$ is an invertible sheaf on X, generated by the global sections $s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$, then there exists a unique A-morphism $f\colon X \to \mathbb{P}^n_A$ such that $\mathscr{L} \cong f^*\mathscr{O}(1)$ and $s_i = f^*(x_i)$.*

Given an invertible sheaf $\mathscr{L}$ over X, for any global section $s \in \Gamma(X, \mathscr{L})$ the subset $X_s = \{x \in X \mid s_x \notin \mathfrak{m}_x\mathscr{L}_x\}$ is open. To see this we restrict to the affine case as follows. Let A be a ring and let M be an A-module which we assume to be isomorphic to A. Then there exists an element $m \in M$ such that the homomorphism $\varphi\colon A \to M$ defined by $\varphi(1) = m$ is an isomorphism. A global section of the sheaf $\operatorname{Shf} M$ over $\operatorname{Spec} A$ is just an element $s = am$ of M, and we claim that $X_s = D(a)$. Indeed $D(a)$ clearly contains X_s, because whenever $a \in \mathfrak{p}$ then $s \in \mathfrak{p}M_{\mathfrak{p}}$, and conversely if $s \in \mathfrak{p}M_{\mathfrak{p}}$ then there exists a fraction $b/c \in \mathfrak{p}A_{\mathfrak{p}}$ and an element $t \notin \mathfrak{p}$ such that $mt(ac - b) = 0$, from which follows $atc = bt \in \mathfrak{p}$ and eventually $a \in \mathfrak{p}$.

With this understood, the construction of the morphism $f\colon X \to \mathbb{P}^n_A$ starting from an invertible sheaf $\mathscr{L}$ goes as follows. For any $i = 0, \dots, n$ we consider the open set $X_i = X_{s_i}$ and, covering $\mathbb{P}^n_A = \operatorname{Proj} k[x_0, \dots, x_n]$ with the usual affine patches, define a morphism of schemes $f_i\colon X_i \to D_h(x_i)$. This will be given by a homomorphism of rings $\varphi_i\colon k[x_0/x_i, \dots, x_n/x_i] \to \Gamma(X_i, \mathscr{O}_{X_i})$ that we set to be $\varphi(x_j/x_i) = s_j/s_i$. The fraction s_j/s_i it's a compact notation, the meaning should be clear from what we have seen above. It is the section of $\mathscr{O}_X(X_i)$ given on any open affine subset U of X_i by the fraction a_j/a_i where $s_j|_U = a_j m$, $s_i|_U = a_i m$ and m is the generator of the module $\mathscr{L}(U)$.

Observe that in this way we construct a morphism of schemes from the open subscheme of X given by the union $X_0 \cup \dots \cup X_n$ to $\mathbb{P}^n_A$. The hypothesis that the sections $s_0, \dots, s_n$ generate $\mathscr{L}$ is needed to ensure that the union above be the whole of X.

6.3.3 Very Ample Invertible Sheaves

Keeping in mind that the word projective has a relative meaning, we fix a base S and by definition a scheme over S is projective if it is endowed with a closed immersion c in $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$. We can now compone with the projection $p\colon \mathbb{P}^n_S \to \mathbb{P}^n_{\mathbb{Z}}$ to obtain a morphism from

X to $\mathbb{P}^n_{\mathbb{Z}}$, and using the result above this is uniquely determined by the invertible sheaf $c^*\big(p^*\mathscr{O}_{\mathbb{Z}}(1)\big)$. Therefore for any scheme S, we define the *twisting sheaf* $\mathscr{O}(1)$ *on* $\mathbb{P}^n_S$ to be $p^*\mathscr{O}_{\mathbb{Z}}(1)$.

Definition. If X is any scheme over S, an invertible sheaf $\mathscr{L}$ on X is *very ample* relative to S, if there is an immersion (§6.1.3) of S-schemes $i\colon X \to \mathbb{P}^r_S$ for some r, such that $i^*\mathscr{O}(1) \cong \mathscr{L}$.

Note in particular that a very ample invertible sheaf relative to S is generated by a finite number of global sections. This is clear when S is affine using the theorem above, while the general case reduces to this one by composition with p. Observe also that a scheme X is quasi-projective over S if and only if there exists a very ample sheaf on X relative to S; a similar characterisation in terms of very ample invertible sheaves holds for projective schemes.

Claim (Remark II.5.16.1 in Hartshorne, 1977)**.** *Let S be a Noetherian scheme. Then a scheme X over S is projective if and only if it is proper, and there exists a very ample sheaf on X relative to S.*

Proof. we have seen in §6.1.1 that if X is projective over S then it is proper and there exists a closed immersion $i\colon X \to \mathbb{P}^r_S$, therefore $i^*\mathscr{O}(1)$ is a very ample invertible sheaf relative to S. Conversely if $\mathscr{L}$ is a very ample invertible sheaf relative to S, there exists an immersion $i\colon X \to \mathbb{P}^r_S$ such that $i^*\mathscr{O}(1) \cong \mathscr{L}$. Since X is proper over S and the projection morphism $\mathbb{P}^n_S \to S$ is in particular separated we can conclude that i is proper.

So the proof will be complete if we show that a proper immersion is necessarily a closed immersion. Since every immersion can be written as the composition of an open immersion followed by a closed immersion, we can reduce to prove that a proper open immersion is necessarily a closed immersion. So let $u\colon X \to Y$ be a proper open immersion; since it is open it induces an isomorphism of X with an open subscheme of Y, say $(U, \mathscr{O}_Y|_U)$, and since it is proper U must be also a closed subset of Y. In particular u induces an homeomorphism onto the closed subset U. Finally keeping in mind the topological configuration, with both U and its complement open subsets, we can conclude that the morphism of sheaves $u^\#\colon \mathscr{O}_Y \to u_*\mathscr{O}_X$ is surjective. Indeed every point of Y has an open neighborhood over which this morphism is either zero or an isomorphism. □

Let A be a ring and let X be a projective scheme over A. Looking back at the characterisation given at the end of §6.2.4 we can realise X as $\operatorname{Proj} S/I$ where S is the polynomial ring $A[x_0, \dots, x_n]$ and I is a homogeneous ideal contained in S_+. In this situation the closed immersion of X in $\mathbb{P}^n_A$, which we call i, is induced by the surjective homomorphism of rings $S \to S/I$, and the invertible sheaf $i^*\mathscr{O}(1)$ coincides with the sheaf $\mathscr{O}_X(1)$ defined as $\mathrm{Shf}_h(S/I)(1)$. We see therefore that *choosing a very ample invertible sheaf over X is equivalent to fixing a representation of X as the projective spectrum of a graded ring.*

Proposition (Exercise II.5.12 in Hartshorne, 1977)**.** *Fix a base scheme S and let X be an S-scheme. If $\mathscr{L}$ and $\mathscr{M}$ are two very ample invertible sheaves on X relative to S then so is their tensor product $\mathscr{L} \otimes \mathscr{M}$.*

Proof. Since we have two immersions, the one defined by $\mathscr{L}$ that we call $\ell\colon X \to \mathbb{P}^n_S$, and the one defined by $\mathscr{M}$ that we call $m\colon X \to \mathbb{P}^r_S$, we can define their product to get a unique morphism $i\colon X \to \mathbb{P}^n_S \times \mathbb{P}^r_S$. To see that this is an immersion it is enough to check separately open and closed immersions with a standard argument, moreover by the Segre embedding we obtain in this way an immersion to a projective space. Observe that we are claiming here that the composition of two immersions is an immersion, this is true provided that the external one is also quasi-compact (see §6.1.3) and in this case it is because the Segre embedding, call it σ, is a closed immersion.

It remains to prove that $i^*\left(\sigma^*\mathscr{O}(1)\right)$ is actually the tensor product $\mathscr{L} \otimes \mathscr{M}$, and to this purpose we must first compute the pull-back $\sigma^*\mathscr{O}(1)$. As with any argument of this kind we can assume S to be affine, say $S = \operatorname{Spec} A$. So we can define $R = A[x_0, \dots, x_r]$ and $T = A[y_0, \dots, y_n]$, and the product $\mathbb{P}^n_A \times \mathbb{P}^r_A$ will be given by $\operatorname{Proj} P$ where P is the ring $\bigoplus_{d \geq 0}(R_d \otimes_A T_d)$. In this graded ring $\mathscr{O}(1)$ is generated by the products $x_0y_0, \dots, x_iy_j, \dots, x_ry_n$, and products of global sections like these generate the sheaf $\mathscr{L} \otimes \mathscr{M}$. □

Lemma (Exercise II.5.12 in Hartshorne, 1977)**.** *Let $f\colon X \to Y$ and $g\colon Y \to S$ be two morphisms of schemes, and assume g to be of finite type. Let $\mathscr{L}$ be a very ample invertible sheaf on X relative to Y, and let $\mathscr{M}$ be a very ample invertible sheaf on X relative to S. Then $\mathscr{L} \otimes f^*\mathscr{M}$ is a a very ample invertible sheaf on X relative to S.*

Proof. We have already encountered this situation in §6.1.2, in showing that the composition of two projective morphisms is projective. With reference to the diagrams in there γ is an immersion because it is obtained by base extension from i_g, moreover since g is of finite type i_g is also of finite type, therefore γ is of finite type. In particular γ is quasi-compact so that the composition γi_f is an immersion. Now we conclude using the Segre embedding. □

6.3.4 Veronese Embedding Let X be a projective scheme over a ring A, more precisely let $X = \operatorname{Proj} R/I$ where $R = A[x_0, \dots, x_n]$ and $I \subseteq R_+$ is a homogeneous ideal. Then we have seen that the sheaf $\mathscr{O}_X(1)$ is very ample, generated by the global sections $x_0, \dots, x_n$, and defines the closed immersion of X in $\mathbb{P}^n_A$ given by the surjective homomorphism $R \to R/I$. In this situation we also know that for any integer $d > 0$ the invertible sheaf $\mathscr{O}_X(d)$ is very ample; we are going to describe now how to construct the closed immersion it defines.

Proposition (Exercise II.5.13 in Hartshorne, 1977). *Let S be a graded ring, generated by S_1 as an S_0-algebra. For any integer $d > 0$ consider the subring of S*

$$dS = \bigoplus_{n \geq 0} S_{nd}$$

Then, if $X = \operatorname{Proj} S$, we have $\operatorname{Proj} dS \cong X$.

Proof. The reader may convince himself that the inclusion $dS \to S$ defines a morphism of schemes $f\colon X \to \operatorname{Proj} dS$ in the same fashion as in §2.2.5. In our hypotheses S_1 generates the ideal S_+, therefore we can conclude that dS_1 generates the ideal dS_+. Hence as in §2.2.3 the homogeneous elements $\alpha \in S_d$ define an open affine covering of both X and $\operatorname{Proj} dS$, let us call $X_{(\alpha)}$ the open subset of X and $D_h(\alpha)$ the open subset of $\operatorname{Proj} dS$. Then both $X_{(\alpha)}$ and $D_h(\alpha)$ are defined to be the spectrum of $S_{(\alpha)}$ and f is an isomorphism. □

When $S = R/I$ as above, a set of generators $s_0, \dots, s_N$ for the A-module S_d defines a set of global sections for the sheaf $\mathscr{O}_X(d)$ which generates it, so we can define a homomorphism of graded rings $\varphi\colon A[y_0, \dots, y_N] \to dS$ which is surjective and therefore defines a closed immersion of X in $\mathbb{P}^N_A$, called *Veronese embedding*. Clearly X will be isomorphic to the projective spectrum of $A[y_0, \dots, y_N]/\ker\varphi$, and over this ring the sheaf $\mathscr{O}_X(1)$ corresponds to the sheaf $\mathscr{O}_X(d)$ over R/I.

For example the closed immersion associated with the invertible sheaf $\mathscr{O}(d)$ over $\mathbb{P}^n_A$ will be defined as follows. Let G be a set of indeterminates indexed over those $(n+1)$-tuples of positive integers $(i_0, \dots, i_n)$ such that $\sum i_j = d$, and define a homomorphism of graded rings $A[G] \to R$ sending $y_{i_0 \dots i_n}$ to the product $x^{i_0} \dots x^{i_n}$. This closed immersion embeds $\mathbb{P}^n_A$ in $\mathbb{P}^N_A$, where $N = \binom{n+d}{d} - 1$.

Example. Let X be the conic in the projective plane $\mathbb{P}^2_k = \operatorname{Proj} k[x, y, z]$ defined by the equation $z^2 = xy$, that is

$$X = \operatorname{Proj} k[x, y, z]/(xy - z^2)$$

We want to construct the closed immersion defined by the very ample invertible sheaf $\mathscr{O}_X(2)$. First we observe that $\mathscr{O}_X(2)$ is generated by the global sections $\overline{x}^2, \overline{xy}, \overline{xz}, \overline{y}^2$ and $\overline{z}^2$. Then we define a homomorphism

$$\varphi\colon k[y_0, \dots, y_4] \longrightarrow k[x, y, z]/(xy - z^2)$$

by setting $\varphi(y_0) = \overline{x}^2$, $\varphi(y_1) = \overline{xy}$, $\varphi(y_2) = \overline{xz}$, $\varphi(y_3) = \overline{y}^2$ and $\varphi(y_4) = \overline{z}^2$. By the Proposition above this is actually a closed immersion, because it is surjective on the subring $2S$ of $S = k[x,y,z]/(xy - z^2)$. Direct computation shows that the kernel of φ is the ideal generated by the six quadratic forms

$$\begin{array}{lll} y_0y_2 - y_1y_4, & y_2^2 - y_0y_4, & y_1y_2 - y_3y_4, \\ y_0y_1 - y_2y_3, & y_1^2 - y_0y_3, & y_0^2 - y_3y_4 \end{array}$$

In this new incarnation of the conic as Proj T, where T is the graded algebra $k[y_0,\dots,y_4]/\ker\varphi$, we immediately recognise the very ample invertible sheaf generated by the global sections $\overline{y_0},\dots,\overline{y_4}$. Clearly this is the same sheaf that before we were calling $\mathscr{O}_X(2)$, so apparently we have lost track of the very ample invertible sheaf $\mathscr{O}_X(1)$. We claim that the graded T-module M (in fact an ideal in T) generated by $\overline{y_0}$, $\overline{y_3}$ and $\overline{y_4}$ defines $\mathscr{O}_X(1)$. To see that M generates an invertible sheaf, observe that for each i the homogeneous localisation $M_{(y_i)} = T_{(y_i)}$ (see above, the two relations in the middle). Now we check that $\mathrm{Shf}_h\, M$ is in fact $\mathscr{O}_X(1)$ by proving that it defines the same closed immersion in $\mathbb{P}^2_A$; start with the homomorphism

$$\psi\colon k[x,y,z] \longrightarrow k[y_0,\dots,y_4]/\ker\varphi$$

defined by $\psi(x) = \overline{y_0}$, $\psi(y) = \overline{y_3}$ and $\psi(z) = \overline{y_4}$. Direct computation shows that the kernel of ψ is precisely $(x^2 - yz)$, so that the image of ψ is a graded algebra whose Proj is X, and ψ induces a closed immersion of X in $\mathbb{P}^2_A$.

All computations in this example were preformed with Macaulay 2, a computer algebra system developed by Grayson and Stillman (2001).

6.3.5 Coherent Sheaves and Finitely Generated Graded Modules Our aim is now to prove that, under reasonable hypotheses, every coherent sheaf on a projective scheme $X = \mathrm{Proj}\, S$ is given by the sheaf $\mathrm{Shf}_h M$ associated to a finitely generated graded S-module M. We will need to this purpose the following result.

Serre's Theorem (II.5.17 in Hartshorne, 1977)**.** *Let X be a projective scheme over a Noetherian ring A, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then there is an integer n_0 such that, for all $n \geq n_0$, the sheaf $\mathscr{F}(n)$ can be generated by a finite number of global sections.*

With this understood we are able to prove our main result. In fact we are just going to spell out clearly something that in Hartshorne (1977) is explained between the lines of the proof of Theorem II.5.19.

Proposition. *Let $X = \operatorname{Proj} S$ be a projective scheme over a Noetherian ring A, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then $\mathscr{F} = \mathrm{Shf}_h M$ for some finitely generated graded S-module M.*

Proof. We set notations as follows, R will be the polynomial ring $A[x_0, \ldots, x_n]$ and $I \subseteq R$ will be a homogeneous ideal. The projective scheme X will be defined as $\operatorname{Proj} S$ where $S = R/I$. We consider the graded S-module (which is not necessarily finitely generated) $M = \Gamma_*(\mathscr{F})$, so that $\mathscr{F} = \mathrm{Shf}_h M$. By Serre's Theorem for n sufficiently large $\mathscr{F}(n)$ is generated by a finite number of global sections in $\Gamma\big(X, \mathscr{F}(n)\big) = M_n$, so we define M' to be the submodule generated by these global sections. Observe that $M'_d = 0$ for any $d < n$. Now we have an inclusion of modules which gives rise to a left-exact sequence of sheaves $0 \to \mathrm{Shf}_h M' \to \mathrm{Shf}_h M$, and twisting by n (which is an exact operation) we obtain the left-exact sequence $0 \to \mathrm{Shf}_h M'(n) \to \mathscr{F}(n)$. This is actually an isomorphism because $\mathscr{F}(n)$ is generated by elements of $M'_n = M'(n)_0$. So we can twist by $-n$ to obtain $0 \to \mathrm{Shf}_h M' \to \mathscr{F} \to 0$. □

Corollary (II.5.18 in Hartshorne, 1977)**.** *Let X be a projective scheme over a Noetherian ring A. Then any coherent sheaf $\mathscr{F}$ on X can be written as a quotient of a sheaf $\mathscr{E}$, where $\mathscr{E}$ is a finite direct sum of twisted structure sheaves $\mathscr{O}_X(-d_i)$ for various integers d_i.*

Proof. With notations and definitions as in the proof of the Proposition above, let $\mathscr{F} = \mathrm{Shf}_h M$ for some finitely generated graded S-module M. Let $m_1, \ldots, m_r$ be a system of homogeneous generators for M of degrees $d_1, \ldots, d_r$ respectively and let E be the finitely generated graded S-module

$$E = \bigoplus_{i=1}^{r} S(-d_i)$$

Then clearly we have a surjective homomorphism from E to M which induces a surjective morphism from $\mathrm{Shf}_h E$ to $\mathscr{F}$. It remains to observe that $\mathrm{Shf}_h E$ is precisely a finite direct sum of twisted structure sheaves as required. □

We conclude this section with the actual statement of Theorem II.5.19 in Hartshorne (1977). This is in fact a direct proof that when X is a projective variety then $H^0(X, \mathscr{F})$ is a finite dimensional vector space; as such it is a good example of the power of cohomology, using which Serre (1955) proves a much more general result concerning $H^n(X, \mathscr{F})$ for every n.

Theorem (II.5.19 in Hartshorne, 1977)**.** *Let k be a field, let A be a finitely generated k-algebra, let X be a projective scheme over A, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then $\Gamma(X, \mathscr{F})$ is a finitely generated A-module. In particular, if $A = k$, then $\Gamma(X, \mathscr{F})$ is a finite dimensional k-vector space.*

6.4 Global Proj and Blow-ups

6.4.1 Blowing up the Plane We start with a classic example, we blow up the affine plane $\mathbb{A}^2_k$ at the origin $(0,0)$. Let $R = k[x,y]$ be the coordinate ring of the affine plane; in the product $\mathbb{A}^2_k \times \mathbb{P}^1_k = \operatorname{Proj} R[t_0, t_1]$ we consider the closed subscheme X defined by the ideal $(xt_1 - yt_0)$. We have a natural morphism $\pi\colon X \to \mathbb{A}^2_k$ obtained by restricting the projection map of $\mathbb{A}^2_k \times \mathbb{P}^1_k$ onto the first factor. Observe that there exists an open affine covering of X consisting of two elements, $X = D_h(t_0) \cup D_h(t_1)$ each of which is isomorphic to $\mathbb{A}^2_k$.

$$D_h(t_0) = \operatorname{Spec} R[\,t_1\,]/(xt_1 - y) \cong \operatorname{Spec} k[x, t_1]$$
$$D_h(t_1) = \operatorname{Spec} R[\,t_0\,]/(x - yt_0) \cong \operatorname{Spec} k[y, t_0]$$

These two affine schemes are glued together by the isomorphism

$$k[x, x^{-1}, t_1, t_1^{-1}] \to k[y, y^{-1}, t_0, t_0^{-1}]$$

defined by $x \mapsto yt_0$ and $t_1 \mapsto t_0^{-1}$. The scheme X is the *Blow-up of the affine plane at the origin* and usually it is denoted $\operatorname{Bl}_O(\mathbb{A}^2_k)$. We will now study the properties of X.

Let O be the origin of the affine plane, that is the point $(0,0)$, and let U be the open set $\mathbb{A}^2_k \setminus \{O\}$. Then π restricted to the open set U is an isomorphism. Indeed π is given over the open subset $D_h(t_0)$ by the homomorphism $y \mapsto xt_1$, therefore the open set $V_1 = \mathbb{A}^2_k \setminus \mathcal{V}(x)$ is isomorphic via π to the distinguished open subset of $D_h(t_0)$ defined by x. Similarly $V_2 = \mathbb{A}^2_k \setminus \mathcal{V}(y)$ is isomorphic via π to the distinguished open subset of $D_h(t_1)$ defined by y. Now observe that $U = V_1 \cup V_2$.

The fiber of π over the closed point O is isomorphic to $\mathbb{P}^1_k$. This is immediate once we recall that $\pi^{-1}(O)$ is given by the fibered product of X with $\operatorname{Spec} R/(x,y)$, so that the open covering above will define an open covering of $\pi^{-1}(O)$ consisting of two affine lines glued together to form $\mathbb{P}^1_k$. Observe that this closed subscheme of X is given on each of the affine pieces above by a single equation.

6.4.2 The Inverse Image of a Quasi-Coherent Sheaf Let X be a scheme and let $Z \subseteq X$ be a closed subscheme. If $f\colon W \to X$ is any morphism, we have learned in §3.2.4 that the inverse image $f^{-1}(Z)$, which is defined as the fibered product $W \times_X Z$, is a closed subscheme of W. But now we know that closed subschemes correspond one-to-one to quasi-coherent sheaves of ideals (see §6.2.1), so we would like to characterise $f^{-1}(Z)$ in terms of quasi-coherent sheaves. Starting with $\mathscr{I}_Z$ a natural candidate could be the pull-back $f^*\mathscr{I}_Z$, but we are going to see now that in general this is different from $\mathscr{I}_{f^{-1}(Z)}$.

As usual it is useful to look first at the affine case, thus we assume f to be induced by the ring homomorphism $\varphi\colon A \to B$ and $\mathscr{I}_Z$ to be the ideal $\mathfrak{a} \subseteq A$. In this case $f^{-1}(Z)$ is defined to be the spectrum of the tensor product $B \otimes_A A/\mathfrak{a} \cong B/\mathfrak{a}^e$, where $\mathfrak{a}^e$ is the ideal of B generated by the image of $\mathfrak{a}$ under φ. It follows that $f^{-1}(Z)$ corresponds to the quasi-coherent sheaf $\operatorname{Shf}\mathfrak{a}^e$, in other words $\mathscr{I}_{f^{-1}(Z)} = \operatorname{Shf}\mathfrak{a}^e$.

The pull back $f^*\mathscr{I}_Z$ is defined to be the product $B \otimes_A \mathfrak{a}$ (see §5.3.1), but in general this is different from the ideal $\mathfrak{a}^e$. Indeed we can apply the functor $B \otimes_A -$ to the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$, and in general we obtain only a right-exact sequence $B \otimes_A \mathfrak{a} \to B \to B/\mathfrak{a}^e \to 0$. In terms of sheaves this corresponds to apply the pull-back functor f^* to the exact sequence $0 \to \mathscr{I}_Y \to \mathscr{O}_X \to \mathscr{O}_X/\mathscr{I}_Z \to 0$ to obtain the right-exact sequence $f^*\mathscr{I}_Z \to \mathscr{O}_W \to f^*\big(\mathscr{O}_X/\mathscr{I}_Z\big) \to 0$, which can thus be rewritten as

$$f^*\mathscr{I}_Z \longrightarrow \mathscr{O}_W \longrightarrow \mathscr{O}_W/\mathscr{I}_Z\mathscr{O}_W \longrightarrow 0$$

where $\mathscr{I}_Z\mathscr{O}_W$ denotes (somewhat improperly) the image of $f^*\mathscr{I}_Z$ in $\mathscr{O}_W$. This argument shows that when all our schemes are affine we have the equality $\mathscr{I}_Z\mathscr{O}_W = \mathscr{I}_{f^{-1}(Z)}$, but now it is an easy exercise to extend this result to the general case.

Example (In which $f^*\mathscr{I}_Y$ is different from $\mathscr{I}_Z\mathscr{O}_W$). We start with the short exact sequence $0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$ and we apply $\mathbb{Z}_6 \otimes_{\mathbb{Z}} -$ to obtain the right-exact sequence

$$M \xrightarrow{\varphi} \mathbb{Z}_6 \longrightarrow \mathbb{Z}_6/2\mathbb{Z}_6 \longrightarrow 0$$

where M is the group generated by $\overline{1} \otimes 2$. Observe that $3(\overline{1} \otimes 2) = \overline{3} \otimes 2$ is different from zero in M, while $\varphi(\overline{3} \otimes 2) = \overline{6} = 0$. You may wish to apply the equational characterisation in Eisenbud (1995, Lemma 6.4).

6.4.3 General Definition of Blow-up "We will use these observations as starting points in generalising the definition of a blow-up to that of an arbitrary scheme along an arbitrary subscheme. The essential fact is that, in the blow-up $\pi\colon \operatorname{Bl}_Y(X) \to X$ of a scheme X along the subscheme $Y \subseteq X$ the inverse image of Y is locally principal."

taken from Eisenbud and Harris (2000, §IV.2.1)

Let X be any scheme and $Y \subseteq X$ a closed subscheme. We say that Y is a *Cartier subscheme* in X if it is locally the zero locus of a single non-zero-divisor, or equivalently if the corresponding sheaf of ideals $\mathscr{I}_Y$ is an invertible sheaf. To see the equivalence observe that a principal ideal (a) inside a ring A is generated by a non-zero divisor if and only if the homomorphism $A \to (a)$, sending 1 to a, is an isomorphism of A-modules.

Lemma (IV.19 in Eisenbud and Harris, 2000)**.** *Let X be any scheme and $Y \subseteq X$ a Cartier subscheme. Then $X \setminus Y$ is dense in X.*

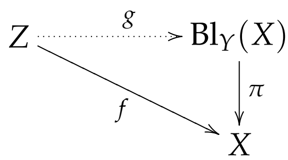
Proof. We may assume that X is affine, say $X = \operatorname{Spec} A$, and that $Y = \mathcal{V}(\alpha)$ for some $\alpha \in A$. Then $X \setminus Y = D(\alpha)$ is affine too, and the open immersion is induced by the localisation homomorphism $j\colon A \to A_\alpha$. Since α is not a zero-divisor in A this homomorphism is injective, and therefore the open immersion is dominant as in §1.4.1. □

Blow-up is a very classical topic, therefore very well understood and widespread in the literature. Here we will follow the account in Eisenbud and Harris (2000), which seems to be the most approachable one from the scheme-theoretic point of view. So before proving the existence of the blow-up, we give a general and abstract definition in terms of the properties we want it to satisfy.

Definition. Let X be a Noetherian scheme and $Y \subseteq X$ a closed subscheme. The *blow-up of X along Y*, denoted $\pi\colon \mathrm{Bl}_Y(X) \to X$, is the morphism to X characterised by the following properties.

(a) the inverse image of Y is a Cartier subscheme in $\mathrm{Bl}_Y(X)$;

(b) the morphism π is universal with respect to this property; that is, if $f\colon Z \to X$ is any morphism such that $f^{-1}(Y)$ is a Cartier subscheme of Z, then there exists a unique morphism $g\colon Z \to \mathrm{Bl}_Y(X)$ factoring f.



The inverse image $E = \pi^{-1}(Y)$ of Y is called the *exceptional divisor* of the blow-up, and Y the *center* of the blow-up.

In this degree of generality, without knowing yet if blow-ups exist at all, we can prove some of their most important properties. The reason for doing this is that proofs are much cleaner, and consequently the results stand out more clearly.

Proposition. *The blow-up is an isomorphism away from its center.*

Proof. Let $i\colon X \setminus Y \to X$ be the open immersion and observe that $i^{-1}(Y)$ is a Cartier subscheme. Indeed since topologically $i^{-1}(Y) = \varnothing$ we can say immediately that $\mathscr{I}_{i^{-1}(Y)} = \mathscr{O}_X|_{X \setminus Y}$. So by the universal property of the blow-up there exists a unique morphism $\varphi\colon X \setminus Y \to \mathrm{Bl}_Y(X)$ such that $\pi\varphi = i$.

Let us consider next the restriction of π to the open set $X \setminus Y$, in other words we consider the following diagram where the external square is a fibered product and p_1 is an open immersion

$$\begin{array}{ccc} \pi^{-1}(X \setminus Y) & \xrightarrow{p_1} & \mathrm{Bl}_Y(X) \\ {\scriptstyle p_2}\downarrow & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle \pi} \\ X \setminus Y & \xrightarrow{i} & X \end{array}$$

We claim that this diagram is commutative. Clearly we have only to prove that $\varphi p_2 = p_1$, and for this we are going to use again the universal property of the blow-up. Indeed the inverse image of Y under ip_2 is a Cartier subscheme of $\pi^{-1}(X \setminus Y)$, So there exists a unique morphism $\pi^{-1}(X \setminus Y) \to \mathrm{Bl}_Y(X)$ which makes a commutative diagram with π and ip_2, and since this happens for both p_1 and φp_2 they must be equal.

We said that the external square in the diagram above is a fibered product, so we can do the following construction. We have the commutative square $\pi\varphi = i\mathrm{id}_{X \setminus Y}$ so there exists a unique morphism $\xi\colon X \setminus Y \to \pi^{-1}(X \setminus Y)$ such that $p_1\xi = \varphi$ and $p_2\xi = \mathrm{id}_{X \setminus Y}$. Observe that the restriction of φ to the open set $\pi^{-1}(X \setminus Y)$ also commutes like that, so $\xi = \varphi|^{\pi^{-1}(X \setminus Y)}$. Now we have

$$\begin{aligned} p_1(\xi p_2) &= \varphi p_2 = p_1 = p_1 \mathrm{id}_{\pi^{-1}(X \setminus Y)} \\ p_2(\xi p_2) &= (p_2\xi) p_2 = p_2 = p_2 \mathrm{id}_{\pi^{-1}(X \setminus Y)} \end{aligned}$$

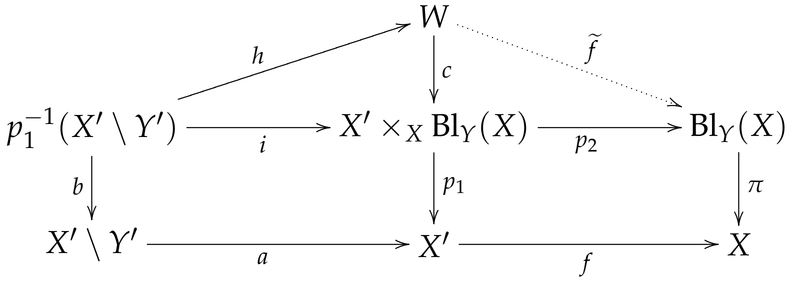
from which we see that p_2 and ξ are inverse to each other, or in other words that φ is an open immersion. So π is an isomorphism away from Y. □

Probably the most important consequence of the definition is the following result, dealing with the behaviour of blow-ups under fibered products (or pull-backs), that will also allow us to define the *strict transform* of Y. In Hartshorne (1977) it is presented as Corollary II.7.15 but a more precise statement can be found in (Eisenbud and Harris, 2000, Proposition IV-21).

Theorem. *Let X be a Noetherian scheme, Y a closed subscheme, and $\pi\colon \mathrm{Bl}_Y(X) \to X$ the blow-up of X along Y. Let $f\colon X' \to X$ be any morphism of Noetherian schemes and set $Y' = f^{-1}(Y)$. If W is the scheme theoretic image of the open immersion*

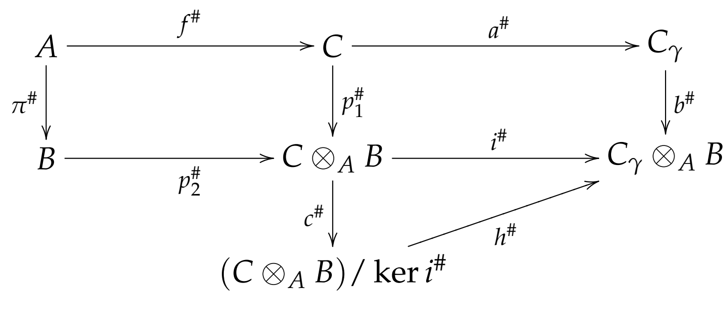
$p_1^{-1}(X' \setminus Y') \subseteq X' \times_X \mathrm{Bl}_Y(X)$, *then the restriction* $p_1 \colon W \to X'$ *is the blow-up of* X' *along* Y'.

Proof. The statement is better understood if we draw a diagram, starting with the fibered product of X' with $\mathrm{Bl}_Y(X)$ and completing with the construction of the scheme theoretic image of $p_1^{-1}(X' \setminus Y')$



The first remark is that a is a morphism of finite type, because it is an open immersion of Noetherian schemes, therefore i is a morphism of finite type. In particular i is quasi-compact and we can construct its scheme theoretic image W as in §4.1.2.

Next we need to prove that $c^{-1}\big(p_1^{-1}(Y')\big)$ is a Cartier subscheme of W. The question is local so let $X = \operatorname{Spec} A$ be affine and consider affine open subsets $\operatorname{Spec} C$ of X' and $\operatorname{Spec} B$ of $\mathrm{Bl}_Y(X)$. The closed subscheme Y will be given by an ideal I in A, and Y' will be the ideal generated by $f^\#(I)$ inside C, which we call J. Moreover we are assuming that $\pi^\#(I)$ generates a principal ideal (β) in B where β is not a zero divisor. To complete the picture we take an open basic subset contained in $X' \setminus Y'$, which will be the spectrum of a localisation C_γ where γ is in fact an element of J.



In this situation the ideal generated by $p_1^\#(J)$ is principal, in fact generated by $p_2^\#(b) = 1 \otimes b$. Now we know that $a^\#(J)$ is the unit ideal, therefore $i^\#(1 \otimes b)$ is a unit and we can conclude that the ideal generated by $c^\#\big(p_1^\#(J)\big)$ is the unit ideal.

If $g\colon Z \to X'$ is a morphism of schemes such that $g^{-1}(Y')$ is a Cartier subscheme of Z, then the same is true for the composition fg and by the universal property of the blow-up we have a unique $g_1\colon Z \to \mathrm{Bl}_Y(X)$ such that $\pi g_1 = fg$. But now we can apply the universal property of the fibered product and say that there exists $g_2\colon Z \to X' \times_X \mathrm{Bl}_Y(X)$ such that $p_2 g_2 = g_1$ and $p_1 g_2 = g$. Now it should be clear that, topologically, $g_2^{-1}(W) = Z$: indeed since $g^{-1}(Y')$ is a Cartier subscheme we can say that $g^{-1}(X' \setminus Y')$ is dense in Z and on the other hand we have

$$g^{-1}(X' \setminus Y') = g_2^{-1}\big(p_1^{-1}(X' \setminus Y')\big) \subseteq g_2^{-1}(W)$$

We claim that this is enough to say that there exists a unique morphism of schemes $g_3\colon Z \to W$ such that $cg_3 = g_2$. □

Claim. *Let S be a scheme and $c\colon W \to S$ be a closed immersion. Let $g\colon Z \to S$ be a morphism of schemes and assume that, topologically, $g^{-1}\big(\mathrm{sp}(W)\big) = \mathrm{sp}(Z)$. Then there exists a unique morphism of schemes $r\colon Z \to W$ such that $cr = g$.*

Proof. We know from §3.2.4 that the restriction of g to W is constructed via the fibered product $W \times_S Z$, moreover when $g^{-1}\big(\mathrm{sp}(W)\big) = \mathrm{sp}(Z)$ the first projection is the identity and we can take r to be simply the second projection. Uniqueness then follow by the universal property of fibered products. □

With the reference to the first diagram in the proof of the Theorem, observe that when f is a closed immersion $\widetilde{f}$ is as well a closed immersion. In this situation we call W the *strict transform* of X' under the blow-up, and usually we denote it by $\widetilde{X'}$.

6.4.4 Sheaves of Graded Algebras and Global Proj In order to prove the existence of the blow-up we need to introduce the general construction of global Proj. This is very similar to the construction of global Spec we have seen in §5.3.8, in fact there is the same sort of analogy as between affine and projective schemes.

Let X be a Noetherian scheme. A quasi-coherent sheaf $\mathscr{S}$ over X is a *sheaf of graded $\mathscr{O}_X$-algebras* if it is a sheaf of rings and there exists a decomposition $\mathscr{S} = \bigoplus_{d \geq 0} \mathscr{S}_d$ into a direct sum of sheaves of $\mathscr{O}_X$-modules, compatible with the ring structure. For simplicity we will always assume furthermore that $\mathscr{S}_0 = \mathscr{O}_X$, that $\mathscr{S}_1$ is a coherent $\mathscr{O}_X$-module, and that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathscr{O}_X$-algebra.

In order to better understand the definition we can look first at the affine case. Let $X = \operatorname{Spec} A$ be affine, then $\mathscr{S} = \operatorname{Shf} S$ where S is the A-algebra of the global sections $\Gamma(X, \mathscr{S})$, and the direct sum decomposition defines a structure of graded ring over S. The assumptions above imply in particular that $S_0 = A$ and that S_1 is a finitely generated A-module.

Claim. *Let X be a Noetherian scheme and $\mathscr{S}$ a sheaf of graded $\mathscr{O}_X$-algebras as above. Then we have*

i) $\mathscr{S}$ is generated by $\mathscr{S}_1$ over any open affine subset, and

ii) the sheaf $\mathscr{S}_d$ is coherent for any $d \geq 0$.

Proof. We can clearly assume X is affine, so with notations as above there exist elements $\alpha_1, \ldots, \alpha_t \in A$, which generate the unit ideal, such that the graded structure on the localisations S_{α_i} is given by A_{α_i} in degree zero, $(S_1)_{\alpha_i}$ in degree one, and the rest is generated by these two. We want to prove that S is generated by S_1 as an A-algebra.

Let θ be any homogeneous element of S of degree d. Then for $i = 1, \ldots, t$, the same θ can be written as a polynomial on the generators of S_1 with coefficients in A_{α_i}, that is $\theta = P_i(m_1, \ldots, m_q)/\alpha_i^r$ where $P_i \in A[x_1, \ldots, x_q]$ and $m_1, \ldots, m_q$ is a system of generators for S_1. Then we have an expression of the form $\alpha_i^r \theta - P_i(m_1, \ldots, m_q) = 0$ in S (up to a power of α_i), and with the usual "partition of unity" kind of argument we find a polynomial expression for θ with coefficients in A.

To prove part *ii*) we must show that the graded decomposition of $\mathscr{S}$ is compatible with localisation, that is $(S_\alpha)_d = (S_d)_\alpha$ for every $\alpha \in A$ and $d \geq 2$ (since we already know that this is the case for $d = 0, 1$). By part *i*), for every $\alpha \in A$ the localisation $S_\alpha = \mathscr{S}\big(D(\alpha)\big)$ is generated by its elements of degree one. Then we have

$$(S_\alpha)_d = \operatorname{Sym}^d\big((S_\alpha)_1\big) = \operatorname{Sym}^d\big((S_1)_\alpha\big) = (\operatorname{Sym}^d S_1)_\alpha = (S_d)_\alpha$$

which is precisely what we wanted. □

The graded A-algebra S is therefore isomorphic to a quotient of the polynomial ring $A[x_1, \ldots, x_q]/I$ where I is a homogeneous ideal. We define $\mathbf{Proj}\, \mathscr{S}$ to be the homogeneous spectrum of S, observe that with our assumptions the natural morphism $\pi \colon \operatorname{Proj} S \to \operatorname{Spec} A$ is a projective morphism.

Let now $\alpha \in A$ and consider the open affine subset $D(\alpha) \subseteq X$. We can make the same construction using the algebra $\mathscr{S}|_{D(\alpha)} = \operatorname{Shf} S_\alpha$, in other words we can define a projective scheme over $\operatorname{Spec} A_\alpha$ as $\mathbf{Proj}\, \mathscr{S}|_{D(\alpha)} = \operatorname{Proj} S_\alpha$. The localisation homomorphism $j \colon S \to S_\alpha$ preserves degrees, therefore it defines a morphism $g \colon \mathbf{Proj}\, \mathscr{S}|_{D(\alpha)} \to \mathbf{Proj}\, \mathscr{S}$ as in §2.2.5, which is clearly compatible with the two structure morphisms. We will prove that g is an open immersion, then a standard gluing argument as in §2.3.6 will give the following.

Proposition. *Let X be a Noetherian scheme and $\mathscr{S}$ a sheaf of graded $\mathscr{O}_X$-algebras as above. For any open affine subset $U \subseteq X$ consider the scheme $\operatorname{Proj}\mathscr{S}(U)$ endowed with the natural morphism $\pi_U\colon \operatorname{Proj}\mathscr{S}(U) \to U$. Then this family of schemes glue together into a scheme $\mathbf{Proj}\,\mathscr{S}$ endowed with a natural proper morphism $\pi\colon \mathbf{Proj}\,\mathscr{S} \to X$.*

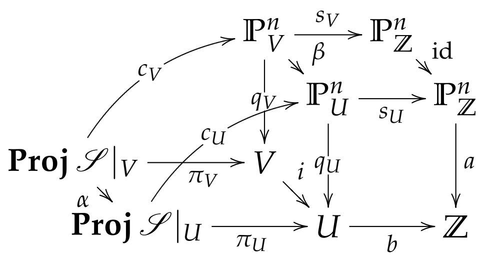
Proof. The condition for a morphism to be an open immersion is *local on the base*, hence it is enough to prove that for any element U of an open covering of $\operatorname{Proj} S$ the restriction $g|^U$ is an open immersion. To this purpose we consider any homogeneous $\beta \in S_+$ and let $U = D_h(\beta) \subseteq \operatorname{Proj} S$. Then the restriction $g|^U$ is a morphism of affine schemes induced by the homogeneous localisation $j_{(\beta)}\colon S_{(\beta)} \to (S_\alpha)_{(\beta)}$. Observe however that we can also form the localisation of $S_{(\beta)}$ in α, and the homomorphism $\xi\colon S_{(\beta)} \to (S_{(\beta)})_\alpha$ induces an open immersion. Now $j_{(\beta)}(\alpha)$ is an invertible element of $(S_\alpha)_{(\beta)}$, thus by the universal property of the localisation (Atiyah and Macdonald, 1969, Proposition 3.1) there exists a unique $\psi\colon (S_{(\beta)})_\alpha \to (S_\alpha)_{(\beta)}$, which turns out to be an isomorphism. This proves the existence of the scheme $\mathbf{Proj}\,\mathscr{S}$ endowed with a natural morphism π. We have already seen that when X is affine π is projective (hence proper), recall now that the condition for a morphism to be proper is local on the base (see §3.3.6). □

Observe the obvious identity $\mathbb{P}^n_X = \mathbf{Proj}\operatorname{Sym}\left(\mathscr{O}_X^{n+1}\right)$. We can also use this remark to see another construction of the product of two projective spaces over X. Indeed we have $\mathbb{P}^n_X \times \mathbb{P}^m_X = \mathbf{Proj}\operatorname{Sym}\left(\mathscr{O}_n^{m+1}\right)$, where $\mathscr{O}_n$ denotes the structure sheaf of $\mathbb{P}^n_X$.

Since Proj is not a functor we don't have any reason to expect **Proj** to be a functor, but it certainly will satisfy some functorial properties derived by the ones satisfied by Proj. The question is too delicate to be answered here, in fact it is somewhat more instructive to keep a case-by-case philosophy.

Lemma. *Let X be a Noetherian scheme, $\mathscr{S}$ a sheaf of graded $\mathscr{O}_X$-algebras as above, and $P = \mathbf{Proj}\,\mathscr{S}$. Then P comes naturally equipped with an invertible sheaf $\mathscr{O}_P(1)$ such that for any open affine subset U of X the restriction $\mathscr{O}_P(1)|_U$ is the twisted sheaf $\mathscr{O}_{P(U)}(1)$, where $P(U)$ is the projective scheme $\mathbf{Proj}\,\mathscr{S}|_U$.*

Proof. We have seen above that when $X = \operatorname{Spec} A$ is affine the scheme P is projective over A and as such it is endowed with the invertible sheaf $\mathscr{O}_P(1)$, which in turn is given by the pull-back $c^*\mathscr{O}(1)$ where $c\colon P \to \mathbb{P}^n_A$ is a closed immersion. In the general case we just need to prove that the schemes $\mathscr{O}_{P(U)}(1)$ glue together. To this purpose consider the following commutative diagram where i is just an inclusion of open affine subsets, and β is the open immersion given both by the construction of **Proj** Sym $\left(\mathscr{O}_U^{n+1}\right)$ and by the product of i with the identity of $\mathbb{P}^n_{\mathbb{Z}}$.



We see from here that the pull-back $c_U^*\mathscr{O}(1)$ commutes with the restriction $\beta^*\mathscr{O}(1)$, in other words $\mathscr{O}_{P(U)}(1)|_V = \mathscr{O}_{P(V)}(1)$. □

6.4.5 General Construction of the Blow-up Let X be a Noetherian scheme and let Y be any closed subscheme of X. If $\mathscr{I}_Y$ is the sheaf of ideals associated to Y, we define a sheaf of graded $\mathscr{O}_X$-algebras by taking as graded parts the powers of $\mathscr{I}_Y$. More precisely we set

$$\mathscr{S} = \bigoplus_{n=0}^{\infty} \mathscr{I}_Y^n = \mathscr{O}_X \oplus \mathscr{I}_Y \oplus \mathscr{I}_Y^2 \oplus \cdots$$

Clearly this sheaf satisfies the hypotheses of §6.4.4 above, so we can consider **Proj** $\mathscr{S}$ which comes naturally endowed with a proper morphism to X. Now we have the following result.

Theorem (IV-23 in Eisenbud and Harris, 2000)**.** *With notations and definitions as above, the proper morphism $\pi\colon$ **Proj** $\mathscr{S} \to X$ is the blow-up of X along Y.*

Proof. The following differs only slightly from the proof in Hartshorne (1977, Propositions II.7.13 and II.7.14). First it is immediate to observe that the sheaf of ideals associated to $\pi^{-1}(Y)$ is just $\mathscr{O}(1)$, therefore it is an invertible sheaf or in other words $\pi^{-1}(Y)$ is a Cartier subscheme. Next we consider the affine case, so $X = \operatorname{Spec} A$ for some Noetherian ring A and $\mathscr{I}_Y = \operatorname{Shf} I$ for some finitely generated ideal $I = (a_1, \ldots, a_n)$ of A. In this situation we can realise the algebra S as a subalgebra of the polynomial ring, more precisely we have $S = A[ta_1, \ldots, ta_n]$ where t is just an indeterminate. There is obviously a surjective homomorphism from the polynomial ring $A[x_1, \ldots, x_n]$ to S, therefore there is a closed immersion $c\colon \operatorname{Proj} S \to \mathbb{P}_A^{n-1}$ compatible with π. Now let Z be any Noetherian scheme and let f be a morphism from Z to X such that $\mathscr{I}_{f^{-1}(Y)}$ is an invertible sheaf of ideals, which we call $\mathscr{L}$. By adjunction f is induced by a homomorphism $\varphi\colon A \to \Gamma(Z, \mathscr{O}_Z)$ and it is easy to see that the global sections $\varphi(a_i)$ generate $\mathscr{L}$, so we have the following commutative diagram where s is the morphism associated to $\mathscr{L}$.

$$\begin{array}{ccc} \operatorname{Proj} S & \xrightarrow{c} & \mathbb{P}^n_A \\ {\scriptstyle g}\uparrow & \nearrow^{s} & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[f]{} & X \end{array}$$

To prove the existence of the morphism g, we restrict the factorisation $s = cg$ to the open basic subset $D_h(x_i)$ of $\mathbb{P}^n_A$ to obtain the diagram

$$\begin{array}{ccc} s^{-1}\big(D_h(x_i)\big) & \xrightarrow{s} & D_h(x_i) \\ {\scriptstyle g}\downarrow & \nearrow_{c} & \\ D_h(ta_i) & & \end{array}$$

Such a restriction will correspond by adjunction to the following commutative diagram of homomorphisms of rings, where U is the open subset of Z defined by $s^{-1}\big(D_h(x_i)\big)$

$$\begin{array}{ccc} A[\frac{x_1}{x_i}, \dots \widehat{i} \dots, \frac{x_n}{x_i}] & \xrightarrow{c^\#} & A_{a_i} \\ & \searrow_{s^\#} & \downarrow{\scriptstyle g^\#} \\ & & \Gamma\big(U, \mathscr{O}_Z|_U\big) \end{array}$$

Now it is clear that there exists a unique $g^\#$ if and only if $\ker c^\# \subseteq \ker s^\#$, but this is obvious by construction. Finally there is to convince ourselves that these morphisms $g^\#$ glue together and that proving the universal property for affine X is enough. □

Appendix A

Algebra

A.1 Rings and Modules of Finite Length

Modules of finite length are a standard topic in commutative algebra, but for some reason they are always treated in great generality. Fulton (1998) uses them to define multiplicity, and from his point of view it is enough to consider finite modules over Noetherian rings. This is a complement to his appendix, my attempt to describe the results he needs. In what follows R will always denote a ring, not necessarily Noetherian.

A.1.1 Composition Series Let M be an R-module. A *normal series* in M is a descending (but not necessarily strictly descending) finite chain of submodules

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = (0)$$

beginning with M and ending with (0); the integer n is called the *length* of the normal series. If all the inclusions are proper the normal series is said to be *without repetitions*. A *refinement* of a normal series is a normal series obtained by inserting additional terms.

Definition. A *composition series* of M is a normal series without repetitions for which every refinement has repetitions.

A normal series without repetitions is a composition series if and only if for each $i = 0, \ldots, n-1$ the quotient M_i / M_{i+1} is *simple*. Where an R-module is said to be simple if it has exactly two submodules, in particular (0) is not a simple module.

Example. Let k be a field and let $R = k[\,x\,]$. The R-module $M = k[\,x\,]/(x^2 - 1)$ has a composition series of length two, namely

$$M \supseteq (\overline{x-1}) \supseteq (0)$$

Indeed the quotient $M/\overline{(x-1)}M$ is isomorphic to k and thus it is simple. While a submodule N such that $(0) \subseteq N \subseteq \overline{(x-1)}$ would correspond to an ideal $\mathfrak{a}$ in $k[\,x\,]$ such that $(x^2-1) \subseteq \mathfrak{a} \subseteq (x-1)$, and since $k[\,x\,]$ is a principal ideal domain $\mathfrak{a}$ must be either (x^2-1) or $(x-1)$.

Slightly more generally for any polynomial $F \in k[\,x\,]$ the module $k[\,x\,]/(F)$ has a composition series of length n, where n is the number of irreducible factors of F.

For a module to have a composition series is a rather special property, observe for instance that $\mathbb{Z}$ or $k[x_1, \ldots, x_n]$ don't have any. Indeed in both cases any finite descending chain of ideals can be extended. But if an R-module M has a composition series, then all of its composition series have the same length. This result can be found in Atiyah and Macdonald (1969, Proposition 6.7), in Eisenbud (1995, Theorem 2.13), or in Zariski and Samuel (1958).

Jordan's Theorem (III.11.19 in Zariski and Samuel, 1958)**.** *If an R-module M has one composition series of length n, then every composition series of M has length n, and every normal series without repetitions can be refined to a composition series.*

The next step is to understand more precisely what properties a module must satisfy in order to have a composition series. Not surprisingly the characterisation is in terms of chain conditions, references are again Atiyah and Macdonald (1969, Proposition 6.8), Eisenbud (1995, Theorem 2.13), or Zariski and Samuel (1958).

Proposition (Theroem III.11.21 in Zariski and Samuel, 1958)**.** *A module M has a composition series if and only if it satisfies both chain conditions.*

A module satisfying both chain conditions is therefore called a *module of finite length*. The common length of all composition series in M will be called the *length* of M and will be denoted $\ell_R(M)$ or simply $\ell(M)$. The simplest examples of modules of finite length are finite groups (groups are modules over $\mathbb{Z}$) and finite dimensional k-vector spaces (where k is any field).

Lemma (A.1.1 in Fulton, 1998)**.** *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, whenever two of the modules have finite length the third also will have finite length. Moreover the length $\ell(M)$ is an additive function on the class of all R-modules of finite length.*

A stronger result than Jordan's Theorem also holds (see Zariski and Samuel, 1958). We say that two normal series (M_i) and (M'_j) of M are equivalent if the set of modules (M_i/M_{i+1}) can be put in bijection with the set of modules (M'_j/M'_{j+1}) so that corresponding quotients are isomorphic over R. The reader can check that this is an equivalence relation, in particular two equivalent normal series have the same length.

Hölder's Theorem (III.11.22 in Zariski and Samuel, 1958)**.** *If an R-module M has one composition series, then any two composition series of M are equivalent.*

A.1.2 Associated Primes We now want to characterise modules of finite length over a Noetherian ring. Before we can state the result we need to recall what an associated prime is and some properties of the set $\mathrm{Ass}(M)$. The main reference for this is Chapter IV in Bourbaki (1998), and what follows is in fact just a rearrangement of it.

Definition. Let M be an R-module. A prime ideal $\mathfrak{p}$ is said to be *associated* with M if there exists an element $m \in M$ such that $\mathfrak{p}$ is equal to the annihilator of m. The set of prime ideals associated with M is denoted by $\mathrm{Ass}_R(M)$ or simply $\mathrm{Ass}(M)$.

To say that a prime ideal $\mathfrak{p}$ is associated with M amounts to saying that M contains a submodule *isomorphic* to $R/\mathfrak{p}$, namely the submodule generated by m. This can also be described as the image of the morphism $R \to M$ given by multiplication by m. Observe that an element $m \in M$ whose annihilator is a prime ideal is necessarily nonzero, because $1 \notin \mathrm{ann}(m)$.

Example. When $R = k[\,x\,]$ and $M = k[\,x\,]/(x^3 + 2x^2 - x - 2)$ the three prime ideals $(x+1)$, $(x-1)$ and $(x+2)$, generated by the linear factors of the polynomial $x^3 + 2x^2 - x - 2$, are associated with M. Indeed

$$(x+1) = \mathrm{ann}(\overline{x^2 + x - 2}),\ (x-1) = \mathrm{ann}(\overline{x^2 + 3x + 2})$$
$$\text{and } (x+2) = \mathrm{ann}(\overline{x^2 - 1}).$$

The submodule generated by $\overline{x^2 + 3x + 2}$ is isomorphic to $k[\,x\,]/(x-1)$. Observe that, although the former is an ideal in $k[\,x\,]/(x^3 + 2x^2 - x - 2)$ and the latter is a ring, this is just an isomorphism of $k[\,x\,]$-modules.

Claim. *Let M be a module over a ring R and let $\mathfrak{S}$ be the set*

$$\mathfrak{S} = \{\, \mathrm{ann}(m) \subseteq R \mid m \in M \setminus \{0\} \,\}$$

Then every maximal element of $\mathfrak{S}$ is a prime ideal, and therefore belongs to $\mathrm{Ass}(M)$.

Proof. Let $\mathfrak{a} = \mathrm{ann}(m)$ be a maximal element of $\mathfrak{S}$. Let b, c be elements of R such that $bc \in \mathfrak{a}$ and assume $c \notin \mathfrak{a}$. Then $cm \neq 0$, $b \in \mathrm{ann}(cm)$ and $\mathfrak{a} \subseteq \mathrm{ann}(cm)$. Since $\mathfrak{a}$ is maximal, $\mathrm{ann}(cm) = \mathfrak{a}$, therefore $b \in \mathfrak{a}$. □

Lemma. *Let M be a module over a Noetherian ring R. Then $M \neq 0$ if and only if* $\mathrm{Ass}(M) \neq \varnothing$.

Proof. If $M = 0$ clearly $\mathrm{Ass}(M)$ is empty (without any hypothesis on R). If $M \neq 0$, then the set $\mathfrak{S}$ is non-empty and consists of proper ideals. Since R is Noetherian, this set has a maximal element . □

Proposition. *Let M be a module over a Noetherian ring R, and let S be a multiplicatively closed subset of R. If Σ is the set of prime ideals of R which do not meet S then* $\mathrm{Ass}_{S^{-1}R}(S^{-1}M)$ *is in bijection with* $\mathrm{Ass}_R(M) \cap \Sigma$ *via the usual map* $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$.

Proof. If $\mathfrak{p} \in \mathrm{Ass}_R(M) \cap \Sigma$, then $\mathfrak{p} = \mathrm{ann}(m)$ for some $m \in M$ and hence $S^{-1}\mathfrak{p} = \mathrm{ann}(m/1)$. Conversely, assume that $S^{-1}\mathfrak{p}$ is the annihilator of m/t in $S^{-1}M$, where $\mathfrak{p}$ is a prime ideal in Σ, m is an element of M and $t \in S$. Since R is Noetherian every ideal is finitely generated, so let $\mathfrak{p} = (a_1, \ldots, a_r)$; then $(a_i/1)(m/t) = 0$ for every i and so there exist elements $s_1, \ldots, s_r \in S$ such that $s_i a_i m = 0$ (for $i = 1, \ldots, r$). We claim that $\mathfrak{p}$ is the annihilator of sm, where s is the product $s_1 s_2 \ldots s_r$. Indeed for any $a \in \mathfrak{p}$, $sam = 0$, and conversely if $b \in R$ satisfies $bsm = 0$, then $b/1$ annihilates m/t so that $b \in \mathfrak{p}$. □

A.1.3 Support of a Module The set of prime ideals $\mathfrak{p}$ of R such that $M_\mathfrak{p} \neq 0$ is called the *support* of M and is denoted by $\mathrm{Supp}(M)$. This is coherent with the language of sheaves, where the support of a sheaf $\mathscr{F}$ over a topological space X is the set of points $x \in X$ such that $\mathscr{F}_x \neq 0$. The analogy is of course due to M giving rise to a quasi-coherent sheaf over $\mathrm{Spec}\, R$, so that the support of the module M is in fact the support of the sheaf $\mathrm{Shf}M$.

Lemma. *Let R be a Noetherian ring, $\mathfrak{p}$ a prime ideal of R and M an R-module. Then* $\mathfrak{p} \in \mathrm{Supp}(M)$ *if and only if* $\mathfrak{p}$ *contains an element of* $\mathrm{Ass}(M)$.

Proof. If $\mathfrak{q}$ is an element of $\mathrm{Ass}(M)$ contained in $\mathfrak{p}$ then $\mathfrak{q} \cap (R \setminus \mathfrak{p}) = \varnothing$ and so it defines an element of $\mathrm{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$ (§A.1.2 above). Then, since the localisation $R_\mathfrak{p}$ is a Noetherian ring, we can conclude that $M_\mathfrak{p} \neq 0$. Conversely, if $M_\mathfrak{p} \neq 0$ then $\mathrm{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq \varnothing$ (see above) and hence there exists $\mathfrak{q} \in \mathrm{Ass}_R(M)$ such that $\mathfrak{q} \cap (R \setminus \mathfrak{p}) = \varnothing$. □

Proposition. *Let R be a Noetherian ring and M a finitely generated R-module. Then* $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$ *and these two sets have the same minimal elements. Moreover* $\mathrm{Supp}(M)$ *coincides with the set of prime ideals containing* $\mathrm{ann}(M)$.

Proof. The first part of the statement is immediate from the previous Proposition. It is also clear that any associated prime contains $\mathrm{ann}(M)$, hence the same is true for any element of $\mathrm{Supp}(M)$. If $\mathfrak{p}$ is any prime ideal then $M_\mathfrak{p} = 0$ if and only if for every $m \in M$ there exists $a_m \in \mathrm{ann}(m) \setminus \mathfrak{p}$, that is $\mathfrak{p} \in \mathrm{Supp}(M)$ if and only if there exists $m \in M$ such that $\mathrm{ann}(m) \subseteq \mathfrak{p}$. Let now $m_1, \ldots, m_r$ be a set of generators for M, so that $\mathrm{ann}(M) = \bigcap \mathrm{ann}(m_i)$; assuming a prime ideal contains $\mathrm{ann}(M)$ we can conclude that it contains one of the annihilators and eventually that it is an element of $\mathrm{Supp}(M)$. □

A.1.4 Normal Series in Noetherian Modules Let R be a Noetherian ring and M a finitely generated R-module. We are interested in normal series inside M, in particular we want to highlight a special class of normal series. The next result is in fact the starting point for Fulton (1998).

Theorem. *Let R be a Noetherian ring and M a finitely generated R-module. There exists a normal series without repetitions in M*

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = (0)$$

such that, for $i = 0, \ldots, n-1$, the quotient M_i/M_{i+1} is isomorphic to $R/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of R. Moreover

$$\mathrm{Ass}(M) \subseteq \{\mathfrak{p}_0, \ldots, \mathfrak{p}_{n-1}\} \subseteq \mathrm{Supp}(M)$$

and the minimal elements of these three sets are the same and coincide with the minimal elements of the set of prime ideals containing $\mathrm{ann}(M)$.

Proof. We can assume the module M to be different from 0, otherwise the statement is trivially true. In order to construct a normal series as above we start with $M \supsetneq (0)$, and take $\mathfrak{p}_1 \in \mathrm{Ass}(M)$ that is $\mathfrak{p}_1 = \mathrm{ann}(m_1)$. If we denote by N_1 the submodule generated by m_1 we have the normal series $M \supseteq N_1 \supsetneq (0)$ with $N_1 \cong R/\mathfrak{p}_1$. The submodule N_1 is either equal to M or strictly contained in it. In the first case we are done, in the second we have $\mathrm{Ass}\,(M/N_1) \neq \varnothing$. So we can take $\mathfrak{p}_2 = \mathrm{ann}(\overline{m_2})$ and let N_2 be the submodule generated by m_1, m_2. We have the normal series $M \supseteq N_2 \supsetneq N_1 \supsetneq (0)$ with $N_2/N_1 \cong R/\mathfrak{p}_2$ and $N_1 \cong R/\mathfrak{p}_1$. Now we can go on; note that $N_1 \subsetneq N_2 \subsetneq \cdots$ is an increasing chain of submodules of M, therefore the process has to end with some N_i being equal to M. In this way the set of elements $m_1, m_2, \ldots$ will be eventually a set of generators for M.

Now that we have the existence of the normal series, in order to prove the second statement, we need a couple of preliminary remarks. First observe that $\mathrm{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$, moreover in this case $\mathfrak{p}$ is the annihilator of every non-zero element of $R/\mathfrak{p}$ (just because the quotient is an integral domain). Next we claim that if N is any submodule of M then

$$\mathrm{Ass}(N) \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N)$$

Indeed the first inclusion is obvious, while for any $\mathfrak{p} \in \mathrm{Ass}(M)$, with $\mathfrak{p} = \mathrm{ann}(m)$, either $Rm \cap N = 0$ or not. In the former case $\overline{m} \neq 0$ in the quotient

M/N and $\mathfrak{p} = \mathrm{ann}(\overline{m})$, in the latter any $y \in Rm \cap N$ is the product $y = am$ with $a \notin \mathfrak{p}$ so that $\mathfrak{p} = \mathrm{ann}(y)$.

Starting from the normal series we have in M and using the tower of inclusions above we can say that $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M/M_1)$. But recalling that $M/M_1 \cong R/\mathfrak{p}_0$ we obtain $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M_1) \cup \{\mathfrak{p}_0\}$. Going on we have the inclusion $\mathrm{Ass}(M) \subseteq \{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\}$.

If N is any submodule of M, and $\mathfrak{p}$ is any prime ideal of R, by localising the short exact sequence $0 \to N \to M \to M/N \to 0$ we derive

$$0 \longrightarrow N_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow (M/N)_{\mathfrak{p}} \longrightarrow 0$$

from which one sees immediately that for $M_{\mathfrak{p}}$ to be reduced to 0 it is necessary and sufficient that $N_{\mathfrak{p}}$ and $(M/N)_{\mathfrak{p}}$ be so. Therefore we have the equality

$$\mathrm{Supp}(M) = \mathrm{Supp}(N) \cap \mathrm{Supp}(M/N)$$

For $i = 0, \dots, n-1$ we have $\mathfrak{p}_i \in \mathrm{Supp}(M_i/M_{i+1})$, because associated primes belong to the support (see above). In particular $\mathfrak{p}_{n-1} \in \mathrm{Supp}(M_{n-1})$ but clearly $\mathrm{Supp}(M_{n-1}) \subseteq \mathrm{Supp}(M)$, moreover using the previous equality we see that for $i = 0, \dots, n-2$

$$\mathfrak{p}_i \in \mathrm{Supp}(M_{i+1}) \subseteq \mathrm{Supp}(M)$$

The last assertion on the minimal elements of the three sets follows immediately from the Proposition in §A.1.3 above. □

Example. Let k be a field and let $R = k[x,y]$. We consider the finitely generated module $M = k[x,y]/(xy)$, which geometrically is the union of the coordinate axes. Then we have

$$\begin{aligned} \mathrm{Ass}(M) &= \big\{(x), (y), (x,y)\big\} \\ \mathrm{Supp}(M) &= \mathcal{V}(xy) = \mathcal{V}(x) \cup \mathcal{V}(y) \end{aligned}$$

In order to find a normal series without repetitions like in the Theorem above we can in the first place just perform the construction in the proof. So we start by taking $(y) = \mathrm{ann}(\overline{x})$, and construct the normal series $M \supseteq \langle \overline{x} \rangle \supseteq (0)$. The quotient $M/\langle \overline{x} \rangle$ is clearly different from zero, so we can go on taking $(x) = \mathrm{ann}(\overline{y})$ and constructing the normal series

$$M \supseteq \langle \overline{x}, \overline{y} \rangle \supseteq \langle \overline{x} \rangle \supseteq (0)$$

Now we are done because $M/\langle \overline{x}, \overline{y} \rangle$ is isomorphic to $R/(x,y)$. In fact, with notations as in the theorem, we have

$$\mathfrak{p}_1 = (x,y), \;\; \mathfrak{p}_2 = (x), \;\; \mathfrak{p}_3 = (y)$$

We can find other normal series in M with the required properties, in fact we can construct normal series as long as we like. The following is an example where the set of primes is strictly bigger than $\mathrm{Ass}(M)$

$$M \supseteq \langle \overline{x}, \overline{y} \rangle \supseteq \langle \overline{x(x-1)}, \overline{y} \rangle \supseteq \langle \overline{y} \rangle \supseteq \langle \overline{y}^2 \rangle \supseteq (0)$$

Here the set of primes is given as follows

$$\mathfrak{p}_1 = (x, y), \;\; \mathfrak{p}_2 = (x-1, y), \;\; \mathfrak{p}_3 = (y), \;\; \mathfrak{p}_4 = \mathfrak{p}_5 = (x)$$

In particular the set of primes is not determined uniquely by the module M; note also that they need not be distinct.

A.1.5 Noetherian Modules of Finite Length Let M be a finitely generated module over a Noetherian ring R. In order to better understand the picture in this situation, we start by putting together many of the previous results. First let us assume that M is simple. In this case M will be isomorphic to $R/\mathfrak{m}$, where $\mathfrak{m} = \mathrm{ann}(M)$ is a maximal ideal. Indeed $\mathrm{Ass}(M)$ is not empty because $M \neq 0$ and if we pick any associated prime $\mathfrak{p} = \mathrm{ann}(m)$ the submodule generated by m is isomorphic to $R/\mathfrak{p}$ and has to be the whole of M. But then $\mathfrak{p} = \mathrm{ann}(M)$ and $R/\mathfrak{p} \cong M$, in particular $R/\mathfrak{p}$ has to be simple so that the ideal is maximal. Now assume M to be of finite length, then we have a composition series

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = (0)$$

where in particular each module M_i/M_{i+1} ($i = 0, \ldots, n-1$) is simple and therefore isomorphic to $R/\mathfrak{m}_i$ for some *maximal* ideal $\mathfrak{m}_i$.

Theorem. *Let M be a finitely generated module over a Noetherian ring R. Then the following properties are equivalent.*

(a) *M is of finite length;*

(b) *Every ideal $\mathfrak{p} \in \mathrm{Ass}(M)$ is a maximal ideal of R;*

(c) *Every ideal $\mathfrak{p} \in \mathrm{Supp}(M)$ is a maximal ideal of R.*

Proof. Let $(M_i)_{0 \leq i \leq n}$ be a normal series without repetitions of M such that, for $i = 0, \ldots, n-1$, the module M_i/M_{i+1} is isomorphic to $R/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal. If M is of finite length, so is each of the R-modules $R/\mathfrak{p}_i$, in particular each of them is an Artinian ring; being also an integral domain each of them is a field.

> Let A be an integral domain which is an Artinian ring. Let $x \in A$, $x \neq 0$. By the d.c.c. we have $(x^s) = (x^{s+1})$ for some s, hence $x^s = x^{s+1}y$ for some $y \in A$. Since A is an integral domain and $x \neq 0$ we can cancel x^s, hence $xy = 1$.

Using the Theorem in §A.1.4 above, we conclude that (a) implies (b) and that (b) implies (c). Finally if all the ideals in $\mathrm{Supp}(M)$ are maximal so are the $\mathfrak{p}_i$, hence all the modules $R/\mathfrak{p}_i$ are simple and the normal series is a composition series. □

This Theorem is the key to understand the definition in Fulton (1998). In §A.1.4 we have seen that for any finitely generated R-module M there is a chain of submodules

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = (0)$$

with $M_i/M_{i+1} \cong R/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of R. And now we know that M is of finite length if and only if the $\mathfrak{p}_i$ which occur in such a chain are all maximal ideals. Moreover this is equivalent to $\mathrm{Supp}(M)$ containing only a finite number of maximal ideals, indeed we have the following.

Corollary. *Let R be a Noetherian ring. Let M be a finitely generated R-module of finite length. Then* $\mathrm{Ass}(M) = \mathrm{Supp}(M)$.

Proof. This is actually contained already in the proof of the Theorem. Since $\mathrm{Ass}(M)$ consists of only maximal ideals it coincides with the set of its minimal elements. These are the minimal primes above $\mathrm{ann}(M)$ and therefore there are only finitely many of them. Now $\mathrm{Supp}(M)$ is the set of all the prime ideals above $\mathrm{ann}(M)$. □

A word of caution: although $\mathrm{Ass}(M)$ and $\mathrm{Supp}(M)$ are the same finite set, the cardinality of this set doesn't give any information on the length of the module. The obvious example is the vector space k^n, which have length n but for which this set has cardinality one.

The last remark is that $\ell_R(M) = \ell_{R/I}(M)$ for any ideal containing $\mathrm{ann}(M)$. This is clear because every submodule of M will be also a module over R/I, and composition series over M will be also composition series over R/I.

A.1.6 Properties of Length We conclude this section proving the results in Fulton (1998), therefore we adopt the same conventions. All rings will be Noetherian, and in saying that a module is of finite length we will mean additionally that it is finitely generated.

Lemma (A.1.2 in Fulton, 1998)**.** *Let M be a module of finite length over R. Then*

$$\ell_R(M) = \sum \ell_{R_\mathfrak{p}}(M_\mathfrak{p})$$

where the sum is taken over all the prime ideals of R.

The following proof of this Lemma is obtained from the proof of a more general statement in Eisenbud (1995) (§2.4, Theorem 2.13).

Proof. Note first that $\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$ if and only if $\mathfrak{p} \in \mathrm{Supp}(M)$. Since $\mathrm{Supp}(M)$ is a finite set and consists only of maximal ideals, the sum in the statement is finite and runs in fact through a subset of the maximal ideals of R.

If M is a simple module then $M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, and for any other maximal ideal $\mathfrak{p}$ we either have $\mathfrak{p} = \mathfrak{m}$, in which case the localisation $(R/\mathfrak{m})_\mathfrak{p} = R/\mathfrak{m}$, or $\mathfrak{p} \not\subseteq \mathfrak{m}$, in which case $(R/\mathfrak{m})_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{m}^e = 0$.

We are now ready to prove the Lemma. Let $\ell_R(M) = n$, and $\mathfrak{p}$ be any maximal ideal of R. Any composition series for M localises to a normal series for $M_\mathfrak{p}$

$$M_\mathfrak{p} = (M_0)_\mathfrak{p} \supseteq (M_1)_\mathfrak{p} \supseteq \dots (M_n)_\mathfrak{p} = (0)$$

the modules M_i/M_{i+1} have length one, so by the previous discussion we have the following situation

$$(M_i)_\mathfrak{p}/(M_{i+1})_\mathfrak{p} = (M_i/M_{i+1})_\mathfrak{p} = \begin{cases} M_i/M_{i+1} & \text{if } \mathfrak{p} = \mathrm{ann}(M_i/M_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

The sequence above is therefore a redundant composition series for $M_\mathfrak{p}$ over $R_\mathfrak{p}$ for any maximal ideal $\mathfrak{p}$ of R. The number of non-trivial inclusions in it is given by $\ell_{R_\mathfrak{p}}(M_\mathfrak{p})$. If $\mathfrak{p}$ and $\mathfrak{q}$ are different maximal ideals then inclusions which are non-trivial in the sequence determined by $\mathfrak{p}$ are trivial in the sequence determined by $\mathfrak{q}$, thus the Lemma is proved. □

Lemma (A.1.3 in Fulton (1998))**.** *Let $A \to B$ be a local homomorphism of local rings. Let d be the degree of the residue field extension. A non-zero B-module M has finite length over A if and only if $d < \infty$ and M has finite length over B, in which case*

$$\ell_A(M) = d \cdot \ell_B(M)$$

Proof. Start assuming that M has finite length over A. Since M is finitely generated over A, it is also finitely generated over B and there is a normal series without repetitions

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = (0)$$

where each M_i is a B-module (thus also an A-module) and for $i = 0, \dots, n-1$ the quotient M_i/M_{i+1} is isomorphic as a B-module (thus also as an A-module) to $B/\mathfrak{p}_i$. Being M of finite length so it is each A-module M_i/M_{i+1}, in particular $B/\mathfrak{p}_i$ is an Artinian A-module. Therefore $B/\mathfrak{p}_i$ satisfies the descending chain condition on ideals (which are particular submodules over A), that is $B/\mathfrak{p}_i$ is an Artinian ring. Being it a domain it is a field, thus each ideal $\mathfrak{p}_i$ is maximal and M is of finite length over B.

In this case for $i = 0, \dots, n-1$ we have the following short exact sequence (which is exact both over A and over B)

$$0 \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow M_i/M_{i+1} \longrightarrow 0$$

we can therefore say the following

$$\ell_A(M_i) = \ell_A(M_{i+1}) + \ell_A(M_i/M_{i+1})$$

and since M_i/M_{i+1} is isomorphic to $B/\mathfrak{m}_B$, we can put all things together to get the formula

$$\ell_A(M) = \ell_B(M) \cdot \ell_A(B/\mathfrak{m}_B)$$

Note now that $\operatorname{ann}_A(B/\mathfrak{m}_B) = \mathfrak{m}_A$, therefore we have the equalities

$$\ell_A(B/\mathfrak{m}_B) = \ell_{A/\mathfrak{m}_A}(B/\mathfrak{m}_B) = d$$

the latter holding because for vector spaces length and dimension are the same (it should be clear by now, but you can see Atiyah and Macdonald, 1969, Proposition 6.10). Therefore if we assume M to be of finite length over A then d is necessarily finite. Conversely if we assume d to be finite then $\ell_A(B/\mathfrak{m}_B) = d$ and if M is of finite length over B then the same argument as above shows that M is of finite length over A. □

Appendix B

Sheaf Theory

It is well known that one cannot possibly read Hartshorne (1977) without at least attempting to solve the exercises, as they contain fundamental results needed in the text. This is in particular true for sheaf theory, introduced in a brief section and developed mostly in the exercises. For this reason I have collected here many of them, but without writing any introduction to sheaves myself. Therefore this appendix should be viewed as a complement of Hartshorne (1977), and the reader must be familiar with definitions and notation in there. To gain a broader vision of sheaf theory as a natural tool in geometry we suggest to read Tennison (1975), to learn everything related with sheaves from a topological point of view there is also Bredon (1997).

B.1 Basic properties

B.1.1 The constant sheaf Let G be a group. On a topological space X the *constant sheaf* determined by G is defined as follows: give G the discrete topology, and for any open set $U \subseteq X$, let $\underline{G}(U)$ be the group of all continuous functions from U into G, with the usual restriction maps. The basic remark here is that whenever U is connected one such a function must be constant and therefore $\underline{G}(U) = G$.

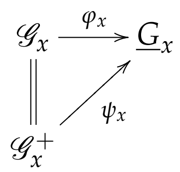
We can describe the same sheaf starting from the *constant presheaf* $\mathscr{G}$, defined by $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity.

Proposition (Exercise II.1.1 in Hartshorne, 1977)**.** *With notations and definitions as above, there exists an isomorphism between $\underline{G}$ and $\mathscr{G}^+$.*

Proof. The associated sheaf $\mathscr{G}^+$ has the following universal property: there is a morphism $\theta\colon \mathscr{G} \to \mathscr{G}^+$, such that for any sheaf $\mathscr{F}$, and any morphism $\varphi\colon \mathscr{G} \to \mathscr{F}$, there is a unique morphism $\psi\colon \mathscr{G}^+ \to \mathscr{F}$ with $\varphi = \psi \circ \theta$. So we define $\varphi\colon \mathscr{G} \to \underline{G}$ over any open subset $U \subseteq X$ as the homomorphism associating to each element of G the corresponding constant function

$$\varphi_U\colon \mathscr{G}(U) \longrightarrow \underline{G}(U)$$

Note that for any $x \in X$ the homomorphism $\varphi_x\colon \mathscr{G}_x \to \underline{G}_x$ is an isomorphism. Injectivity follows immediately from the injectivity of φ_U, so we check surjectivity. It is enough to observe that given a pair (U, s), where U is an open neighborhood of x and s is a continuous function from U to G, the subset of X given by the preimage $s^{-1}\big(s(x)\big)$ is an open subset over which the function s is constant. Now there is a unique morphism $\psi\colon \mathscr{G}^+ \to \underline{G}$ such that $\varphi = \psi \circ \theta$, so the following diagram on the stalks is commutative



This proves that ψ_x is an isomorphism for all x, and this is enough to conclude that ψ is an isomorphism. □

B.1.2 Injectivity and Surjectivity A morphism of sheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ is injective if for every open subset $U \subseteq X$ the homomorphism φ_U is injective. The kernel of φ is the subsheaf of $\mathscr{F}$ defined by $\ker\varphi(U) = \ker\varphi_U$, thus φ is injective if and only if $\ker\varphi = 0$. The image of φ is the sheaf associated to the presheaf $U \mapsto \operatorname{Im}\varphi_U$, we say φ is surjective if $\operatorname{Im}\varphi = \mathscr{G}$.

Claim (Exercise II.1.4 in Hartshorne, 1977)**.** *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi_U\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Then the induced map $\varphi^+\colon \mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.*

In particular, if $\varphi\colon \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then Im φ *can be naturally identified with a subsheaf of $\mathscr{G}$.*

Proof. To see this one can fix an open set U and work with the homomorphism $\varphi_U^+\colon \mathscr{F}^+(U) \to \mathscr{G}^+(U)$. Recalling that this is defined by sending a section $s\colon U \to \coprod \mathscr{F}_x$ to the composition $(\coprod \varphi_x) \circ s$, injectivity follows at once.

To check the final statement, let $\mathscr{H}$ be the presheaf $U \to \operatorname{Im}\varphi_U$ and observe that the morphism of presheaves $\psi\colon \mathscr{H} \to \mathscr{G}$ defined by the inclusion $\operatorname{Im}\varphi_U \subseteq \mathscr{G}(U)$ is injective for all U. □

Proposition (Exercise II.1.2 in Hartshorne, 1977)**.** *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves over a topological space X. Then*

(a) *for any* $x \in X$*, we have* $(\ker \varphi)_x = \ker(\varphi_x)$ *and* $(\operatorname{Im} \varphi)_x = \operatorname{Im}(\varphi_x)$*.*

(b) φ *is injective (respectively, surjective) if and only if the induced map on the stalks* φ_x *is injective (respectively, surjective) for all* x*.*

Moreover, a sequence $\dots \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \dots$ *of sheaves and morphisms is exact if and only if for each* $x \in X$ *the corresponding sequence of stalks is exact as a sequence of Abelian groups.*

Proof. Since $\ker \varphi$ is a subsheaf of $\mathscr{F}$ and $\operatorname{Im} \varphi$ is a subsheaf of $\mathscr{G}$, the stalk $(\ker \varphi)_x$ is a subgroup of $\mathscr{F}_x$ and the stalk $(\operatorname{Im} \varphi)_x$ is a subgroup of $\mathscr{G}_x$. This allows us to prove the equalities above in the category of sets. The first, $(\ker \varphi)_x = \ker(\varphi_x)$, holds because both groups can be described as the set of couples (U, t), where U is an open neighborhood of x and t is an element of $\mathscr{F}(U)$ such that $\varphi_U(t) = 0$, under the same equivalence relation. Similarly the second holds because both groups can be described in the same way, only one have to be careful and recall that a presheaf and the sheaf associated to it have the same stalks.

Statement (b) is a consequence of (a). The morphism φ is injective if and only if $\ker \varphi = 0$, which is the case if and only if $(\ker \varphi)_x = 0$ for all $x \in X$. Applying (a) we can say that φ is injective if and only if $\ker(\varphi_x) = 0$ for all x, that is φ_x is injective for all x. In the same way $\operatorname{Im} \varphi = \mathscr{G}$ if and only if $(\operatorname{Im} \varphi)_x = \mathscr{G}_x$ for all $x \in X$, and by (a) if and only if φ_x is surjective for all x.

The last assertion about the exact sequence is immediate now. □

Corollary (Exercise II.1.5 in Hartshorne, 1977)**.** *A morphism of sheaves is an isomorphism if and only if it is both injective and surjective.*

Proof. According to (Hartshorne, 1977, Proposition II.1.1), a morphism of sheaves is an isomorphism if and only if the induced maps on the stalks are all isomorphisms, that is are all both injective and surjective. By the Proposition above this is equivalent to say that the morphism itself is both injective and surjective. □

Example (In which φ is surjective but φ_U is not). Let $X = [0, 1] \subset \mathbb{R}$ and let $\mathscr{F}$ be the constant sheaf with stalk $\mathbb{Z}$. Let $\mathscr{G}$ be the sheaf whose stalks are

$$\mathscr{G}_x = \begin{cases} \mathbb{Z} & \text{if } x = 0 \text{ or } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

so that, for instance, $\Gamma(X, \mathscr{G}) = \mathbb{Z} \oplus \mathbb{Z}$. Let $\varphi \colon \mathscr{F} \to \mathscr{G}$ be the unique morphism such that $\varphi_x = \mathrm{id}_{\mathbb{Z}}$ if $x = 0$ or $x = 1$. Then φ is surjective, but the homomorphism $\varphi_X \colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ cannot be surjective.

Lemma (Exercise II.1.3 in Hartshorne, 1977)**.** *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Then φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$ for all i.*

Proof. Assume φ is surjective, then φ_x is surjective for all x. Let U be an open set in X and let s be an element of $\mathscr{G}(U)$; for any $x \in U$ let σ_x be the germ of s in the stalk $\mathscr{G}_x$. There exist elements $\tau_x \in \mathscr{F}_x$ such that $\varphi_x(\tau_x) = \sigma_x$, and for each one of them there exists an open set U_x containing x and elements $s_x \in \mathscr{G}(U_x)$ and $t_x \in \mathscr{F}(U_x)$ such that $\varphi(t_x) = s_x = s|_{U_x}$. Thus the condition holds.

Conversely if the condition holds we prove that φ is surjective by proving that φ_x is surjective for all x. Every $\sigma \in \mathscr{G}_x$ is the germ of some $s \in \mathscr{G}(U)$; in the open set U there is a covering such as described above, in particular there exists an open set $V \subseteq U$ containing x and an element $t \in \mathscr{F}(V)$ such that $\varphi_V(t) = s|_V$, in other words $\varphi_x(\tau) = \sigma$ where τ is the germ of t in $\mathscr{F}_x$. □

B.1.3 Exact sequences If $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$, the *quotient sheaf* $\mathscr{F}/\mathscr{F}'$ is defined to be the sheaf associated to the presheaf $U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)$. One sees immediately that the stalks of the quotient sheaf are given by the quotients of the stalks, $(\mathscr{F}/\mathscr{F}')_x = \mathscr{F}_x/\mathscr{F}'_x$.

Proposition (Exercise II.1.6 in Hartshorne, 1977)**.**

(*a*) *Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Then the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{F}' \longrightarrow 0$$

(*b*) *Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, then $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$, and $\mathscr{F}''$ is isomorphic to the quotient of $\mathscr{F}$ by this subsheaf.*

Proof. The natural map is surjective because it induces surjective maps on the stalks. Its kernel is a subsheaf of $\mathscr{F}$ whose stalks are the same as those of $\mathscr{F}'$, so it is $\mathscr{F}'$.

Conversely the map $\mathscr{F}' \to \mathscr{F}$ is injective (because so it is on the stalks), hence $\mathscr{F}'$ is isomorphic to its image, that is a subsheaf of $\mathscr{F}$, say $\mathscr{H}$. We define $\bar{g}\colon \mathscr{F}/\mathscr{H} \to \mathscr{F}''$ via the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{H} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}/\mathscr{H} & \longrightarrow & 0 \\
 & & \downarrow\wr & & \| & & \downarrow \bar{g} & & \\
0 & \longrightarrow & \mathscr{F}' & \xrightarrow{f} & \mathscr{F} & \xrightarrow{g} & \mathscr{F}'' & \longrightarrow & 0
\end{array}$$

It turns out that $\bar{g}$ is an isomorphism. □

Corollary (Exercise II.1.7 in Hartshorne, 1977)**.** *Let* $\varphi\colon \mathscr{F} \to \mathscr{G}$ *be a morphism of sheaves. Then*

i) $\operatorname{Im}\varphi \cong \mathscr{F} / \ker\varphi$.

ii) $\operatorname{coker}\varphi \cong \mathscr{G} / \operatorname{Im}\varphi$.

Proof. We have exact sequences

$$0 \longrightarrow \ker\varphi \longrightarrow \mathscr{F} \xrightarrow{\varphi} \operatorname{Im}\varphi \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Im}\varphi \longrightarrow \mathscr{G} \longrightarrow \operatorname{coker}\varphi \longrightarrow 0$$

According to the Proposition the first gives *i*) while the second gives *ii*). □

B.2 Functorial constructions

B.2.1 The Global Sections Functor For any open subset $U \subseteq X$, the association $\mathscr{F} \mapsto \mathscr{F}(U)$ defines a functor from the category of sheaves over X to the category of Abelian groups. When $U = X$ this is particularly important, so that it gets a special name and notation; it's called the *global sections functor* and is denoted $\Gamma(X, \cdot\,)$ or also $H^0(X, \cdot\,)$.

Proposition (Exercise II.1.8 in Hartshorne, 1977)**.** *The global sections functor is* left exact, *i.e. if* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ *is an exact sequence of sheaves, then* $0 \to \Gamma(X,\mathscr{F}') \to \Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{F}'')$ *is an exact sequence of groups.*

Proof. We consider the following sequence

$$0 \longrightarrow \Gamma(X,\mathscr{F}')\underset{(1)}{\xrightarrow{\varphi_X}} \Gamma(X,\mathscr{F})\underset{(2)}{\xrightarrow{\psi_X}} \Gamma(X,\mathscr{F}'')$$

To show exactness in (1) it is enough to remind that a morphism is injective if and only if it is injective on any open set. Now we have to prove that it is exact in (2) i.e. that $\ker\psi_X = \operatorname{Im}\varphi_X$. One inclusion is clear, since we have

$$\operatorname{Im}\varphi_X \subseteq \Gamma(X,\operatorname{Im}\varphi) = \Gamma(X,\ker\psi) = \ker\psi_X$$

It remains to prove the other. Let $s \in \ker\psi_X$ and for every $P \in X$ let σ_P be its germ in the stalk $\ker\psi_P = \operatorname{Im}\varphi_P$, then there exists $\tau_P \in \mathscr{F}'_P$ such that $\varphi_P(\tau_P) = \sigma_P$. From this it follows that there are an open covering $\{U_i\}$ of X

and elements $t_i \in \mathscr{F}'(U_i)$ such that $\varphi_{U_i}(t_i) = s|_{U_i}$; now we use the injectivity of φ to say that the family $\{t_i\}$ is coherent. If $i \neq j$ we have

$$\varphi_{U_{ij}}(t_i|_{U_{ij}}) = \varphi_{U_i}(t_i)|_{U_{ij}} = s|_{U_{ij}} = \varphi_{U_j}(t_j)|_{U_{ij}} = \varphi_{U_{ij}}(t_j|_{U_{ij}})$$

Since $\varphi_{U_{ij}}$ is injective we can now say that $t_i|_{U_{ij}} = t_j|_{U_{ij}}$ and since $\mathscr{F}'$ is a sheaf there exist a unique element $t \in \Gamma(X, \mathscr{F}')$ such that $t|_{U_i} = t_i$. Now it's easy to conclude that $\varphi_X(t) = s$, that is $s \in \operatorname{Im} \varphi_X$. □

B.2.2 Direct Sum of Sheaves Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves over X. The presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is already a sheaf. To see this observe that restriction maps are defined componentwise, hence uniqueness and gluing axioms are verified in the same way. This sheaf is called the *direct sum* of $\mathscr{F}$ and $\mathscr{G}$, and is denoted by $\mathscr{F} \oplus \mathscr{G}$.

Proposition (Exercise II.1.9 in Hartshorne, 1977)**.** *Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves over X. The direct sum $\mathscr{F} \oplus \mathscr{G}$ plays the role of direct sum and of direct product in the category of sheaves of Abelian groups over X.*

Proof. For each U note that $\mathscr{F}(U) \oplus \mathscr{G}(U)$ it is the direct sum and the direct product of $\mathscr{F}(U)$ and $\mathscr{G}(U)$ in the category of Abelian groups, with projections and injections defined in the usual way. The following diagram chases, where $f \in \mathscr{F}(U)$, $g \in \mathscr{G}(U)$ and vertical arrows are restrictions, also show that they are morphisms of sheaves:

$$\begin{array}{ccc} f & \longrightarrow & (f,0) \\ \downarrow & & \downarrow \\ f|_V & \longrightarrow & (f|_V,0) \end{array} \qquad \begin{array}{ccc} (f,g) & \longrightarrow & f \\ \downarrow & & \downarrow \\ (f|_V,g|_V) & \longrightarrow & f|_V \end{array}$$

We have to check universal properties and again $\mathscr{F}(U) \oplus \mathscr{G}(U)$ verifies them in the category of Abelian groups. If $\varphi \colon \mathscr{F} \to \mathscr{H}$ and $\psi \colon \mathscr{G} \to \mathscr{H}$ are morphisms of sheaves, for all U there exists a unique morphism of Abelian groups

$$\varphi \oplus \psi \colon \mathscr{F}(U) \oplus \mathscr{G}(U) \to \mathscr{H}(U)$$

such that $(\varphi \oplus \psi) \circ i_1 = \varphi$ and $(\varphi \oplus \psi) \circ i_2 = \psi$, moreover this morphism is defined by $(f,g) \mapsto \varphi(f) + \psi(g)$. Now we can chase the diagram

$$\begin{array}{ccc} (f,g) & & \\ \downarrow & & \\ (f|_V,g|_V) & \longrightarrow & \varphi(f|_V) + \psi(g|_V) \end{array} \qquad \begin{array}{ccc} (f,g) & \longrightarrow & \varphi(f) + \psi(g) \\ & & \downarrow \\ & & (\varphi(f) + \psi(g))|_V \end{array}$$

and conclude that $\varphi \oplus \psi$ is a morphism of sheaves, because of the following

$$(\varphi(f) + \psi(g))|_V = \varphi(f)|_V + \psi(g)|_V = \varphi(f|_V) + \psi(g|_V)$$

In the same way if $\varphi\colon \mathscr{H} \to \mathscr{F}$ and $\psi\colon \mathscr{H} \to \mathscr{G}$ are morphisms of sheaves, for all U there exist a unique morphism of Abelian groups

$$\varphi \times \psi\colon \mathscr{H}(U) \to \mathscr{F}(U) \oplus \mathscr{G}(U)$$

such that $p_1 \circ (\varphi \times \psi) = \varphi$ and $p_2 \circ (\varphi \times \psi) = \psi$, moreover this morphism is defined by $h \mapsto (\varphi(h), \psi(h))$. Now we can chase the diagram

$$\begin{array}{ccc} h & & \\ \downarrow & & \\ h|_V & \longrightarrow & (\varphi(h|_V), \psi(h|_V)) \end{array} \qquad \begin{array}{ccc} h & \longrightarrow & (\varphi(h), \psi(h)) \\ & & \downarrow \\ & & (\varphi(h)|_V, \psi(h)|_V) \end{array}$$

and conclude that $\varphi \times \psi$ is a morphism of sheaves, because of the following

$$(\varphi(h)|_V, \psi(h)|_V) = (\varphi(h|_V), \psi(h|_V))$$ □

B.2.3 Direct and Inverse Limits Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms over X. We define the *direct limit* of the system $\{\mathscr{F}_i\}$, denoted $\varinjlim \mathscr{F}_i$, to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$. The reader not familiar with limits in categories will find all the relevant definitions in (Berrick and Keating, 2000, Chapter 5).

Lemma (Exercise II.1.10 in Hartshorne, 1977)**.** *Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms over X. The construction of the sheaf $\varinjlim \mathscr{F}_i$ is a direct limit in the category of sheaves over X.*

Proof. As above the only thing we have to do is to verify the universal property, knowing that for any open set U it is satisfied by $\varinjlim \mathscr{F}_i(U)$. So let $\psi^i\colon \mathscr{F}_i \to \mathscr{G}$ be morphisms of sheaves such that for any $i \leq j$ the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}_i & \xrightarrow{\psi^i} & \mathscr{G} \\ {\scriptstyle f^{ij}}\downarrow & \nearrow{\scriptstyle \psi^j} & \\ \mathscr{F}_j & & \end{array}$$

If we denote $\mathscr{H}$ the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$, for each open set U there is a unique morphism $\beta_U\colon \mathscr{H}(U) \to \mathscr{G}(U)$ such that for all i the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}_i(U) & \xrightarrow{\psi_U^i} & \mathscr{G}(U) \\ {\scriptstyle f_U^i}\downarrow & \nearrow {\scriptstyle \beta_U} & \\ \mathscr{H}(U) & & \end{array}$$

Since we know exactly how the maps work, it is straightforward to show that β is a morphism of presheaves. Now it defines a unique morphism of sheaves $\beta^+ \colon \mathscr{H}^+ \to \mathscr{G}$, and we are done. □

Claim (Exercise II.1.11 in Hartshorne, 1977). *Let $\{\mathscr{F}_i\}$ be a direct system of sheaves over a Noetherian topological space X. In this case the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$ is already a sheaf.*

Proof. Let $U \subseteq X$ be open. In §1.2.5 we have seen that every open subset of a Noetherian topological space X is quasi-compact, hence we can assume every open covering of U to be finite. Let then $U = U_1 \cup \cdots \cup U_r$ and consider a coherent family of sections, that is elements $\tau_1 \in \varinjlim \mathscr{F}_i(U_1), \ldots, \tau_r \in \varinjlim \mathscr{F}_i(U_r)$ such that

$$\tau_k|_{U_k \cap U_h} = \tau_h|_{U_k \cap U_h}$$

For each $k = 1, \ldots, r$ the section τ_k is an equivalence class, a representative is given by some element $t_k \in \mathscr{F}_{i_k}(U_k)$. Since the family of sections is coherent for every pair h, k there exists an index ℓ, bigger than both i_h and i_k, such that

$$f^{i_k \ell}_{U_k \cap U_h}(t_k|_{U_k \cap U_h}) = f^{i_h \ell}_{U_k \cap U_h}(t_h|_{U_k \cap U_h})$$

where f^{ij} are the structure morphisms of the direct system. We can in fact assume ℓ bigger than each index $i_1, \ldots, i_r$, so that we've got a family

$$f^{i_1 \ell}_{U_1}(t_1) \in \mathscr{F}_\ell(U_1), \ldots, f^{i_r \ell}_{U_r}(t_r) \in \mathscr{F}_\ell(U_r)$$

which is a coherent family of sections for the sheaf $\mathscr{F}_\ell$. Thus there exists a unique element $s \in \mathscr{F}_\ell(U)$ whose equivalence class $\sigma \in \varinjlim \mathscr{F}_i(U)$ is the unique section such that $\sigma|_{U_k} = \tau_k$ for any $k = 1, \ldots, r$. □

Proposition (Exercise II.1.12 in Hartshorne, 1977). *Let $\{\mathscr{F}_i\}$ be an inverse system of sheaves over X. Then*

i) the presheaf $U \mapsto \varprojlim \mathscr{F}_i(U)$ is already a sheaf;

ii) it is an inverse limit in the category of sheaves over X.

It is called the inverse limit *of the system $\{\mathscr{F}_i\}$, and is denoted by $\varprojlim \mathscr{F}_i$.*

Proof. Let U be an open subset of X and let $\{U_k\}$ be an open covering of U. Consider a coherent family of sections, that is elements $\tau_k \in \varprojlim \mathscr{F}_i(U_k)$ such that

$$\tau_k|_{U_k \cap U_h} = \tau_h|_{U_k \cap U_h}$$

For each k the section τ_k can be realised as an element $t^k \in \prod \mathscr{F}_i(U_k)$ such that whenever $i \leq j$

$$f^{ij}_{U_k}(t^k_i) = t^k_j$$

where f^{ij} are the structure morphisms of the inverse system. The restriction of τ_k to the intersection $U_h \cap U_k$ is then given by the family

$$\tau_k|_{U_k \cap U_h} = (t^k_i|_{U_k \cap U_h})_i$$

which is automatically an element of $\varprojlim \mathscr{F}_i(U_h \cap U_k)$ because f^{ij} is a morphism of sheaves. Thus the gluing condition applies componentwise, that is for every i we have a coherent family of sections $\{t^k_i\}$ that glue into a unique section $t_i \in \mathscr{F}_i(U)$. The element $t \in \prod \mathscr{F}_i(U)$ obtained in this way is in fact a section in $\varprojlim \mathscr{F}_i(U)$ since for every k

$$f^{ij}_U(t_i)|_{U_k} = f^{ij}_{U_k}(t^k_i) = t^k_j = t_j|_{U_k}$$

This proves part $i)$, while for part $ii)$ the reader will not have any difficulty in applying the same argument as in the Lemma above. □

B.2.4 Flasque Sheaves A sheaf $\mathscr{F}$ on a topological space X is *flasque* if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective. A very easy but incredibly relevant example of flasque sheaf is a constant sheaf on an irreducible topological space. Indeed, since the intersection of two open subsets is in this case always nonempty, a constant presheaf is already a sheaf. Restriction maps are the identity so they are surjective. If $f\colon X \to Y$ is a continuous map, and if $\mathscr{F}$ is a flasque sheaf on X, then $f_*\mathscr{F}$ is a flasque sheaf on Y. Indeed restriction maps of $f_*\mathscr{F}$ are particular restriction maps of $\mathscr{F}$.

Proposition (Exercise II.1.16 in Hartshorne, 1977)**.** *We describe here how flasque sheaves behave with respect to exact sequences.*

i) *If* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence of sheaves over* X*, and if* $\mathscr{F}'$ *is flasque, then for any open subset* U *of* X*, the sequence of Abelian groups* $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ *is also exact.*

ii) *If* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence of sheaves, and if* $\mathscr{F}'$ *and* $\mathscr{F}$ *are flasque, then* $\mathscr{F}''$ *is flasque.*

Proof. In §B.2.1 above we have seen that whenever $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves then for any open set U the sequence of Abelian groups $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U)$ is exact; to have $i)$ we have then to show that the last map is surjective. Without loss of generality we can deal with global sections, since $\mathscr{F}'|_U$ is also a flasque sheaf. So what we have to prove is that in the sequence

$$0 \to \Gamma\left(X, \mathscr{F}'\right) \xrightarrow{\psi_X} \Gamma\left(X, \mathscr{F}\right) \xrightarrow{\varphi_X} \Gamma\left(X, \mathscr{F}''\right) \to 0$$

the map φ_X is surjective. Let $s \in \Gamma\left(X, \mathscr{F}''\right)$ and consider the collection Σ of all pairs (U, t) where $U \subseteq X$ is an open set and $t \in \mathscr{F}(U)$ is a section such that $\varphi_U(t) = s|_U$. Observe that such a collection is not empty because φ is a surjective morphism of sheaves (applying the Lemma in §B.1.2 above). Order Σ by $(U, t) < (U', t')$ if $U \subseteq U'$ and $t'|_U = t$. Then Σ is inductively ordered (i.e. a chain in Σ has an upper bound, its union) and by Zorn's lemma has a maximal element, say (V, t). If we prove that $V = X$ then we are done, because in that case $\varphi_X(t) = s$. Suppose $V \neq X$, let $x \notin V$ and let W be a neighborhood of x such that $(W, t') \in \Sigma$ for some $t' \in \mathscr{F}(W)$; note that the existence of such a neighborhood is guaranteed by the Lemma in §B.1.2 above. Now $t|_{V \cap W} - t'|_{V \cap W}$ is in the kernel of the map $\varphi_{(V \cap W)}$ which is equal to the image of $\psi_{(V \cap W)}$ hence extends to some $t'' \in \mathscr{F}(W)$, since $\mathscr{F}'$ is flasque. Then t and $t' + t''$ agree on $V \cap W$, so that together they define an element of $\mathscr{F}(V \cup W)$, extending t and contradicting the maximality of (V, t).

Statement *ii*) follows from the following commutative diagram (which is so by the previous part), where $V \subset U$ is an inclusion of open sets and all sequences are exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F}'(U) & \longrightarrow & \mathscr{F}(U) & \longrightarrow & \mathscr{F}''(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}'(V) & \longrightarrow & \mathscr{F}(V) & \longrightarrow & \mathscr{F}''(V) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and a diagram chase. In fact in this way one can prove that, being the sheaf $\mathscr{F}'$ flasque, then $\mathscr{F}$ is flasque if and only if so is $\mathscr{F}''$. □

Another example of flasque sheaf is the *sheaf of discontinuous functions*. Let $\mathscr{F}$ be any sheaf on X, for each open set $U \subseteq X$, we define $\mathscr{G}(U)$ to be the set of maps $s\colon U \to \bigcup_{P \in U} \mathscr{F}_P$ such that for each $P \in U$, $s(P) \in \mathscr{F}_P$. It is clear that $\mathscr{G}$ is a sheaf, that restriction homomorphisms are surjective, and also that there is a natural injective morphism of $\mathscr{F}$ to $\mathscr{G}$.

B.3 Topological attributes

B.3.1 Support of a sheaf Let $\mathscr{F}$ be a sheaf over X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The *support* of s, denoted $\operatorname{Supp} s$, is defined to be $\{P \in U \mid s_P \neq 0\}$ where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. We define the *support* of $\mathscr{F}$, $\operatorname{Supp} \mathscr{F}$ to be $\{P \in U \mid \mathscr{F}_P \neq 0\}$.

Lemma (Exercise II.1.14 in Hartshorne, 1977)**.** *Let $\mathscr{F}$ be a sheaf over X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. Then the support* $\operatorname{Supp} s$ *is a closed subset of U.*

Proof. We show that the complement in U of $\operatorname{Supp} s$ is an open set. First note that this is $\{P \in U \mid s_P = 0\}$, then let $P \in U \setminus \operatorname{Supp} s$. Since $s_P = 0$ there exists an open neighborhood $V_P \subseteq U$ of P such that $s|_{V_P} = 0$; this proves that $U \setminus \operatorname{Supp} s$ contains V_P, and thus it is an open set. □

Example (In which $\operatorname{Supp} \mathscr{F}$ is not closed). Let $\mathscr{F}$ be the sheaf over $[0,1] \subset \mathbb{R}$ defined as the sheaf associated to the presheaf

$$\mathscr{F}(U) = \begin{cases} \mathbb{Z} & \text{if } U \subseteq [\frac{1}{3}, \frac{2}{3}) \text{ or } \frac{1}{3} \in U \\ 0 & \text{otherwise} \end{cases}$$

with the obvious restriction maps. It is clear that the stalks of $\mathscr{F}$ are

$$\mathscr{F}_x = \begin{cases} \mathbb{Z} & \text{if } x \in [\frac{1}{3}, \frac{2}{3}) \\ 0 & \text{otherwise} \end{cases}$$

therefore the support $\operatorname{Supp} \mathscr{F} = [\frac{1}{3}, \frac{2}{3})$ is not a closed nor an open set.

B.3.2 Skyscraper sheaves Let G be an Abelian group. We fix a point $x \in X$ and construct the *skyscraper sheaf* over x, denoted $\underline{G}^x$, by setting $\underline{G}^x(U) = G$ if $x \in U$, otherwise $\underline{G}^x(U) = 0$. It is clearly a sheaf, and we have the following characterisation.

Proposition (Exercise II.1.17 in Hartshorne, 1977)**.** *The stalks of the skyscraper sheaf are given by $\underline{G}^x_y = G$ if $y \in \overline{\{x\}}$, otherwise $\underline{G}^x_y = 0$. The same sheaf is given by $i_*\underline{G}$, where $\underline{G}$ is the constant sheaf over the closed subspace $\overline{\{x\}}$ and $i\colon \overline{\{x\}} \to X$ is the inclusion.*

Proof. To prove this result it is enough to make a couple of remarks. If $y \in \overline{\{x\}}$ then any open neighborhood of y contains also x, otherwise there exists an open neighborhood of y that doesn't contain x. In the same way, for any open subset U of X, the preimage $i^{-1}(U)$ is empty if and only if $x \notin U$. To conclude we need to observe that $\underline{G}$ coincides with the constant *presheaf* because $\overline{\{x\}}$ is an irreducible space. □

B.3.3 Extending a Sheaf by Zero Let $Z \subseteq X$ be a closed subset of X, and let $i\colon Z \to X$ be the inclusion. If $\mathscr{F}$ is a sheaf over Z the stalks of the direct image sheaf $i_*\mathscr{F}$ are given by $(i_*\mathscr{F})_P = \mathscr{F}_P$ for any $P \in Z$, otherwise $(i_*\mathscr{F})_P = 0$. For this reason we say that $i_*\mathscr{F}$ is obtained by extending $\mathscr{F}$ by zero outside the closed subset Z.

Let now U be any open subset of X and let $j\colon U \to X$ be the inclusion. If $\mathscr{G}$ is a sheaf over U we define $j_!\mathscr{G}$ to be the sheaf associated to the presheaf $V \mapsto \mathscr{G}(V)$ if $V \subseteq U$, otherwise $V \mapsto 0$. Clearly the stalks of $j_!\mathscr{G}$ are given by $(j_!\mathscr{G})_P = \mathscr{G}_P$ if $P \in U$, otherwise $(j_!\mathscr{G})_P = 0$.

Proposition (Exercise II.1.19 in Hartshorne, 1977)**.** *Let $Z \subseteq X$ be a closed subset of X, and let $U = X \setminus Z$. If $\mathscr{F}$ is a sheaf over X, with the same notations as above, there is an exact sequence of sheaves*

$$0 \longrightarrow j_!(\mathscr{F}|_U) \longrightarrow \mathscr{F} \longrightarrow i_*(\mathscr{F}|_Z) \longrightarrow 0$$

Proof. We can define an injective morphism of presheaves by taking, for any open subset V of X, the identity of $\mathscr{F}(V)$ if $V \subseteq U$, otherwise the inclusion of zero. This will extend uniquely to a morphism of sheaves $j_!(\mathscr{F}|_U) \to \mathscr{F}$, that will be injective as we have seen in §B.1.2.

We define a surjective morphism $\mathscr{F} \to i_*(\mathscr{F}|_Z)$ by taking, for any open subset V of X, a limit of restriction homomorphisms. Indeed we have

$$i_*(\mathscr{F}|_Z)(V) = \mathscr{F}|_Z(V \cap Z) = i^{-1}\mathscr{F}(V \cap Z) = \varinjlim_{V \cap Z \subseteq W \subseteq V} \mathscr{F}(W)$$

The resulting sequence is exact because it induces exact sequences on the stalks (see §B.1.2 above). □

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