Database tools for the large scale computation of Maximally Mutable Laurent polynomials

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Motivation

Context

Fanosearch Project: study Fano varieties using Mirror Symmetry.
[Coates, Corti, Galkin, Golyshev, Kasprzyk (arXiv:1212.1722)]

- Two sides of the mirror:

$$\left\{\begin{matrix}\text{some Laurent}\\ \text{polynomials}\end{matrix}\right\}\Big/\{\text{mutations}\} \Leftrightarrow \left\{\begin{matrix}\text{Fano}\\ \text{varieties}\end{matrix}\right\}\Big/\{\text{deformations}\}$$

- Laurent polynomials are simpler than Fano manifolds.

Motivation

Definition

Fix a lattice $N \equiv \mathbb{Z}^d$.

A *canonical Fano polytope* $P \subset N$ is a convex lattice polytope such that:

- the vertices $\mathcal{V}(P)$ are integral lattice points.
- The only lattice point in $\mathring{P}$ is the origin of the lattice.
- $\dim(P) = d$.

A special case

- If the polar polytope P^* is itself an integral lattice polytope, we call P *reflexive*.

Motivation

“Interesting” Laurent polynomials f:

- Given a canonical Fano polytope P,

$$\operatorname{Newt} f = P, \qquad f = \sum_{\gamma \in N \cap P} a_\gamma x^\gamma$$

Definition

Given a vector $w \in \mathrm{Hom}(N, \mathbb{Z})$, and $u \in \mathbb{C}[w^\perp]$, a *mutation* $\Phi : \mathbb{C}[N] \to \mathbb{C}[N]$ defined by:

$$\Phi : x^\gamma \mapsto x^\gamma u^{\langle w, \gamma \rangle}.$$

$$\Phi : f \mapsto \sum_{\gamma \in N \cap P} a_\gamma x^\gamma u^{\langle w, \gamma \rangle}.$$

$\Phi f \in \mathbb{C}[N]$ might constrain some a_γ.

Akhtar, Coates, Galkin, Kasprzyk [arXiv:1212.1785]

Motivation

Definition (Kasprzyk, Tveiten)

Let P be a canonical Fano polytope.
Consider

$$f = \sum_{\gamma \in N \cap P} a_\gamma x^\gamma$$

such that there exist $\Phi_1, \ldots, \Phi_r$ such that

$$\Phi_i f \in \mathbb{C}[N] \qquad \text{for } i = 1, \ldots, r$$

$$\text{fixes uniquely all values of } a_\gamma, \quad \gamma \in \partial P \setminus \mathcal{V}(P),$$

then we call f a *(rigid) Maximally Mutable* Laurent polynomial.

- if $\gamma \in \mathcal{V}(P)$, then $a_\gamma = 1$.
- If $\gamma = (0, \ldots, 0)$, then $a_\gamma = 0$.

Akhtar, Coates, Galkin, Kasprzyk [arXiv:1212.1785]

Motivation

Definition

Given a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the *period* of f is:

$$\pi_f(t) = \frac{1}{(2\pi \mathrm{i})^n} \int_{|x_1|=1,\cdots,|x_n|=1} \frac{1}{1 - tf(x_1, \ldots, x_n)} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}, \qquad t \in \mathbb{C},$$

Moreover, if f is mirror to a Fano manifold,

$$\pi_f(t) = \sum_{k \geq 0} c_k t^k \qquad c_k \in \mathbb{Z}, \quad \text{for all } k.$$

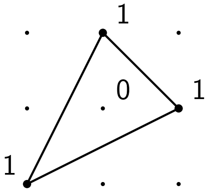
The sequence $(c_0, c_1, \ldots)$ is called the *period sequence* of f.

$$c_k = \operatorname{coeff}_1 f^k.$$

[Akhtar, Coates, Galkin, Kasprzyk (arXiv:1212.1785)]

Motivation

example

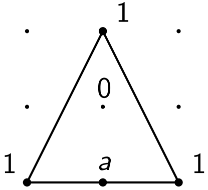


$$f = x + y + \frac{1}{xy}$$

$$\pi_f(t) = 1 + 6t^3 + 90t^6 + 1680t^9 + 34650t^{12} + 756756t^{15} + \dots$$

Motivation

example



$$f = y + \frac{1}{xy}\left(x^2 + ax + 1\right)$$

f is Maximally Mutable iff $a = 2$, and we have:

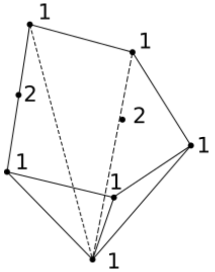
$$f = y + \frac{1}{xy}(x+1)^2 .$$

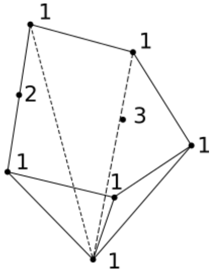
$$\pi_f(t) = 1 + 4t^2 + 36t^4 + 400t^6 + 4900t^8 + 63504t^{10} + \dots$$

Motivation

If $\dim P > 2$, P may have more than one MM Laurent polynomials.

example





$$f=\frac{1}{z}+z\left(\frac{(x+1)^2}{xy}+\frac{x^2+2x+1}{x}+y\right) \qquad g=\frac{1}{z}+z\left(\frac{(x+1)^2}{xy}+\frac{x^2+3x+1}{x}+y\right)$$

(left) $\pi_f(t) = 1 + 4t^2 + 60t^4 + 1120t^6 + 24220t^8 + \dots$

(right) $\pi_g(t) = 1 + 6t^2 + 90t^4 + 1860t^6 + 44730t^8 + \dots$

Computational aspects

Use case

Given a class of polytopes $\mathcal{U}$, for all $P \in \mathcal{U}$:

- find all Maximally-Mutable polynomials supported on P.
- Make a list of the period sequences.
- Compute other interesting quantities.

$\mathcal{U}$ can be very large

3-reflexives	3-canonicals	4-reflexives
4 319	674 688	473 800 776

Performance Requirements

- Ideally, as many cores as polytopes in $\mathcal{U}$.
- High single-core performance.
- High disk read/write rate.

Computational aspects

IO infrastructure

- Text/binary file dump.
  - As simple as possible.
  - Naturally concurrent.
  - Slow postprocessing.
- SQLite database.
  - High quality, simple to use SQL database.
  - Not designed for concurrency.
- Dedicated database server.
  - High performance, high concurrency.
  - Flexible data structures, SQL (Postgres, CockroachDB) or JSON (Mongo).
  - Non negligible set up and maintenance cost.
  - The HPC-database interface is of fundamental importance.

Our solution: in-house tooling for workers/database interaction [Coates, Kasprzyk (`https://bitbucket.org/pcas`)]

Example: canonical Fano 3-topes

Canonical Fano polytopes in 3 dimensions were classified by Kasprzyk [arXiv:0806.2604].
Up to $\mathrm{GL}(\mathbb{Z}, 3)$-equivalence, there are 674 688 canonical Fano 3-topes.
Target For each canonical Fano 3-tope P:

1. find all Maximally Mutable polynomials f supported on P.
2. For every such f, compute the period sequence.
3. Can we analise and understand this data?

Example: code metrics

The Maximally Mutables code is quite complex

The most demanding computational tasks include:

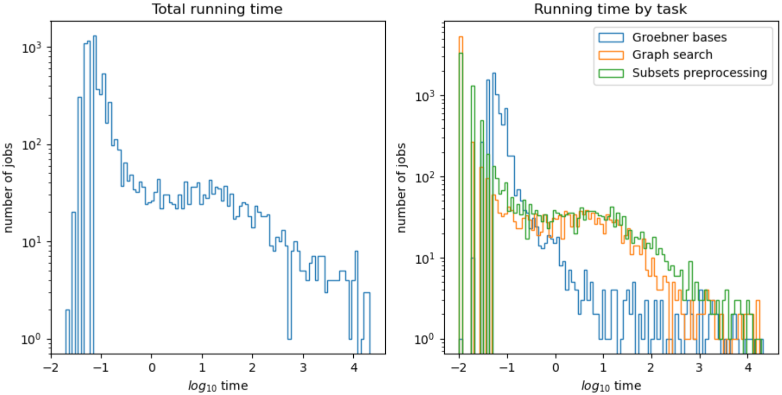
- Minkowski decompositions.
- Gröbner basis computations.
- Graph search algorithm.
- Several performance tricks.

Performance analysis and optimisation

- Code profiling (development).
- Runtime metrics (deployment).

Example: code metrics

Random sample of 10 000 polytopes.



Example: canonical Fano 3-topes

As of today, the search is 99.8% complete.
We found $\sim 170\,000$ Maximally Mutable Laurent polynomials.
These polynomials originate $\sim 8\,200$ period sequences.

- We can leverage the wealth of open source libraries for Data Science.
- Example: basic analysis of the period sequence data.
- Tools: `pandas` and `scikit-learn` libraries for python.

Example: canonical Fano 3-topes

Data Analysis

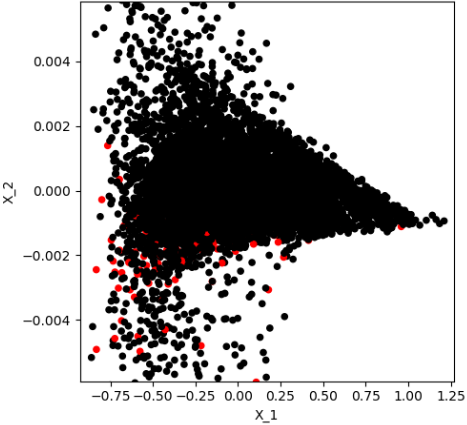
Difficulties in dealing with period sequences data:

- the coefficients of the period sequences grow quickly.
  $\hookrightarrow$ Work instead with $\log c_k$, then rescale so that
  $\log c_k \in [-1, 1]$
- The data is high-dimensional (in this computation, each datapoint is in $\mathbb{Z}^{14}$ ($[-1, 1]^{14}$ after rescaling).
  $\hookrightarrow$ Principal Component Analysis: project to linear subspaces that explain most of the variance in the data.
- Beware of low-dimensional projections of high-dimensional data.

Example: canonical Fano 3-topes

Low-dimensional projection

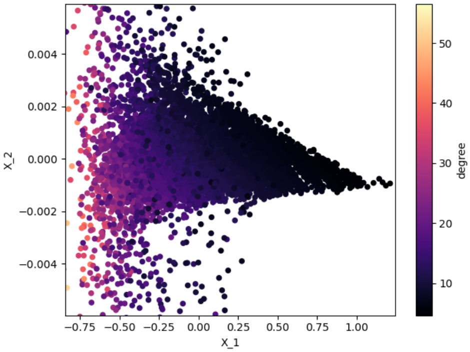
Every point represents a period sequence.
Period sequences of smooth manifolds are in red.



Example: canonical Fano 3-topes

Low-dimensional projection

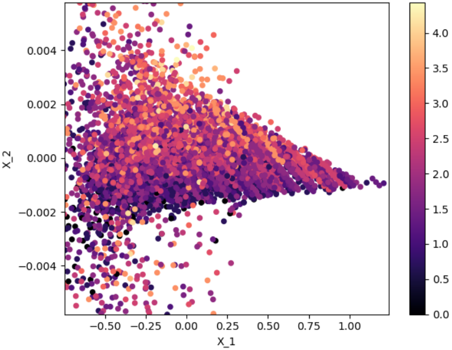
Every point represents a period sequence.
Anticanonical degree.



Example: canonical Fano 3-topes

Low-dimensional projection

Every point represents a period sequence.
Gorenstein index (log scale).



Perspectives

Opportunities and challenges ahead

- Databases are evolving quickly.
  - ↪ Automatic sharding (CockroachDB).
  - ↪ Optimised performance for SSDs.
  - ↪ Memory caches (e.g. Redis).
- Data analysis on $\sim$1 Tb.