ON THE FINE INTERIOR OF THREE-DIMENSIONAL CANONICAL FANO POLYTOPES

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Abstract. The Fine interior Δ^{FI} of a d-dimensional lattice polytope Δ is a rational subpolytope of Δ which is important for constructing minimal birational models of non-degenerate hypersurfaces defined by Laurent polynomials with Newton polytope Δ. This paper presents some computational results on the Fine interior of all 674,688 three-dimensional canonical Fano polytopes.

1. Introduction

Let $M \cong \mathbb{Z}^d$ be a free abelian group of rank d. We set $M_{\mathbb{Q}} := M \otimes \mathbb{Q}$ and denote by N the dual group $\mathrm{Hom}(M, \mathbb{Z})$ in the dual $\mathbb{Q}$-linear vector space $N_{\mathbb{Q}} := \mathrm{Hom}(M, \mathbb{Q})$. Let $\langle \cdot, \cdot \rangle : M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$ be the natural pairing.

A convex compact d-dimensional polytope $\Delta \subseteq M_{\mathbb{Q}}$ is called *lattice d-tope* if all vertices of Δ belong to the lattice $M \subseteq M_{\mathbb{Q}}$, *i.e.*, Δ equals the convex hull $\mathrm{conv}(\Delta \cap M)$ of all lattice points in Δ. The usual interior Δ° of Δ is the complement $\Delta \setminus \partial\Delta$, where $\partial\Delta$ is the boundary of Δ. Another interior of a lattice polytope Δ was introduced by J. Fine [Fin83, Rei87, Ish99, Bat17]:

Definition 1.1. Let $\Delta \subseteq M_{\mathbb{Q}}$ be a lattice d-tope. Denote by ord_Δ the piecewise linear function $N_{\mathbb{Q}} \to \mathbb{Q}$ with

$$\mathrm{ord}_\Delta(y) := \min_{x \in \Delta} \langle x, y \rangle \quad (y \in N_{\mathbb{Q}}).$$

Then the convex subset

$$\Delta^{\mathrm{FI}} := \bigcap_{n \in N \setminus \{0\}} \{x \in M_{\mathbb{Q}} \mid \langle x, n \rangle \geq \mathrm{ord}_\Delta(n) + 1\}$$

is called *Fine interior* of Δ.

One can show that only finitely many linear inequalities $\langle x, n \rangle \geq \mathrm{ord}_\Delta(n) + 1$ are necessary to define Δ^{FI}. Therefore, Δ^{FI} is a convex hull of finitely many rational points $p \in M_{\mathbb{Q}}$. Moreover, any lattice point $p \in \Delta^\circ \cap M$ in the usual interior of Δ is contained in Δ^{FI}. Therefore, Δ^{FI} contains the convex hull of $\Delta \cap M$, *i.e.*, we get the inclusion $\mathrm{conv}(\Delta^\circ \cap M) \subseteq \Delta^{\mathrm{FI}}$. In particular, Δ^{FI} is non-empty if $\Delta^\circ \cap M$ is non-empty. Moreover, for any lattice polytope Δ of dimension $d \leq 2$ one has the equality $\mathrm{conv}(\Delta^\circ \cap M) = \Delta^{\mathrm{FI}}$ [Bat17]. The Fine interior Δ^{FI} of a lattice polytope Δ of dimension $d \geq 3$ may happen to be strictly larger than the convex hull $\mathrm{conv}(\Delta^\circ \cap M)$. The simplest famous example of such a situation is due to M. Reid. Other similar examples based on hollow 3-topes can be found in Appendix B:

Example 1.2 [Rei87, Example 4.15]. Let $M \subseteq \mathbb{Q}^4$ be 3-dimensional affine lattice defined by

$$M := \{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 \mid \sum_{i=1}^{4} m_i = 5, \sum_{i=1}^{4} i m_i \equiv 0 \pmod 5\}.$$

Consider the M-lattice 3-tope $\Delta \subseteq M_{\mathbb{Q}}$ defined as the convex hull of 4 lattice points

$$(5, 0, 0, 0),\ (0, 5, 0, 0),\ (0, 0, 5, 0),\ \text{and } (0, 0, 0, 5) \in M.$$

Then $\mathrm{conv}(\Delta^\circ \cap M) = \varnothing$, but Δ^{FI} is the 3-dimensional M-rational simplex

$$\mathrm{conv}((2, 1, 1, 1), (1, 2, 1, 1), (1, 1, 2, 1), (1, 1, 1, 2))$$

and $\Delta^{\mathrm{FI}} \cap M$ is empty.

In this paper, we are interested in lattice d-topes $\Delta \subseteq M_{\mathbb{Q}}$ obtained as Newton polytopes of Laurent polynomials f_Δ in d variables $x_1, \dots, x_d$, *i.e.*,

$$f_\Delta(\mathbf{x}) = \sum_{m \in \Delta \cap M} a_m \mathbf{x}^m,$$

where $a_m \in \mathbb{C}$ are sufficiently general complex numbers. The importance of Fine interior is explained by the following theorem [Rei87, Ish99, Bat17]:

Theorem 1.3. *Let $\mathcal{Z}_\Delta \subseteq \mathbb{T}^d$ be a non-degenerate affine hypersurface in the d-dimensional algebraic torus $\mathbb{T}^d$ defined by a Laurent polynomial f_Δ with Newton d-tope Δ. Then the following conditions are equivalent:*

(i) *a smooth projective compactification $\mathcal{V}_\Delta$ of $\mathcal{Z}_\Delta$ has non-negative Kodaira dimension,* i.e., $\kappa \geq 0$*;*
(ii) *$\mathcal{Z}_\Delta$ is birational to a minimal model $\mathcal{S}_\Delta$ with abundance;*
(iii) *the Fine interior Δ^{FI} of Δ is non-empty.*

Remark 1.4. By well known results of Khovanskii [Kho78], one has vanishing of the cohomology groups

$$h^i(\mathcal{O}_{\mathcal{V}_\Delta}) = 0 \ \ (1 \leq i \leq d-2)$$

and the equation $h^{d-1}(\mathcal{O}_{\mathcal{V}_\Delta}) = |\Delta^\circ \cap M|$. The numbers $h^i(\mathcal{O}_{\mathcal{V}_\Delta})$ are birational invariants of $\mathcal{Z}_\Delta$, because they do not depend on a smooth projective compactification $\mathcal{V}_\Delta$ of $\mathcal{Z}_\Delta$. In particular, the number $|\Delta^\circ \cap M|$ is the geometric genus p_g of the affine hypersurface $\mathcal{Z}_\Delta \subseteq \mathbb{T}^d$.

Smooth projective compactifications of non-degenerate hypersurfaces in $\mathbb{T}^d$ can be obtained using the theory of toric varieties [Kho78].

Let $\Delta \subseteq M_{\mathbb{Q}}$ be a lattice d-tope. We consider the *normal fan* Σ^Δ of Δ in the dual space $N_{\mathbb{Q}}$, *i.e.*, $\Sigma^\Delta := \left\{\sigma^\theta \,\middle|\, \theta \preceq \Delta\right\}$, where σ^θ is the cone generated by all inward-pointing facet normals of facets containing the face $\theta \preceq \Delta$ of Δ. One has $\dim(\sigma^\theta) + \dim(\theta) = d$ for any face $\theta \preceq \Delta$. We denote by X_Δ the normal projective toric variety constructed via the normal fan Σ^Δ. In particular, the above function $\mathrm{ord}_\Delta : N_{\mathbb{Q}} \to \mathbb{Q}$ is a piecewise linear function with respect to this fan defining an ample Cartier divisor on X_Δ. In particular, the cones $\sigma^\theta \in \Sigma^\Delta$ are be defined as

$$\sigma^\theta = \{y \in N_{\mathbb{Q}} \,|\, \mathrm{ord}_\Delta(y) = \langle x, y\rangle \quad \text{for all } x \in \theta\}\,.$$

Remark 1.5. Using the normal fan Σ^Δ, one can compute the fundamental group $\pi_1(\mathcal{V}_\Delta)$ of a smooth projective birational model $\mathcal{V}_\Delta$ of a non-degenerate affine hypersurface $\mathcal{Z}_\Delta$ (given as in Theorem 1.3). The fundamental group $\pi_1(\mathcal{V}_\Delta)$ does not depend on the choice of the smooth birational model and it is isomorphic to the quotient of the lattice N modulo the sublattice N' generated by all lattice points in $(d-1)$-dimensional cones σ^θ of the normal fan Σ^Δ [BK06].

Example 1.6. The minimal model $\mathcal{S}_\Delta$ of a non-degenerate affine surface $\mathcal{Z}_\Delta$ defined by a Laurent polynomial with the Newton polytope Δ from Example 1.2 is a *Godeaux surface*. It is a surface of general type with $p_g = q = 0$, $K^2 = 1$, and $\pi_1(\mathcal{S}_\Delta) \cong \mathbb{Z}/5\mathbb{Z}$.

Definition 1.7. A lattice d-tope Δ is called *canonical Fano d-tope* if $|\Delta^\circ \cap M| = 1$. Up to a shift by a lattice vector, we will assume without loss of generality that $0 \in M$ is the single lattice point in the interior Δ° of the canonical Fano d-tope Δ, *i.e.*, $\Delta^\circ \cap M = \{0\}$.

All canonical Fano 3-topes have been classified [Kas10]. There exists exactly 674,688 canonical Fano 3-topes Δ. The aim of this paper is to present computational results of their Fine interiors Δ^{FI} and some related combinatorial invariants. These data are important for computing minimal smooth projective surfaces $\mathcal{S}_\Delta$ with $p_g = 1$ and $q = 0$ which are birational to affine non-degenerate hypersurfaces $\mathcal{Z}_\Delta \subseteq \mathbb{T}^3 \cong (\mathbb{C}^\times)^3$.

The simplest description of the minimal surface $\mathcal{S}_\Delta$ has been obtained when Δ is a reflexive 3-tope [Bat94].

Definition 1.8. A d-dimensional lattice polytope $\Delta \subseteq M_{\mathbb{Q}}$ containing the origin $0 \in M$ in its interior is called *reflexive* if the dual polytope

$$\Delta^* := \{y \in N \,|\, \langle x, y\rangle \geq -1 \ \text{ for all } x \in \Delta\} \subseteq N_{\mathbb{Q}}$$

is a lattice polytope.

There exists exactly 4,319 reflexive 3-topes classified by Kreuzer and Skarke [KS98] and they form a small subset in the list of all 674,688 canonical Fano 3-topes [Kas10]. Reflexive 4-topes are also classified by Kreuzer and Skarke [KS00]. There exist 473,800,776 reflexive 4-topes, but the complete list of all canonical Fano 4-topes is unknown and expected to be much bigger.

If Δ is a reflexive d-tope, then X_Δ is a Gorenstein toric Fano d-fold and the Zariski closure $\overline{Z}_\Delta$ in X_Δ is a Gorenstein Calabi-Yau $(d-1)$-fold. If $d = 3$, then $\overline{Z}_\Delta$ is a $K3$-surface with at worst finitely many Du Val singularities of type A_k. The minimal surface $\mathcal{S}_\Delta$ is a smooth $K3$-surface which is obtained as the minimal (crepant) desingularization of $\overline{Z}_\Delta$ [Bat94].

One motivation for the present paper is due to Corti and Golyshev, who have found 9 interesting examples of canonical Fano 3-simplices Δ such that the affine surfaces $\mathcal{Z}_\Delta$ are birational to elliptic surfaces of Kodaira dimension $\kappa = 1$ [CG11].

The computation of the Fine interior Δ^{FI} for all canonical Fano 3-topes $\Delta \subseteq M_\mathbb{Q}$ has shown that the dimension of the Fine interior Δ^{FI} has only three values: 0, 1, and 3. It is rather surprising that there are no canonical Fano 3-topes Δ with $\dim(\Delta^{\mathrm{FI}}) = 2$.

The condition $\dim(\Delta^{\mathrm{FI}}) = 0$ holds if and only if Δ^{FI} equals the lattice point $0 \in M$. There exist exactly 665,599 canonical Fano 3-topes with $\Delta^{\mathrm{FI}} = \{0\}$, where $0 \in M$ is the only interior lattice point of Δ. These polytopes are characterized in [Bat17, Proposition 3.4] by the condition that $0 \in N$ is an interior lattice point of the 3-dimensional lattice polytope

$$[\Delta^*] := \mathrm{conv}(\Delta^* \cap N).$$

Remark 1.9. If Δ is a canonical Fano 3-tope, then $\Delta^{\mathrm{FI}} = \{0\}$ if and only if the non-degenerate affine surface $\mathcal{Z}_\Delta$ is birational to a $K3$-surface [Bat17, Theorem 2.26].

The case $\dim(\Delta^{\mathrm{FI}}) = 1$ splits in two subcases. There exists exactly 20 canonical Fano 3-topes Δ such that $0 \in M$ is the midpoint of the Fine interior Δ^{FI}. Therefore, we call this Fine interior *symmetric*. Canonical Fano 3-topes with 1-dimensional symmetric Fine interior are characterized by the condition that $[\Delta^*]$ is a 2-dimensional reflexive polytope. The Fine interior of the remaining 9,020 canonical Fano 3-topes with $\dim(\Delta^{\mathrm{FI}}) = 1$ contains $0 \in M$ as a vertex. Therefore, we call this Fine interior *asymmetric*. Canonical Fano 3-topes with 1-dimensional asymmetric Fine interior are combinatorially characterized by the condition that $0 \in N$ is contained in the relative interior of a facet $\Theta \preceq [\Delta^*]$ of the lattice 3-tope $[\Delta^*]$. The minimal surfaces $\mathcal{S}_\Delta$ corresponding to canonical Fano 3-topes with 1-dimensional Fine interior (symmetric and asymmetric) are elliptic surfaces of Kodaira dimension $\kappa = 1$.

There exist exactly 49 canonical Fano 3-topes with $\dim(\Delta^{\mathrm{FI}}) = 3$. These polytopes are characterized by the condition that $0 \in N$ is a vertex of the 3-dimensional lattice polytope $[\Delta^*]$. The surfaces $\mathcal{S}_\Delta$ corresponding to canonical Fano 3-topes Δ with 3-dimensional Fine interior Δ^{FI} are of general type (*i.e.*, $\mathcal{S}_\Delta$ has maximal Kodaira dimension $\kappa = \dim(\mathcal{S}_\Delta) = 2$).

Remark 1.10. The Fine interior computations were done using

$$\Delta^{\mathrm{FI}} = \bigcap_{\theta \preceq \Delta} \ \bigcap_{n \in \mathcal{H}(\sigma^\theta)} \{x \in M_\mathbb{Q} \,|\, \langle x, n\rangle \geq \mathrm{ord}_\Delta(n) + 1\},$$

where $\mathcal{H}(\sigma^\theta)$ denotes the set of all irreducible elements in the monoid $\sigma^\theta \cap N$. It is the minimal generating set of the monoid $\sigma^\theta \cap N$ and is called *Hilbert basis* of $\sigma^\theta \cap N$.

In the next sections we consider examples and discuss additional properties of canonical Fano 3-topes Δ in dependence of their Fine interiors Δ^{FI}. All computations were done using the Graded Ring Database[1], including the data of all 674,688 canonical Fano 3-topes and MAGMA[2]. Therefore, all statements have been checked by computer calculations. The canonical Fano 3-topes used as examples in this chapter appear with an ID that is the example's ID in the Graded Ring Database.[3]

[1] http://www.grdb.co.uk

[2] http://magma.maths.usyd.edu.au/magma/

[3] http://www.grdb.co.uk/forms/toricf3c

2. Almost reflexive polytopes of dimension 3 and 4

Definition 2.1. A canonical Fano d-tope $\Delta \subseteq M_{\mathbb{Q}}$ is called *almost reflexive* if the convex hull of all N-lattice points in the dual polytope Δ^* is reflexive.

It is easy to show the following statement:

Proposition 2.2. *If a canonical Fano d-tope Δ is almost reflexive, then*

$$\Delta^{FI} = \{0\}.$$

Proof. If $[\Delta^*]$ is reflexive, then $\Delta = (\Delta^*)^*$ is contained in the dual reflexive polytope $[\Delta^*]^*$. Therefore, the Fine interior of Δ is contained in the Fine interior of the reflexive polytope $[\Delta^*]^*$ and $([\Delta^*]^*)^{\mathrm{FI}} = \{0\}$. Thus, $\Delta^{\mathrm{FI}} = \{0\}$. □

The converse statement is not true in general for $d \geq 5$, but there exist many equivalent characterizations of reflexive and almost reflexive d-topes among canonical Fano d-topes if $d = 3$ or $d = 4$.

Let us recall some combinatorial invariants of arbitrary lattice d-topes.

Definition 2.3. The *Ehrhart power series* of an arbitrary lattice d-tope $\Delta \subseteq M_{\mathbb{Q}}$ is defined as

$$P_\Delta(t) := \sum_{k \geq 0} |k\Delta \cap M|\, t^k,$$

where $|k\Delta \cap M|$ denotes the number of lattice points in the k-th dilate $k\Delta$ of Δ.

This Ehrhart series is a rational function of the form

$$P_\Delta(t) = \frac{\psi_d(\Delta)t^d + \cdots + \psi_1(\Delta)t + \psi_0(\Delta)}{(1-t)^{d+1}},$$

where $\psi_i(\Delta)$ are non-negative integers for all $0 \leq i \leq d$ [Sta80] such that $\psi_0(\Delta) = 1$ and $\psi_1(\Delta) = |\Delta \cap M| - d - 1$. Moreover, $\sum_{i=0}^d \psi_i(\Delta) = v(\Delta)$, where $v(\Delta) := d! \cdot \mathrm{vol}(\Delta)$ denotes the normalized volume of Δ.

One has the following characterization of reflexive d-topes:

Proposition 2.4 [BR15, Theorem 4.6]. *A canonical Fano d-tope Δ is reflexive if and only if*

$$\psi_i(\Delta) = \psi_{d-i}(\Delta) \ \ (0 \leq i \leq d).$$

The Ehrhart reciprocity implies that the power series

$$Q_\Delta(t) := \sum_{k \geq 1} |(k\Delta)^\circ \cap M|\, t^k$$

is a rational function

$$Q_\Delta(t) = \frac{\varphi_{d+1}(\Delta)t^{d+1} + \cdots + \varphi_2(\Delta)t + \varphi_1(\Delta)t + \varphi_0(\Delta)}{(1-t)^{d+1}},$$

where $\varphi_0(\Delta) = 0$ and $\varphi_1(\Delta) = |\Delta^\circ \cap M|$. Using Serre duality, one obtains

$$\varphi_i(\Delta) = \psi_{d+1-i}(\Delta) \ \ (1 \leq i \leq d+1),$$

i.e., in particular

$$\psi_d(\Delta) = \varphi_1(\Delta) = |\Delta^\circ \cap M|$$

and

$$\psi_{d-1}(\Delta) = \varphi_2(\Delta) = |2\Delta^\circ \cap M| - (d+1)|\Delta^\circ \cap M|$$

[DK86, Section 4, 5.11]. Therefore, the lattice d-tope Δ is a canonical Fano d-tope if and only if $\psi_d(\Delta) = 1$. Moreover,

$$\psi_{d-1}(\Delta) = |(2\Delta)^\circ \cap M| - (d+1)$$

if Δ is a canonical Fano d-tope.

Applying the above equations, one immediately obtains the following criterion for reflexivity of canonical Fano d-topes in the case $d = 3, 4$:

Proposition 2.5. *Let $\Delta \subseteq M_{\mathbb{Q}}$ be a canonical Fano d-tope with $d \in \{3,4\}$. Then for $d = 3$, one has*

$$P_\Delta(t) = \frac{t^3 + (|(2\Delta)^\circ \cap M| - 4)t^2 + (|\Delta \cap M| - 4)t + 1}{(1-t)^4}$$

and for $d = 4$, one obtains

$$P_\Delta(t) = \frac{t^4 + (|(2\Delta)^\circ \cap M| - 5)t^3 + \psi_2(\Delta)t^2 + (|\Delta \cap M| - 5)t + 1}{(1-t)^5}.$$

In particular, Δ is reflexive if and only if

$$|\Delta \cap M| = |(2\Delta)^\circ \cap M|.$$

Proposition 2.6. *Let $\Delta \subseteq M_{\mathbb{Q}}$ be a canonical Fano d-tope with $d \in \{3,4\}$ such that $0 \in N$ is an interior lattice point of $[\Delta^*]$. Then $[\Delta^*]$ is reflexive,* i.e., *Δ is almost reflexive.*

Proof. Let $n \in N$ be an interior lattice point of $[\Delta^*]$. Then $\langle x, n\rangle \geq 0$ for all $x \in \Delta \cap M$ because

$$\Delta^* = \{y \in N_{\mathbb{Q}} \mid \langle x, y\rangle \geq -1 \quad \text{for all } x \in \Delta\}$$

and $\langle x, n\rangle$ is an integer. Since $0 \in \Delta^\circ \cap M$, $M_{\mathbb{Q}}$ is the set of all non-negative $\mathbb{Q}$-linear combinations of all lattice points in $\Delta \cap M$. This implies $\langle x', n\rangle \geq 0$ for all $x' \in M_{\mathbb{Q}}$, *i.e.*, $n = 0$. Therefore, $[\Delta^*]$ has only one interior lattice point $0 \in N$, *i.e.*, $[\Delta^*]$ is a canonical Fano d-tope.

It is clear that $[\Delta^*]$ is contained in the interior of $2[\Delta^*]$. Therefore, we have $[\Delta^*] \cap N \subseteq (2[\Delta^*])^\circ \cap N$. On the other hand, for any lattice point $n \in (2[\Delta^*])^\circ$, $\langle x, n\rangle > -2$ for all $x \in \Delta \cap M$. Since $\langle x, n\rangle$ is an integer, $n \in \Delta^* \cap N$, *i.e.*,

$$[\Delta^*] \cap N = (2[\Delta^*])^\circ \cap N.$$

Using Proposition 2.5, $[\Delta^*]$ is reflexive. □

Corollary 2.7. *Let $\Delta \subseteq M_{\mathbb{Q}}$ be a canonical Fano d-tope with $d \in \{3,4\}$ such that $0 \in N$ is an interior lattice point of $[\Delta^*]$. Then $[\Delta^*]^*$ is the smallest (referring to inclusion) reflexive polytope containing Δ.*

Proof. Let $\Delta' \subseteq M_{\mathbb{Q}}$ be a reflexive d-tope such that $\Delta \subseteq \Delta'$. Then $(\Delta')^* \subseteq \Delta^*$. Since $(\Delta')^*$ is a lattice polytope, it is contained in $[\Delta^*]$. Thus, $[\Delta^*]^*$ is contained in $((\Delta')^*)^* = \Delta'$. □

Remark 2.8. If Δ is a reflexive d-tope, then $[2\Delta^\circ] = \Delta$. If Δ is a canonical Fano d-tope with $d \in \{3,4\}$ such that $\Delta^{\mathrm{FI}} = \{0\}$ and Δ is contained in a reflexive d-tope Δ', then $[2\Delta^\circ]$ is contained in $[(2\Delta')^\circ] = \Delta'$. Therefore, $[2\Delta^\circ]$ is contained in the smallest reflexive polytope $[\Delta^*]^*$ containing Δ, *i.e.*,

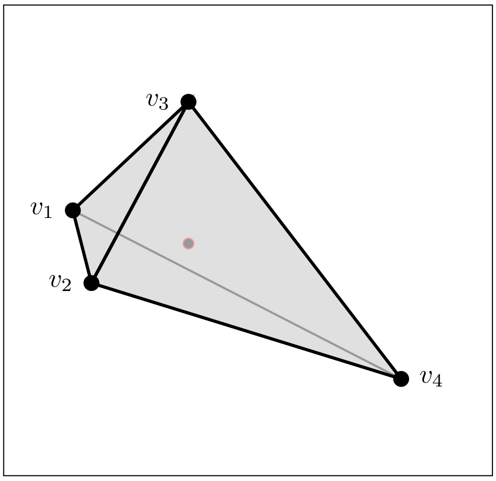
$$[2\Delta^\circ] \subseteq [\Delta^*]^*.$$

Computations showed that among all 665,599 canonical Fano 3-topes Δ with $\Delta^{\mathrm{FI}} = \{0\}$ there exist exactly 211,941 canonical Fano 3-tops such that $[2\Delta^\circ]$ is reflexive. For the remaining canonical Fano 3-topes Δ the lattice 3-topes $[2\Delta^\circ]$ are larger than Δ, but are not equal to the reflexive hull $[\Delta^*]^*$.

Remark 2.9. Let Δ be an almost reflexive 3-tope. We denote by $\tau(\Delta)$ the lattice d-tope $[2\Delta^\circ]$. If $\tau(\Delta)$ is not reflexive, then it is almost reflexive and we can consider the larger lattice d-tope $\tau^2(\Delta) := \tau(\tau(\Delta)) \subseteq [\Delta^*]^*$. After at morst five steps, $\tau^k(\Delta)$ is equal to the reflexive hull $[\Delta^*]^*$ of Δ.

In dimension 4, the situation is comparable:

Example 2.10. Let $\Delta \subseteq \mathbb{R}^4$ be the almost reflexive 4-tope defined by the inequalities $x_i \geq -1$ $(1 \leq i \leq 4)$, $x_1 \leq 2$, and $x_1 + x_2 + x_3 + x_4 \leq 1$. Then $\Delta^{FI} = \{0\}$ and the smallest reflexive 4-tope containing Δ is the 4-simplex $[\Delta^*]^*$ defined by the inequalities $x_i \geq -1$ $(1 \leq i \leq 4)$ and $x_1 + x_2 + x_3 + x_4 \leq 1$. It is easy to see that $\tau(\Delta)$ is not the reflexive 4-tope $[\Delta^*]^*$ because the vertex $(4,-1,-1,-1) \in \mathrm{vert}([\Delta^*]^*)$ is not in $2\Delta^\circ$. However, $\tau^2(\Delta) = [\Delta^*]^*$.



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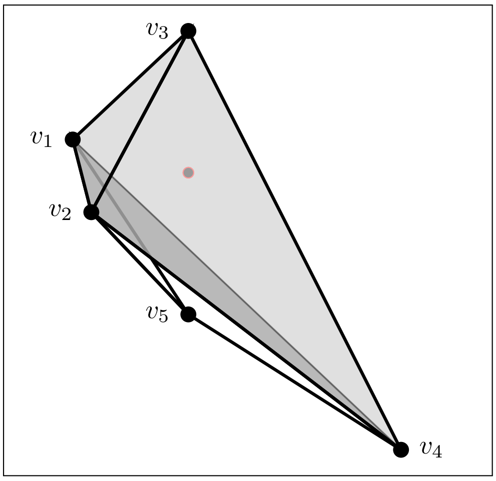

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FIGURE 1. **Canonical Fano 3-topes Δ with $\Delta^{\mathrm{FI}} = \{0\}$.** Shaded faces are occluded and the Fine interior $\{0\}$ is coloured grey with a red margin. The whole polytope is the canonical hull Δ^{can} as well as the reflexive hull Δ^{ref} and the grey coloured polytope is Δ. ***(a)*** Reflexive polytope $\Delta = \mathrm{conv}(v_1, v_2, v_3, v_4)$ with $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$, $v_4 = (-1, -1, -1)$, and $\Delta^{\mathrm{ref}} = \Delta^{\mathrm{can}} = \Delta$. All facets of Δ have lattice distance 1 to the origin. ***(b)*** Almost reflexive polytope $\Delta = \mathrm{conv}(v_1, v_2, v_3, v_4)$ with $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$, $v_4 = (-1, -1, -2)$, and $\Delta^{\mathrm{ref}} = \Delta^{\mathrm{can}} = \mathrm{conv}(v_1, v_2, v_3, v_4, v_5)$ with $v_5 = (0, 0, -1)$ reflexive. The dark grey coloured facet of Δ has lattice distance 2 and all other facets have lattice distance 1 to the origin.

3. Canonical Fano 3-topes with $\Delta^{\mathrm{FI}} = \{0\}$

We note that the set of all reflexive 3-topes forms a rather small part of the set of all canonical Fano 3-topes. The majority of canonical Fano 3-topes belongs to the subset of almost reflexive 3-topes. The proof of the following statement is based on the result of Skarke [Ska96] and the explanations in the previous section:

Proposition 3.1. *A canonical Fano 3-tope Δ is almost reflexive if one of the following equivalent conditions is satisfied:*

(i) $\Delta^{\mathrm{FI}} = \{0\}$;
(ii) $0 \in N$ *is an interior lattice point of* $[\Delta^*]$*;*
(iii) Δ *is contained in some reflexive* 3*-tope;*
(iv) $\tau^k(\Delta)$ *is the reflexive* 3*-tope* $[\Delta^*]^*$ *for some sufficiently large* k $(1 \le k \le 5)$*;*
(v) *the lattice* 3*-tope* $[2\Delta^\circ]$ *has exactly one interior lattice point;*
(vi) *the non-degenerate affine hypersurface* $\mathcal{Z}_\Delta$ *defined by a Laurent polynomial with Newton polytope* Δ *is birational to a smooth* $K3$*-surface.*

Computations show that there exist exactly 665,599 almost reflexive canonical Fano 3-topes. The set of almost reflexive 3-topes includes all 4,319 reflexive 3-topes. We have shown that for any almost reflexive 3-tope Δ, the reflexive polytope $\Delta^{\mathrm{ref}} := [\Delta^*]^*$ is the smallest reflexive 3-tope containing Δ. We call Δ^{ref} the *reflexive hull* of Δ. Thus we obtain a natural surjective map $\Delta \mapsto \Delta^{\mathrm{ref}}$ from the set of almost reflexive 3-topes to the set of reflexive 3-topes, which is the identity on the set of reflexive 3-topes. The minimal surface $\mathcal{S}_\Delta$ is a $K3$-surface if and only if Δ is an almost reflexive 3-tope. If Δ is an almost reflexive 3-tope, but not reflexive, then the minimal surface $\mathcal{S}_\Delta$ is a crepant desingularization of the Zariski closure of $\mathcal{Z}_\Delta$ in the Gorenstein toric Fano threefold $X_{\Delta^{\mathrm{ref}}}$ defined by the reflexive hull of Δ.

A generalization of the reflexive hull of almost reflexive 3-topes for arbitrary lattice d-topes with non-empty Fine interior can be obtained using the notion of support of the Fine interior Δ^{FI}.

Definition 3.2. Let $\Delta \subseteq M_{\mathbb{Q}}$ a lattice d-tope with $\Delta^{\mathrm{FI}} \neq \varnothing$. Then the set

$$\mathrm{supp}(\Delta^{\mathrm{FI}}) := \{n \in N \mid \text{ there exists } x \in \Delta^{\mathrm{FI}} \text{ with } \langle x, n \rangle = \mathrm{ord}_{\Delta}(n) + 1\}$$

is called *support of the Fine interior* of Δ.

Example 3.3. If Δ is a reflexive d-tope, then the support of the Fine interior of Δ is the set of all non-zero lattice points in $\Delta^* \cap N$.

Remark 3.4. It is easy to show that one always has

$$\Delta^{\mathrm{FI}} = \bigcap_{n \in \mathrm{supp}(\Delta^{\mathrm{FI}})} \{x \in M_{\mathbb{Q}} \mid \langle x, n \rangle \geq \mathrm{ord}_{\Delta}(n) + 1\}.$$

Definition 3.5. Let $\Delta \subseteq M_{\mathbb{Q}}$ a lattice d-tope with $\Delta^{\mathrm{FI}} \neq \varnothing$. Then the rational polytope

$$\Delta^{\mathrm{can}} := \bigcap_{n \in \mathrm{supp}(\Delta^{\mathrm{FI}})} \{x \in M_{\mathbb{Q}} \mid \langle x, n \rangle \geq \mathrm{ord}_{\Delta}(n)\}$$

contains Δ and is called *canonical hull* of Δ.

Example 3.6. If Δ is an almost reflexive 3-tope, then $\mathrm{supp}(\Delta^{\mathrm{FI}})$ is the set $(\Delta^* \cap N) \setminus \{0\}$ of boundary lattice points in the reflexive 3-tope $[\Delta^*]$ and the canonical hull Δ^{can} equals the reflexive hull Δ^{ref} of the polytope Δ, *i.e.*, $\Delta^{\mathrm{can}} = \Delta^{\mathrm{ref}} = [\Delta^*]^*$. In particular, in this case Δ^{can} is always a lattice 3-tope.

There exist a smooth projective toric variety X_{Σ} defined by a fan Σ whose 1-dimensional cones are generated by all lattice vectors from the finite set $\mathrm{supp}(\Delta^{\mathrm{FI}})$. Then the minimal surface $\mathcal{S}_{\Delta}$ is a $K3$-surface which is the Zariski closure of $\mathcal{Z}_{\Delta}$ in X_{Σ} [Bat17].

Example 3.7. Let us consider the (almost) reflexive canonical Fano 3-tope $\Delta = \mathrm{conv}(v_1, v_2, v_3, v_4) \subseteq M_{\mathbb{Q}}$ (ID 547386, Figure 1(a)) with vertices

$$v_1 := (1, 0, 0),\ v_2 := (0, 1, 0),\ v_3 := (0, 0, 1), \text{ and } v_4 := (-1, -1, -1)$$

and $\Delta^{\mathrm{FI}} = \{0\}$. Moreover,

$$\Delta^{\mathrm{ref}} = \mathrm{conv}(2\Delta^{\circ} \cap M) = \mathrm{conv}(\Delta \cap M) = \Delta$$

and

$$\Delta^{\mathrm{can}} = [\Delta^*]^* = (\Delta^*)^* = \Delta$$

because Δ is reflexive, *i.e.*, $\Delta^{\mathrm{ref}} = \Delta^{\mathrm{can}} = \Delta$ reflexive (Figure 1(a)).

Example 3.8. Let us consider the almost reflexive canonical Fano 3-tope $\Delta = \mathrm{conv}(v_1, v_2, v_3, v_4) \subseteq M_{\mathbb{Q}}$ (ID 547385, Figure 1(b)) with vertices

$$v_1 := (1, 0, 0),\ v_2 := (0, 1, 0),\ v_3 := (0, 0, 1), \text{ and } v_4 := (-1, -1, -2)$$

and $\Delta^{\mathrm{FI}} = \{0\}$. Moreover,

$$\Delta^{\mathrm{ref}} = \mathrm{conv}((\Delta \cap M) \cup \{v_5\}) = \mathrm{conv}(v_1, v_2, v_3, v_4, v_5)$$

and

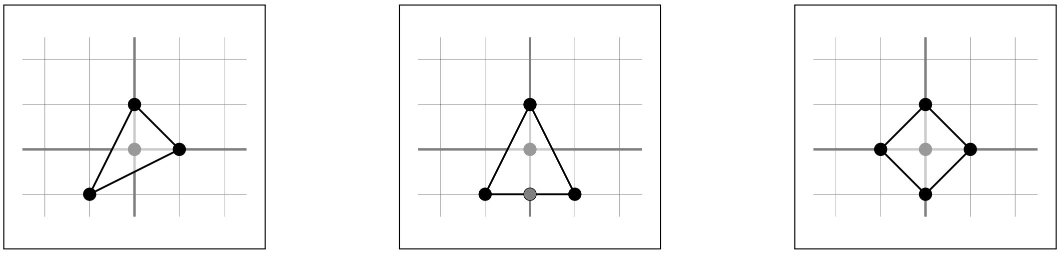
$$\Delta^{\mathrm{can}} = [\Delta^*]^* = \mathrm{conv}(v_1, v_2, v_3, v_4, v_5)$$

with $v_5 := (0, 0, -1)$ because Δ is almost reflexive, *i.e.*, $\Delta^{\mathrm{ref}} = \Delta^{\mathrm{can}} = \Delta$ reflexive (Figure 1(b)).

4. Asymmetric Fine interior of dimension 1

There exist exactly 9,020 canonical Fano 3-topes Δ with 1-dimensional Fine interior such that $0 \in N$ belongs to a facet $\Theta \preceq [\Delta^*]$ of the lattice 3-tope $[\Delta^*]$. This class of canonical Fano 3-topes is characterized by the property that the lattice 3-tope $[2\Delta^{\circ}]$ has exactly 2 interior lattice points.

The corresponding minimal surfaces $\mathcal{S}_{\Delta}$ are *simply connected* (*i.e.*, have trivial fundamental group $\pi_1(\mathcal{S}_{\Delta})$) elliptic surfaces of Kodaira dimension $\kappa = 1$. We observed that the facet $\Theta \preceq [\Delta^*]$ is a reflexive 2-tope corresponding to one of the three types pictured in Figure 2. All N-lattice points on the boundary of Θ belong to $\mathrm{supp}(\Delta^{\mathrm{FI}})$. It was checked that for all these 3-topes Δ the canonical hull Δ^{can} is again a lattice 3-tope. Moreover, the Fine interior Δ^{FI} is contained in the ray generated by the primitive lattice



(A) (B) (C)

FIGURE 2. **Reflexive Facets of Δ Containing $\pm v_\Delta$.** Three types of reflexive facets $\theta_\pm \preceq \Delta$ of Δ containing $\pm v_\Delta$ for all 9,020+20 canonical Fano 3-topes Δ with $\dim(\Delta^{\mathrm{FI}}) = 1$. Vertices are coloured black, boundary points that are not vertices grey, and the origin light grey. ***(a)*** $\operatorname{conv}((1,0),(0,1),(-1,-1))$. ***(b)*** $\operatorname{conv}((1,0),(-1,1),(-1,-1))$. ***(c)*** $\operatorname{conv}((\pm 1,0),(0,\pm 1))$.

vector $v_\Delta \in M$ which is the primitive inward-pointing facet normal of Θ, *i.e.*, $\langle x, y\rangle = 0$ for all $x \in \Delta^{\mathrm{FI}}$, $y \in \Theta$. The lattice point $0 \in M$ is a vertex of Δ^{FI}. More precisely, one has

$$\Delta^{\mathrm{FI}} = \operatorname{conv}(0, \lambda v_\Delta),$$

where $\lambda \in \{1/2, 2/3\}$. The primitive lattice vector v_Δ is the unique interior lattice point on a reflexive facet $\theta_+ \preceq \Delta$ of Δ of one of the three possible types pictured in Figure 2. These three reflexive polygons θ_+ are characterized by the condition that the dual reflexive polygons θ_+^* are obtained from θ_+ (Figure 3) by enlarging the lattice $\mathbb{Z}^2$ in the following ways: $\mathbb{Z}^2 + \mathbb{Z}(1/3, 2/3)$ (Figure 3(a)), $\mathbb{Z}^2 + \mathbb{Z}(1/2, 0)$ (Figure 3(b)), and $\mathbb{Z}^2 + \mathbb{Z}(1/2, 1/2)$ (Figure 3(c)). Moreover, the reflexive facet θ_+ of Δ is isomorphic to the facet Θ of $[\Delta^*]$. The projection $M \to M/\mathbb{Z}v_\Delta$ of Δ or of θ_+ along v_Δ is a reflexive polygon of one of the three types pictured in Figure 3, which is dual to θ_+ and Θ. The lattice vector v_Δ defines a character of the 3-dimensional torus $\chi : \mathbb{T}^3 \to \mathbb{C}^\times$. For almost all $\alpha \in \mathbb{C}^\times$, the fiber $\chi^{-1}(\alpha)$ is an affine elliptic curve defined by a Laurent polynomial with the reflexive Newton polytope $\Theta^* \cong \theta_+^*$ of one of the three types pictured in Figure 3 with the distribution shown in Table 1. So χ defines birationally an elliptic fibration.

$\theta_\pm$	$\theta_\pm^*$	enlarged lattice	$\#\Delta_{\mathrm{asym}}$	$\#\Delta_{\mathrm{sym}}$
Figure 2(a)	Figure 3(a)	$\mathbb{Z}^2 + \mathbb{Z}(1/3, 2/3)$	3,038	7
Figure 2(b)	Figure 3(b)	$\mathbb{Z}^2 + \mathbb{Z}(1/2, 0)$	4,663	9
Figure 2(c)	Figure 3(c)	$\mathbb{Z}^2 + \mathbb{Z}(1/2, 1/2)$	1,319	4

TABLE 1. **Distribution of the Reflexive Facets of Δ Containing $\pm v_\Delta$.** Table contains: Type of the reflexive facet $\theta_\pm$ containing $\pm v_\Delta$, type of the dual reflexive facet $\theta_\pm^*$, the enlarged lattice used to obtain $\theta_\pm^*$ from $\theta_\pm$, the number of canonical Fano 3-topes $\Delta_{\mathrm{asym}} := \{\Delta \mid \text{1-dim. asym. } \Delta^{\mathrm{FI}}\}$, and the number of canonical Fano 3-topes $\Delta_{\mathrm{sym}} := \{\Delta \mid \text{1-dim. sym. } \Delta^{\mathrm{FI}}\}$ with respect to the facet type of $\theta_\pm$ pictured in Figure 2.

Example 4.1. Let $\Delta \subseteq M_{\mathbb{Q}}$ be a canonical Fano 3-tope given as the convex hull

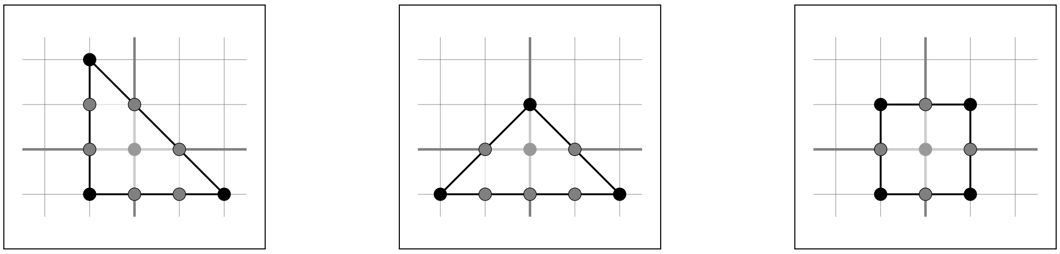
$$v_1 := (2,3,8),\ v_2 := (1,0,0),\ v_3 := (0,1,0), \text{ and } v_4 := (-1,-1,-1)$$

(ID 547324, Figure 4(a), Table A.1 and A.3). Then

$$\Delta^{\mathrm{FI}} = \operatorname{conv}((0,0,0),(1/2,1/2,1)) = \operatorname{conv}(0, 1/2 \cdot v_\Delta),$$

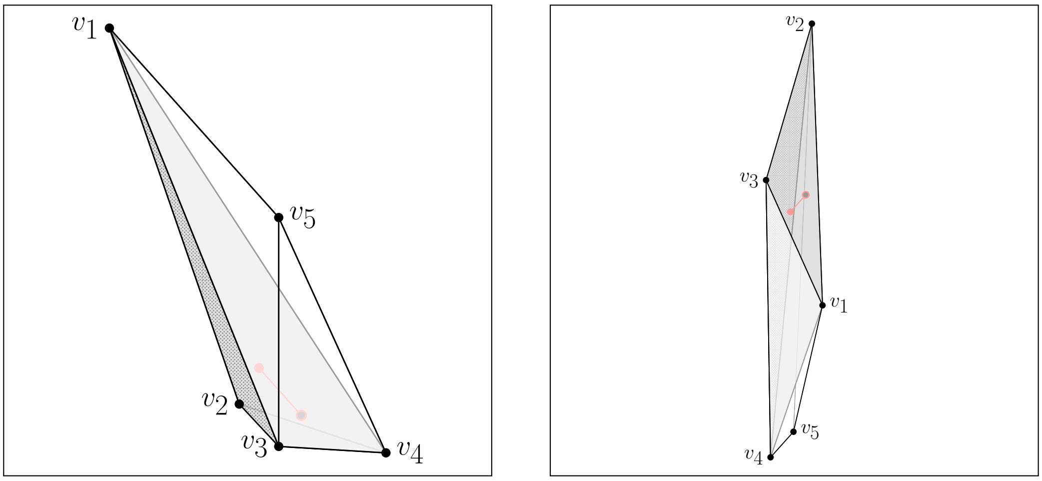
where $v_\Delta = (1,1,2)$. One has $v_1 + 2v_2 + v_3 = 4v_\Delta$. Therefore, v_Δ is the interior lattice point of the reflexive facet θ_+ of Δ (Figure 2(b)) with vertices v_1, v_2, v_3 and the images $\overline{v}_1, \overline{v}_2, \overline{v}_3$ of v_1, v_2, v_3 in $M/\mathbb{Z}v_\Delta$ are vertices of the dual reflexive triangle θ_+^* (Figure 3(b)) satisfying the relation

$$\overline{v}_1 + 2\overline{v}_2 + \overline{v}_3 = 0.$$



(A) (B) (C)

FIGURE 3. **Reflexive Projection Polytopes.** Three types of reflexive polytopes obtained via a projection of Δ along $\pm v_\Delta$ for all $9{,}020 + 20$ canonical Fano 3-topes Δ with $\dim(\Delta^{\mathrm{FI}}) = 1$ Vertices are coloured black, boundary points that are not vertices grey, and the origin light grey. ***(a)*** $\operatorname{conv}((-1,2),(-1,-1),(2,-1))$. ***(b)*** $\operatorname{conv}((-2,-1),(0,1),(2,-1))$. ***(c)*** $\operatorname{conv}((\pm1,\pm1))$.



(A) ID 547324 (B) ID 547323

FIGURE 4. **Canonical Fano 3-topes with Asymmetric Fine Interior of Dimension 1.** Shaded faces are occluded. The Fine interior is coloured red, the origin grey with a red margin, and the facet θ_+ grey dotted. **(a)** The whole polytope is $\Delta = \operatorname{conv}(v_1, v_2, v_3, v_4)$ with $v_1 = (2,3,8)$, $v_2 = (1,0,0)$, $v_3 = (0,1,0)$, $v_4 = (-1,-1,-1)$. Moreover, $\Delta^{\mathrm{FI}} = \operatorname{conv}((0,0,0),(1/2,1/2,1))$, $\theta_+ = \operatorname{conv}(v_1,v_2,v_3)$, and $\Delta^{\mathrm{can}} = \operatorname{conv}(v_1,v_2,v_3,v_4,v_5)$ with $v_5 = (0,1,4)$. **(b)** The whole polytope is $\Delta = \operatorname{conv}(v_1,v_2,v_3,v_4)$ with $v_1 = (-1,1,-2)$, $v_2 = (1,-2,3)$, $v_3 = (1,0,0)$, $v_4 = (-2,5,-3)$. Moreover, $\Delta^{\mathrm{FI}} = \operatorname{conv}((0,0,0),(0,2/3,0))$ and $\theta_+ = \operatorname{conv}(v_2,v_3,v_4)$, and $\Delta^{\mathrm{can}} = \operatorname{conv}(v_1,v_2,v_3,v_4,v_5)$ with $v_5 = (-2,4,-3)$.

To compute the canonical hull Δ^{can} of Δ, we obtain $\operatorname{supp}(\Delta^{\mathrm{FI}}) = \{s_i \,|\, 1 \le i \le 18\}$ with $s_1 \coloneqq (-1,-1,1)$, $s_2 \coloneqq (-1,-1,2)$, $s_3 \coloneqq (-1,-1,3)$, $s_4 \coloneqq (-1,0,1)$, $s_5 \coloneqq (-1,0,2)$, $s_6 \coloneqq (-1,1,0)$, $s_7 \coloneqq (-1,1,1)$, $s_8 \coloneqq (-1,2,0)$, $s_9 \coloneqq (-1,3,-1)$, $s_{10} \coloneqq (0,-1,1)$, …, $s_{18} \coloneqq (-2,-2,1)$, which leads to

$$\Delta^{\mathrm{can}} = \operatorname{conv}(v_1, v_2, v_3, v_4, v_5)$$

with $v_5 \coloneqq (0,1,4)$ (Figure 4(a)).

Example 4.2. Let $\Delta \subseteq M_{\mathbb{Q}}$ be canonical Fano 3-tope given as the convex hull

$$v_1 \coloneqq (-1,1,-2),\ v_2 \coloneqq (1,-2,3),\ v_3 \coloneqq (1,0,0), \text{ and } v_4 \coloneqq (-2,5,-3)$$

(ID 547323, Figure 4(b), Table A.1 and A.3). Then

$$\Delta^{\mathrm{FI}} = \operatorname{conv}((0,0,0),(0,2/3,0)) = \operatorname{conv}(0, 2/3 \cdot v_\Delta),$$

where $v_\Delta = (0,1,0)$. One has $v_2+v_3+v_4 = 3v_\Delta$. Therefore, v_Δ is the interior lattice point of the reflexive facet θ_+ of Δ (Figure 2(a)) with vertices v_2, v_3, v_4 and the images $\overline{v}_2, \overline{v}_3, \overline{v}_4$ of v_2, v_3, v_4 in $M/\mathbb{Z}v_\Delta$ are vertices of the dual reflexive triangle θ_+^* (Figure 3(a)) satisfying the relation

$$\overline{v}_2 + \overline{v}_3 + \overline{v}_4 = 0.$$

To compute the canonical hull Δ^{can} of Δ, we obtain $\operatorname{supp}(\Delta^{\text{FI}}) = \{s_i \,|\, 1 \le i \le 20\}$ with $s_1 \coloneqq (-3,-3,-2)$, $s_2 \coloneqq (-1,0,0)$, $s_3 \coloneqq (-1,0,1)$, $s_4 \coloneqq (-1,1,1)$, $s_5 \coloneqq (-1,2,2)$, $s_6 \coloneqq (-1,3,2)$, $s_7 \coloneqq (-1,4,3)$, $s_8 \coloneqq (-1,6,4)$, $s_9 \coloneqq (0,1,1)$, $s_{10} \coloneqq (0,2,1)$, $\ldots$, $s_{20} \coloneqq (4,1,-1)$, which leads to

$$\Delta^{\text{can}} = \operatorname{conv}(v_1, v_2, v_3, v_4, v_5)$$

with $v_5 \coloneqq (-2,4,-3)$ (Figure 4(b)).

Remark 4.3. The detailed information about a small selection of the 9,020 canonical Fano 3-topes with $\dim(\Delta^{\text{FI}}) = 1$ and $0 \in \operatorname{vert}(\Delta^{\text{FI}})$ can be found in Appendix A. To be precise, it is listed in Table A.1, A.2, and A.3 and can be viewed in [Sch18, Appendix A, Figure A1].

5. Symmetric Fine interior of dimension 1

There exist exactly 20 canonical Fano 3-topes Δ such that 0 is the center of 1-dimensional Fine interior Δ^{FI}. In this case, $\mathcal{S}_\Delta$ is an elliptic surface of Kodaira dimension $\kappa = 1$ with non-trivial fundamental group $\pi_1(\mathcal{S}_\Delta)$ of order 2 or 3. Computations show that one always has $\Delta = \Delta^{\text{can}}$ and

$$\Delta^{\text{FI}} = \operatorname{conv}(-\lambda v_\Delta, \lambda v_\Delta)$$

with $\lambda = \frac{1}{2}$ if and only if $|\pi_1(\mathcal{S}_\Delta)| = 2$ and

$$\Delta^{\text{FI}} = \operatorname{conv}(-\mu v_\Delta, \mu v_\Delta)$$

with $\mu = \frac{2}{3}$ if and only if $|\pi_1(\mathcal{S}_\Delta)| = 3$. The primitive lattice vectors $\pm v_\Delta$ are the two unique interior lattice points in two reflexive facets $\theta_\pm \preceq \Delta$ of one of the three possible types pictured in Figure 2. The reflexive facets $\theta_\pm$ of Δ are isomorphic to the facet Θ of $[\Delta^*]$. The projections $M \to M/\mathbb{Z}(\pm v_\Delta)$ of Δ or of $\theta_\pm$ along $\pm v_\Delta$ reveal a reflexive polygon of one of the three types pictured in Figure 3, which is dual to $\theta_\pm$ and Θ. The lattice vector v_Δ defines a character of the 3-dimensional torus $\chi : \mathbb{T}^3 \to \mathbb{C}^\times$. For almost all $\alpha \in \mathbb{C}^\times$, the fiber $\chi^{-1}(\alpha)$ is an affine elliptic curve defined by a Laurent polynomial with the reflexive Newton polytope $\Theta^* \cong \theta_\pm^*$ of one of the three types pictured in Figure 3 with the distribution shown in Table 1. So χ defines birationally an elliptic fibration. The vertex sets of Δ and these reflexive facets are related via $\operatorname{vert}(\Delta) = \operatorname{vert}(\theta_+) \cup \operatorname{vert}(\theta_-)$. Moreover, every edge of Δ is either an edge of θ_+ or θ_- of these two facets or it is parallel to v_Δ.

Example 5.1. Let $\Delta \subseteq M_\mathbb{Q}$ be canonical Fano 3-tope given as the convex hull

$$v_1 \coloneqq (0,1,0),\ v_2 \coloneqq (2,1,1),\ v_3 \coloneqq (-2,-3,-5), \text{ and } v_4 \coloneqq (2,1,9)$$

(ID 547393, Figure 5(a), Table A.4 and A.5). Then

$$\Delta^{\text{FI}} = \operatorname{conv}((0,0,-1/2),(0,0,1/2)) = (-\lambda v_\Delta, \lambda v_\Delta)$$

with $\lambda = \frac{1}{2}$, where $v_\Delta = (0,0,1)$. One has $2v_1 + v_3 + v_4 = 4v_\Delta$ and $2v_1 + v_2 + v_3 = 4(-v_\Delta)$. Therefore, v_Δ is the interior lattice point of the reflexive facet $\theta_+ = \theta_{134}$ of Δ and $-v_\Delta$ is the interior lattice point of the reflexive facet $\theta_- = \theta_{123}$ of Δ (Figure 2(b)). The images $\overline{v}_1, \overline{v}_3, \overline{v}_4$ of v_1, v_3, v_4 in $M/\mathbb{Z}v_\Delta$ and the images $\overline{v}_1, \overline{v}_2, \overline{v}_3$ of v_1, v_2, v_3 in $M/\mathbb{Z}(-v_\Delta)$ are vertices of the dual reflexive triangle $\theta_\pm^*$ (Figure 3(b)) satisfying the relation

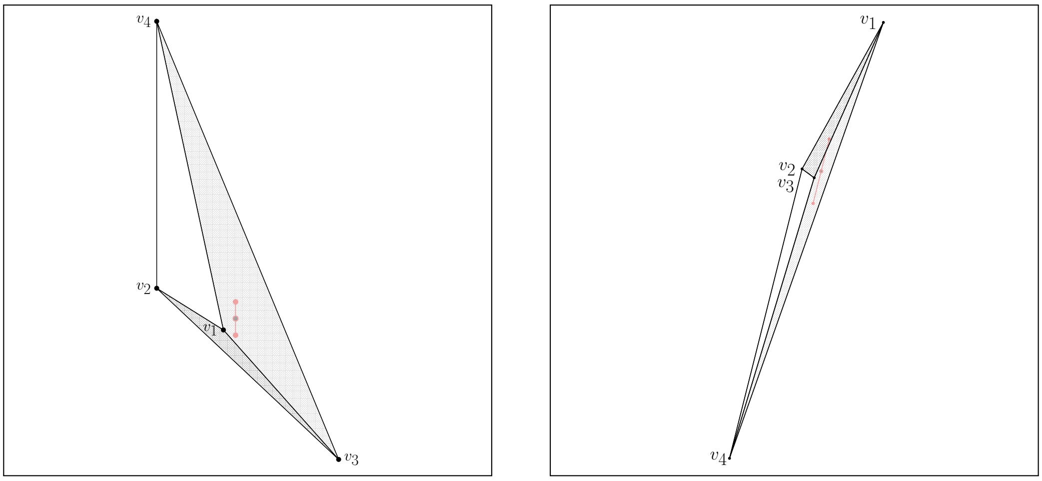
$$2\overline{v}_1 + \overline{v}_3 + \overline{v}_4 = 0$$

and

$$2\overline{v}_1 + \overline{v}_2 + \overline{v}_3 = 0,$$

respectively.

To compute the canonical hull Δ^{can} of Δ, we obtain $\operatorname{supp}(\Delta^{\text{FI}}) = \{s_i \,|\, 1 \le i \le 6\}$ with $s_1 \coloneqq (-1,-2,2)$, $s_2 \coloneqq (-1,1,0)$, $s_3 \coloneqq (0,-1,0)$, $s_4 \coloneqq (1,-1,0)$, $s_5 \coloneqq (2,-1,0)$, and $s_6 \coloneqq (9,-2,-2)$, which leads to $\Delta^{\text{can}} = \Delta$.



(A) ID 547393 (B) ID 547409

FIGURE 5. **Canonical Fano 3-topes with Symmetric Fine Interior of Dimension 1.** Shaded faces are occluded. The Fine interior is coloured red, the origin grey with a red margin, and the facets $\theta_\pm$ grey dotted. **(a)** The whole polytope is $\Delta = \operatorname{conv}(v_1, v_2, v_3, v_4)$ with $v_1 = (1, 0, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (-2, -3, -5)$, $v_4 = (2, 1, 9)$. Moreover, $\Delta^{\mathrm{FI}} = \operatorname{conv}((0, 0, -1/2), (0, 0, 1/2))$, $\theta_+ = \operatorname{conv}(v_1, v_3, v_4)$, $\theta_- = \operatorname{conv}(v_1, v_2, v_3)$, and $\Delta^{\mathrm{can}} = \Delta$. **(b)** The whole polytope is $\Delta = \operatorname{conv}(v_1, v_2, v_3, v_4)$ with $v_1 = (-4, 2, 9)$, $v_2 = (1, 0, 0)$, $v_3 = (0, 1, 0)$, $v_4 = (7, -6, -18)$. Moreover, $\Delta^{\mathrm{FI}} = \operatorname{conv}((-2/3, 2/3, 2), (2/3, -2/3, -2))$, $\theta_+ = \operatorname{conv}(v_1, v_2, v_3)$, $\theta_- = \operatorname{conv}(v_1, v_3, v_4)$, and $\Delta^{\mathrm{can}} = \Delta$.

Example 5.2. Let $\Delta \subseteq M_{\mathbb{Q}}$ be canonical Fano 3-tope given as the convex hull

$$v_1 := (-4, 2, 9),\ v_2 := (1, 0, 0),\ v_3 := (0, 1, 0), \text{ and } v_4 := (7, -6, -18)$$

(ID 547409, Figure 5(b), Table A.4 and A.5). Then

$$\Delta^{\mathrm{FI}} = \operatorname{conv}((-2/3, 2/3, 2), (2/3, -2/3, -2)) = (-\mu v_\Delta, \mu v_\Delta)$$

with $\mu = \frac{2}{3}$, where $v_\Delta = (1, -1, -3)$. One has $v_1 + v_2 + v_3 = -3v_\Delta$ and $v_1 + v_3 + v_4 = -3(-v_\Delta)$. Therefore, v_Δ is the interior lattice point of the reflexive facet $\theta_+ = \theta_{123}$ of Δ and $-v_\Delta$ is the interior lattice point of the reflexive facet $\theta_- = \theta_{134}$ of Δ (Figure 2(b)). The images $\overline{v}_1, \overline{v}_2, \overline{v}_3$ of v_1, v_2, v_3 in $M/\mathbb{Z}v_\Delta$ and the images $\overline{v}_1, \overline{v}_3, \overline{v}_4$ of v_1, v_3, v_4 in $M/\mathbb{Z}(-v_\Delta)$ are vertices of the dual reflexive triangle $\theta^*_\pm$ (Figure 3(b)) satisfying the relation

$$\overline{v}_1 + \overline{v}_2 + \overline{v}_3 = 0,$$

and

$$\overline{v}_1 + \overline{v}_3 + \overline{v}_4 = 0,$$

respectively.

To compute the canonical hull Δ^{can} of Δ, we obtain $\operatorname{supp}(\Delta^{\mathrm{FI}}) = \{s_i \,|\, 1 \leq i \leq 5\}$ with $s_1 := (-3, -3, -1)$, $s_2 := (-1, -1, 0)$, $s_3 := (-1, 2, -1)$, $s_4 := (2, -1, 1)$, and $s_5 := (15, -3, 7)$, which leads to $\Delta^{\mathrm{can}} = \Delta$.

Remark 5.3. The detailed information about all 20 canonical Fano 3-topes with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in (\Delta^{\mathrm{FI}})^\circ$ can be found in Appendix A. To be precise, it is listed in Table A.4 and A.5 and can be viewed in [Sch18, Appendix A, Figure A2].

6. Fine interior of dimension 3

There exist 49 canonical Fano 3-topes Δ such that $\dim(\Delta^{\mathrm{FI}}) = 3$. Exactly 3 of these polytopes Δ define minimal surface $\mathcal{S}_\Delta$ with non-trivial fundamental group of order 2 and $K^2 = 2$. For these 3 polytopes one has $\Delta = \Delta^{\mathrm{can}}$. The surfaces $\mathcal{S}_\Delta$ were investigated by Todorov [Tod81] as well as Catanese and Debarre [CD89].

The remaining 46 canonical Fano 3-topes Δ define simply connected minimal surfaces $\mathcal{S}_\Delta$ with $K^2 = 1$. These surfaces were investigated by Kanev [Kan77], Catanese [Cat79], and Todorov [Tod80]. Among these 46 canonical Fano 3-topes there exist exactly 26 polytopes Δ such that $\Delta = \Delta^{\text{can}}$.

Example 6.1 [Kan77]. Let $M \subseteq \mathbb{Q}^4$ be the 3-dimensional affine lattice defined by

$$M := \{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 \,|\, m_1 + m_2 + m_3 + 2m_4 = 6,\ m_2 + 2m_3 \equiv 0 \pmod 3\}$$

and $\Delta' \subseteq M_\mathbb{Q}$ be the convex hull of 4 lattice points

$$(6, 0, 0, 0),\ (0, 6, 0, 0),\ (0, 0, 6, 0), \text{ and } (0, 0, 0, 3) \in M.$$

Then $(\Delta')^{\text{FI}}$ is the 3-dimensional rational simplex

$$\operatorname{conv}((2, 1, 1, 1), (1, 2, 1, 1), (1, 1, 2, 1), (1, 1, 1, 3/2))$$

and $(\Delta')^{\text{FI}} \cap M = \{(2, 1, 1, 1)\}$.

The canonical Fano 3-tope Δ' is the Newton polytope of the μ_3-cyclic quotient $\overline{Z}_{\Delta'}$ of the projective surface of degree 6 defined by the polynomial $z_1^6 + z_2^6 + z_3^6 + z_4^3 = 0$ in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$, where the cyclic group μ_3 acts via $(z_1 : z_2 : z_3 : z_4) \mapsto (z_1 : \varepsilon z_2 : \varepsilon^2 z_3 : z_4)$. The single interior lattice point in Δ' corresponds to the monomial $z_1^2 z_2 z_3 z_4$. The surface $\overline{Z}_{\Delta'}$ has 3 cyclic quotient singularities of type A_2. The minimal desingularization $\mathcal{S}_{\Delta'}$ of $\overline{Z}_{\Delta'}$ is simply connected surface of general type with $K^2 = 1$.

One can identify Δ' with the canonical Fano 3-simplex Δ given as the convex hull

$$v_1 := (1, 0, 0),\ v_2 := (-2, -4, -5),\ v_3 := (1, 2, 4), \text{ and } v_4 := (1, 4, 2)$$

(ID 547444, Figure 6(a), Table A.6, A.7, and A.8). The primitive inward-pointing facet normals of the facets θ_{124}, θ_{234}, θ_{123}, and $\theta_{134} \preceq \Delta$ of this simplex Δ are

$$n_1 := (-2, -1, 2),\ n_2 := (5, -1, -1),\ n_3 := (-1, 2, -1), \text{ and } n_4 := (-1, 0, 0),$$

respectively. They satisfy the relation

$$n_1 + n_2 + n_3 + 2n_4 = 0.$$

To compute the canonical hull Δ^{can} of Δ, we obtain $\operatorname{supp}(\Delta^{\text{FI}}) = \{s_i \,|\, 1 \le i \le 6\}$ with $s_1 := (-2, -1, 2)$, $s_2 := (-1, 0, 0)$, $s_3 := (-1, 2, -1)$, $s_4 := (1, 1, -1)$, $s_5 := (3, 0, -1)$, and $s_6 := (5, -1, -1)$, which leads to $\Delta^{\text{can}} = \Delta$.

Example 6.2 [Tod81]. Let $M \subseteq \mathbb{Q}^4$ be the 3-dimensional affine lattice defined by

$$M := \{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 \,|\, m_1 + m_2 + 2m_3 + 2m_4 = 8,\ 3m_2 + m_3 + 3m_4 \equiv 0 \pmod 4\}$$

and $\Delta' \subseteq M_\mathbb{Q}$ be the convex hull of 4 lattice points

$$(8, 0, 0, 0),\ (0, 8, 0, 0),\ (0, 0, 4, 0), \text{ and } (0, 0, 0, 4) \in M.$$

Then $(\Delta')^{\text{FI}}$ is the 3-dimensional rational simplex

$$\operatorname{conv}((3, 1, 1, 1), (1, 3, 1, 1), (1, 1, 2, 1), (1, 1, 1, 2))$$

and $(\Delta')^{\text{FI}} \cap M = \{(1, 1, 2, 1)\}$.

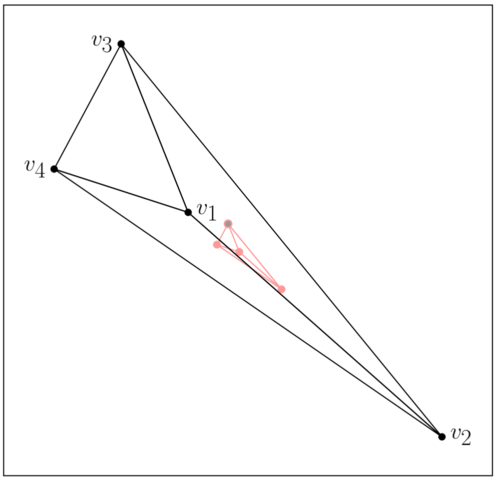
The canonical Fano 3-tope Δ' is the Newton polytope of the μ_4-cyclic quotient $\overline{Z}_{\Delta'}$ of the projective surface of degree 8 defined by the polynomial $z_1^8 + z_2^8 + z_3^4 + z_4^4 = 0$ in the weighted projective space $\mathbb{P}(1, 1, 2, 2)$, where the cyclic group μ_4 acts via $(z_1 : z_2 : z_3 : z_4) \mapsto (z_1 : i^3 z_2 : i z_3 : i^3 z_4)$. The single interior lattice point in this lattice simplex Δ' corresponds to the monomial $z_1 z_2 z_3^2 z_4$. The projective surface $\overline{Z}_{\Delta'}$ has two Gorenstein cyclic quotient singularities of type A_3. The minimal desingularization $\mathcal{S}_{\Delta'}$ of $\overline{Z}_{\Delta'}$ is a surface of general type with K^2 and fundamental group $\pi_1(\mathcal{S}_\Delta)$ of order 2.

One can identify Δ' with the canonical Fano 3-simplex Δ given as the convex hull

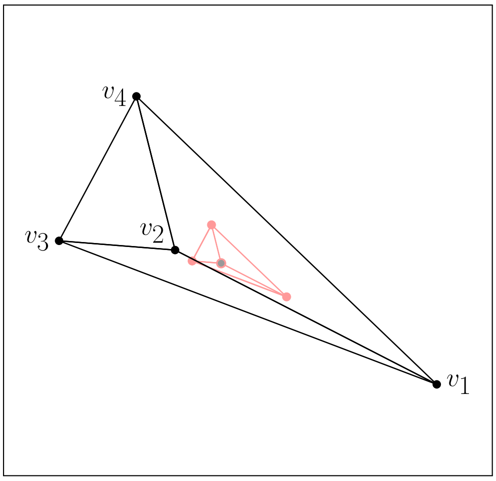
$$v_1 := (-3, -2, -2),\ v_2 := (1, 0, 0),\ v_3 := (1, 3, 1), \text{ and } v_4 := (1, 1, 3)$$

(ID 547465, Figure 6(b), Table A.6, A.7, and A.8). The primitive inward-pointing facet normals of the facets θ_{123}, θ_{124}, θ_{234}, $\theta_{134} \preceq \Delta$ of this simplex Δ are

$$n_1 := (-1, -1, 3),\ n_2 := (-1, 3, -1),\ n_3 := (-1, 0, 0), \text{ and } n_4 := (2, -1, -1),$$



(A) ID 547444



(B) ID 547465

FIGURE 6. **Canonical Fano 3-topes with Fine Interior of Dimension 3.** Shaded faces are occluded. The Fine interior is coloured red and the origin grey with a red margin. **(a)** The whole polytope is $\Delta = \operatorname{conv}(v_1, v_2, v_3, v_4)$ with $v_1 = (1, 0, 0)$, $v_2 = (-2, -4, -5)$, $v_3 = (1, 2, 4)$, $v_4 = (1, 4, 2)$. Moreover, $\Delta^{\mathrm{FI}} = \operatorname{conv}((0, 0, 0), (-1/2, -1, -3/2), (0, -1/3, -2/3), (0, 1/3, -1/3))$ and $\Delta^{\mathrm{can}} = \Delta$. **(b)** The whole polytope is $\Delta = \operatorname{conv}(v_1, v_2, v_3, v_4)$ with $v_1 = (-3, -2, -2)$, $v_2 = (1, 0, 0)$, $v_3 = (1, 3, 1)$, $v_4 = (1, 1, 3)$. Moreover, $\Delta^{\mathrm{FI}} = \operatorname{conv}((0, 0, 0), (-1, -1/2, -1/2), (0, 3/4, 1/4), (0, 1/4, 3/4))$ and $\Delta^{\mathrm{can}} = \Delta$.

respectively. They satisfy the relation

$$n_1 + n_2 + 2n_3 + 2n_4 = 0.$$

To compute the canonical hull Δ^{can} of Δ, we obtain $\operatorname{supp}(\Delta^{\mathrm{FI}}) = \{s_i \mid 1 \le i \le 9\}$ with $s_1 := (-1, -1, 3)$, $s_2 := (-1, 0, 0)$, $s_3 := (-1, 0, 1)$, $s_4 := (-1, 0, 2)$, $s_5 := (-1, 1, 0)$, $s_6 := (-1, 1, 1)$, $s_7 := (-1, 2, 0)$, $s_8 := (-1, 3, -1)$, and $s_9 := (2, -1, -1)$, which leads to $\Delta^{\mathrm{can}} = \Delta$.

Remark 6.3. The detailed information about all 49 canonical Fano 3-topes with $\dim(\Delta^{\mathrm{FI}}) = 3$ can be found in the Appendix A. To be precise, it is listed in Table A.6, A.7, and A.8 and can be viewed in [Sch18, Appendix A, Figure A3].

Appendix A. Computational data

In all the tables, the canonical Fano 3-topes Δ are given by their IDs used in the Graded Ring Database.

ID	$\mathrm{vert}(\Delta)$	$\mathrm{vert}(\Delta^{\mathrm{FI}})$	v_Δ	$(w_i)_{0\le i\le 3}$
547324	$(2,3,8),(1,0,0),(0,1,0),(-1,-1,-1)$	$0, 1/2\cdot v_\Delta$	$(1,1,2)$	$(1,5,6,8)$
547323	$(-1,1,-2),(1,-2,3),(1,0,0),(-2,5,-3)$	$0, 2/3\cdot v_\Delta$	$(0,1,0)$	$(1,4,7,9)$
547311	$(-1,4,2),(-1,-1,0),(0,0,-1),(2,0,1)$	$0, 2/3\cdot v_\Delta$	$(0,1,1)$	$(2,5,8,9)$
547490	$(1,2,4),(1,0,0),(1,-2,3),(-1,1,-2)$	$0, 1/2\cdot v_\Delta$	$(0,1,0)$	$(1,5,8,14)$
547321	$(1,-2,3),(0,1,0),(1,0,0),(-6,3,-8)$	$0, 1/2\cdot v_\Delta$	$(-1,1,-2)$	$(3,7,8,10)$
547305	$(0,1,0),(1,0,0),(1,2,4),(-4,-6,-7)$	$0, 2/3\cdot v_\Delta$	$(-1,-1,-1)$	$(4,7,9,10)$
547526	$(1,0,0),(0,1,0),(-2,1,5),(2,-4,-9)$	$0, 2/3\cdot v_\Delta$	$(1,-1,-3)$	$(5,9,8,11)$
547454	$(2,1,7),(1,0,0),(0,1,0),(-2,-3,-3)$	$0, 1/2\cdot v_\Delta$	$(0,0,1)$	$(3,7,8,18)$
547446	$(0,1,1),(-6,7,-15),(1,-2,3),(1,0,0)$	$0, 1/2\cdot v_\Delta$	$(-1,1,-2)$	$(5,8,9,22)$

Table A.1. **9 Canonical Fano 3-topes with Asymmetric Fine Interior of Dimension 1.** Table contains: vertices $\mathrm{vert}(\Delta)$ of Δ, vertices $\mathrm{vert}(\Delta^{\mathrm{FI}})$ of the Fine interior Δ^{FI}, unique primitive lattice point $v_\Delta \in \theta_+$ in the reflexive facet $\theta_+ \preceq \Delta$, and weights $(w_i)_{0\le i\le 3}$ of the weighted projective 3-space $\mathbb{P}(w_0,\dots,w_3)$ appearing in [CG11].

ID	$(n_i)_{1\le i\le 4}$	$\mathrm{vert}(\theta_+)$	n_{θ_+}
547324	$(-2,-2,1),(-1,-1,3),$ $(-1,3,-1),(7,-3,-1)$	$(2,3,8),(1,0,0),(0,1,0)$	$(-2,-2,1)$
547323	$(-3,-3,-2),(-1,0,1),$ $(-1,6,4),(17,3,-5)$	$(1,-2,3),(1,0,0),(-2,5,-3)$	$(-3,-3,-2)$
547311	$(-1,-1,1),(-1,2,1),$ $(1,2,-5),(7,-2,5)$	$(-1,4,2),(-1,-1,0),(2,0,1)$	$(1,2,-5)$
547490	$(-2,-2,1),(-1,0,0),$ $(-1,6,4),(23,2,-8)$	$(1,2,4),(1,0,0),(-1,1,-2)$	$(-2,-2,1)$
547321	$(-3,-3,-2),(-2,-2,1),$ $(-1,3,2),(9,-5,-8)$	$(0,1,0),(1,0,0),(-6,3,-8)$	$(-2,-2,1)$
547305	$(-7,-7,11),(-2,-2,1),$ $(-1,2,-1),(7,-3,-1)$	$(0,1,0),(1,2,4),(-4,-6,-7)$	$(7,-3,-1)$
547526	$(-5,-5,-2),(-3,-3,1),$ $(-1,2,-1),(25,-8,10)$	$(1,0,0),(0,1,0),(2,-4,-9)$	$(-3,-3,1)$
547454	$(-7,-7,2),(-1,-1,2),$ $(-1,1,0),(7,-2,-2)$	$(2,1,7),(0,1,0),(-2,-3,-3)$	$(7,-2,-2)$
547446	$(-9,21,14),(-5,-3,-2),$ $(-1,-1,0),(9,1,-3)$	$(0,1,1),(-6,7,-15),(1,-2,3)$	$(9,1,-3)$

Table A.2. **9 Canonical Fano 3-topes with Asymmetric Fine Interior of Dimension 1.** Table contains: primitive inward-pointing facet normals $(n_i)_{1\le i\le 4}$ of Δ, vertices $\mathrm{vert}(\theta_+)$ of the reflexive facet $\theta_+ \preceq \Delta$, and primitive inward-pointing facet normal n_{θ_+} of the reflexive facet $\theta_+ \preceq \Delta$.

ID	vert(Θ)	supp(Δ^{FI})	vert(Δ^{can})
547324	$(-1,3,-1),(-1,-1,1),(1,-1,0)$	$(-2,-2,1),(-1,-1,1),S_1$	$\mathrm{vert}(\Delta),(0,1,4)$
547323	$(-1,0,1),(-1,0,0),(2,0,-1)$	$(-3,-3,-2),(-1,0,0),S_2$	$\mathrm{vert}(\Delta),(-2,4,-3)$
547311	$(-1,-1,1),(0,1,-1),(1,0,0)$	$(-1,-1,1),(-1,0,1),S_3$	$\mathrm{vert}(\Delta),(-1,2,0)$
547490	$(-1,0,0),(-1,0,1),(3,0,-1)$	$(-2,-2,1),(-1,0,0),S_4$	$\mathrm{vert}(\Delta),(1,-1,4)$
547321	$(-1,-1,0),(-1,3,2),(1,-1,-1)$	$(-2,-2,1),(-1,-1,0),$ $(-1,0,0),(-1,1,1),$ $(-1,3,2),(0,-1,-1),$ $(1,-1,-1)$	$\mathrm{vert}(\Delta),(1,0,1),$ $(0,-3,4)$
547305	$(-1,2,-1),(1,-1,0),(0,-1,1)$	$(-1,-1,1),(-1,0,0),$ $(-1,2,-1),(0,-1,1),$ $(1,-1,0),(7,-3,-1)$	$\mathrm{vert}(\Delta),(0,-2,-3),$ $(1,2,2)$
547526	$(-1,-1,0),(-1,2,-1),(2,-1,1)$	$(-3,-3,1),(-1,-1,0),$ $(-1,2,-1),(0,-1,0),$ $(2,-1,1)$	$\mathrm{vert}(\Delta)\setminus\{(-2,1,5)\},$ $(0,1,3),(-3,1,6)$
547454	$(-1,1,0),(0,-1,0),(2,-1,0)$	$(-1,-1,1),(-1,-1,2),S_5$	$\mathrm{vert}(\Delta),(2,1,2)$
547446	$(-1,-1,0),(0,2,1),(2,0,-1)$	$(-1,-1,0),(-1,0,0),$ $(0,2,1),(1,1,0),$ $(2,0,-1),(9,1,-3)$	$\mathrm{vert}(\Delta),(1,0,-1),$ $(1,0,3)$

Table A.3. **9 Canonical Fano 3-topes with Asymmetric Fine Interior of Dimension 1.** Table contains: vertices vert(Θ) of the reflexive facet $\Theta \preceq [\Delta^*]$, support supp($\Delta^{\mathrm{FI}}$) of the Fine interior Δ^{FI}, and vertices vert(Δ^{can}) of the canonical hull Δ^{can}.

$$S_1 := (-1,-1,2),(-1,-1,3),(-1,0,1),(-1,0,2),(-1,1,0),(-1,1,1),(-1,2,0),(-1,3,-1),(0,-1,1),$$
$$(0,-1,2),(0,0,1),(0,1,0),(1,-1,0),(1,-1,1),(1,0,0),(2,-1,0)$$
$$S_2 := (-1,0,1),(-1,1,1),(-1,2,2),(-1,3,2),(-1,4,3),(-1,6,4),(0,1,1),(0,2,1),(0,3,2),(0,5,3),$$
$$(1,1,0),(1,2,1),(1,4,2),(2,0,-1),(2,1,0),(2,3,1),(3,2,0),(4,1,-1)$$
$$S_3 := (-1,1,1),(-1,2,1),(0,0,1),(0,1,-1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,2,-5)$$
$$S_4 := (-1,0,1),(-1,1,1),(-1,2,2),(-1,3,2),(-1,4,3),(-1,6,4),(0,1,1),(0,2,1),(0,3,2),(0,5,3),$$
$$(1,0,0),(1,1,0),(1,2,1),(1,4,2),(2,1,0),(2,3,1),(3,0,-1),(3,2,0),(4,1,-1)$$
$$S_5 := (-1,0,1),(-1,1,0),(0,-1,0),(0,-1,1),(1,-1,0),(2,-1,0),(7,-2,-2)$$

ID	vert(Δ)
547393	$(0,1,0),(2,1,1),(-2,-3,-5),(2,1,9)$
547409	$(-4,2,9),(1,0,0),(0,1,0),(7,-6,-18)$
547461	$(0,1,0),(2,1,1),(-2,-3,-5),(0,1,4)$
544442	$(1,0,0),(0,1,0),(3,-6,8),(1,-4,4),(-5,6,-12)$
544443	$(-1,-2,0),(3,-6,8),(0,1,0),(1,0,0),(-3,4,-8)$
544651	$(-4,1,-3),(4,-2,3),(0,1,0),(1,-2,3),(-1,1,-3)$
544696	$(5,-4,-15),(1,0,0),(0,1,0),(-4,2,9),(-3,1,6)$
544700	$(-2,-3,-3),(0,1,0),(1,0,0),(-1,-4,-6),(2,5,9)$
544749	$(-6,-5,-8),(0,1,0),(1,0,0),(-2,-1,0),(3,2,4)$
520925	$(0,1,0),(0,0,1),(-2,-1,0),(-2,0,-1),(8,2,3),(-2,-3,-2)$
520935	$(3,4,6),(2,1,2),(-3,-2,-2),(1,0,0),(0,1,0),(-6,-5,-8)$
522056	$(-1,-1,0),(0,1,0),(1,0,0),(-1,-1,-3),(-5,-3,-6),(6,4,9)$
522059	$(2,5,6),(-2,-3,-3),(0,1,0),(1,0,0),(-1,-4,-6),(0,1,3)$
522087	$(1,0,-3),(1,0,0),(0,1,0),(-4,2,9),(-3,1,6),(5,-4,-12)$
522682	$(2,1,4),(-3,-2,-4),(-2,-3,-4),(1,2,4),(1,0,0),(0,1,0)$
522684	$(-2,-1,-4),(3,2,4),(-2,-1,0),(1,0,0),(0,1,0),(-4,-3,-4)$
526886	$(-3,4,-6),(1,0,0),(0,1,0),(3,-6,8),(0,1,-2),(2,-5,6)$
439403	$(1,2,2),(-1,0,0),(-1,1,-1),(1,0,0),(-1,-2,-2),(1,1,3),$ $(1,-3,-1)$
275525	$(4,1,2),(0,1,0),(-2,-1,0),(1,1,2),(-3,-1,-2),(-2,-1,-2),$ $(1,1,0),(1,-1,0)$
275528	$(-1,0,-1),(-3,-2,1),(-2,-1,2),(0,-1,0),(0,1,0),(1,0,1),$ $(2,1,-2),(3,2,-1)$

TABLE A.4. **20 Canonical Fano 3-topes with Symmetric Fine Interior of Dimension 1.** Table contains: vertices vert(Δ) of Δ.

ID	$\mathrm{vert}(\Delta^{\mathrm{FI}})$	$\pm v_\Delta$	$\mathrm{vert}(\theta_\pm)$	$\mathrm{supp}(\Delta^{\mathrm{FI}})$
547393	$\pm 1/2 \cdot v_\Delta$	$\pm(0,0,1)$	$(0,1,0),(2,1,1),(-2,-3,-5)$	$(-1,-2,2),(-1,1,0),(0,-1,0),$
			$(0,1,0),(-2,-3,-5),(2,1,9)$	$(1,-1,0),(2,-1,0),(9,-2,-2)$
547409	$\pm 2/3 \cdot v_\Delta$	$\pm(1,-1,-3)$	$(-4,2,9),(1,0,0),(0,1,0)$	$(-3,-3,-1),(-1,-1,0),(-1,2,-1),$
			$(-4,2,9),(0,1,0),(7,-6,-18)$	$(2,-1,1),(15,-3,7)$
547461	$\pm 1/2 \cdot v_\Delta$	$\pm(0,0,1)$	$(0,1,0),(2,1,1),(-2,-3,-5)$	$(-3,6,-2),(-1,-2,2),(-1,1,0),$
			$(2,1,1),(-2,-3,-5),(0,1,4)$	$(0,-1,0),(1,-1,0),(2,-1,0)$
544442	$\pm 1/2 \cdot v_\Delta$	$\pm(1,-1,2)$	$(0,1,0),(1,-4,4),(-5,6,-12)$	$(-2,-2,-1),(-1,-1,0),(-1,1,1),$
			$(1,0,0),(0,1,0),(3,-6,8)$	$(1,-1,-1),(3,-1,-2),(10,-2,-5)$
544443	$\pm 1/2 \cdot v_\Delta$	$\pm(1,-1,2)$	$(-1,-2,0),(0,1,0),(-3,4,-8)$	$(-2,-2,-1),(-1,-1,0),(-1,1,1),$
			$(3,-6,8),(0,1,0),(1,0,0)$	$(1,-1,-1),(3,-1,-2),(6,-2,-3)$
544651	$\pm 2/3 \cdot v_\Delta$	$\pm(1,0,0)$	$(-4,1,-3),(0,1,0),(1,-2,3)$	$(-3,-3,1),(0,-1,-1),(0,-1,0),$
			$(4,-2,3),(0,1,0),(-1,1,-3)$	$(0,2,1),(3,-3,-4)$
544696	$\pm 2/3 \cdot v_\Delta$	$\pm(1,-1,-3)$	$(1,0,0),(0,1,0),(-4,2,9)$	$(-3,-3,-1),(-3,12,-4),(-1,-1,0),$
			$(5,-4,-15),(1,0,0),(-3,1,6)$	$(-1,2,-1),(2,-1,1)$
544700	$\pm 2/3 \cdot v_\Delta$	$\pm(1,2,3)$	$(-2,-3,-3),(0,1,0),(-1,-4,-6)$	$(-3,-3,2),(-1,-1,1),(-1,2,-1),$
			$(0,1,0),(1,0,0),(2,5,9)$	$(2,-1,0),(3,-3,2)$
544749	$\pm 1/2 \cdot v_\Delta$	$\pm(1,1,2)$	$(-6,-5,-8),(0,1,0),(1,0,0)$	$(-2,-2,3),(-1,-1,1),(-1,1,0),$
			$(0,1,0),(-2,-1,0),(3,2,4)$	$(-1,3,-1),(1,-1,0),(2,-2,-1)$
520925	$\pm 1/2 \cdot v_\Delta$	$\pm(2,1,1)$	$(-2,-1,0),(-2,0,-1),(-2,-3,-2)$	$(-1,-1,3),(0,-1,1),(0,1,-1),$
			$(0,1,0),(0,0,1),(8,2,3)$	$(1,-2,-2),(1,-1,-1),(1,0,0)$
520935	$\pm 1/2 \cdot v_\Delta$	$\pm(1,1,2)$	$(1,0,0),(0,1,0),(-6,-5,-8)$	$(-2,-2,3),(-1,-1,1),(-1,1,0),$
			$(3,4,6),(2,1,2),(-3,-2,-2)$	$(-1,3,-1),(0,4,-3),(1,-1,0)$
522056	$\pm 2/3 \cdot v_\Delta$	$\pm(2,1,3)$	$(0,1,0),(-1,-1,-3),(-5,-3,-6)$	$(-3,6,-1),(-1,-1,1),(-1,2,0),$
			$(-1,-1,0),(1,0,0),(6,4,9)$	$(0,-3,2),(2,-1,-1)$
522059	$\pm 2/3 \cdot v_\Delta$	$\pm(1,2,3)$	$(-2,-3,-3),(0,1,0),(-1,-4,-6)$	$(-3,3,-2),(-1,-1,1),(-1,2,-1),$
			$(2,5,6),(1,0,0),(0,1,3)$	$(2,-1,0),(3,-3,2)$
522087	$\pm 2/3 \cdot v_\Delta$	$\pm(1,-1,-3)$	$(1,0,0),(0,1,0),(-4,2,9)$	$(-3,-3,-1),(-1,-1,0),(-1,2,-1),$
			$(1,0,-3),(-3,1,6),(5,-4,-12)$	$(2,-1,1),(9,0,4)$
522682	$\pm 1/2 \cdot v_\Delta$	$\pm(1,1,2)$	$(-3,-2,-4),(-2,-3,-4),(1,0,0),(0,1,0)$	$(-2,-2,1),(-2,-2,3),(-1,-1,1),$
			$(2,1,4),(1,2,4),(1,0,0),(0,1,0)$	$(-1,1,0),(1,-1,0),(1,1,-1)$
522684	$\pm 1/2 \cdot v_\Delta$	$\pm(1,1,2)$	$(-2,-1,-4),(1,0,0),(-4,-3,-4)$	$(-2,2,1),(-1,-1,1),(-1,1,0),$
			$(3,2,4),(-2,-1,0),(0,1,0)$	$(-1,3,-1),(1,-1,0),(2,-2,-1)$
526886	$\pm 1/2 \cdot v_\Delta$	$\pm(1,-1,2)$	$(-3,4,-6),(0,1,-2),(2,-5,6)$	$(-2,-2,-1),(-1,-1,0),(-1,1,1),$
			$(1,0,0),(0,1,0),(3,-6,8)$	$(0,4,3),(1,-1,-1),(3,-1,-2)$
439403	$\pm 1/2 \cdot v_\Delta$	$\pm(0,1,1)$	$(-1,1,-1),(1,0,0),(-1,-2,-2),(1,-3,-1)$	$(-2,-1,3),(-1,-1,1),(-1,0,0),$
			$(1,2,2),(-1,0,0),(-1,1,-1),(1,1,3)$	$(1,0,0),(1,1,-1),(2,-1,-1)$
275525	$\pm 1/2 \cdot v_\Delta$	$\pm(1,0,0)$	$(0,1,0),(-2,-1,0),(1,1,2),(-3,-1,-2)$	$(-2,0,3),(0,-1,0),(0,-1,1),$
			$(4,1,2),(-2,-1,-2),(1,1,0),(1,-1,0)$	$\pm(0,1,-1),(0,1,0),(2,-2,-1)$
275528	$\pm 1/2 \cdot v_\Delta$	$\pm(1,1,-1)$	$(-3,-2,1),(-2,-1,2),(0,-1,0),(1,0,1)$	$(-1,1,0),(-1,2,-1),(0,-1,-1),$
			$(-1,0,-1),(0,1,0),(2,1,-2),(3,2,-1)$	$(0,1,1),(1,-2,1),(1,-1,0)$

TABLE A.5. **20 Canonical Fano 3-topes with Symmetric Fine Interior of Dimension 1.** Table contains: vertices $\mathrm{vert}(\Delta^{\mathrm{FI}})$ of the Fine interior Δ^{FI}, unique primitive lattice points $\pm v_\Delta \in \theta_\pm$ in the reflexive facets $\theta_\pm \preceq \Delta$, vertices $\mathrm{vert}(\theta_\pm)$ of the reflexive facets $\theta_\pm \preceq \Delta$, and support $\mathrm{supp}(\Delta^{\mathrm{FI}})$ of the Fine interior Δ^{FI} (here: $\Delta^{\mathrm{can}} = \Delta$).

ID	vert(Δ)
547444	(1, 0, 0), (−2, −4, −5), (1, 2, 4), (1, 4, 2)
547465	(−3, −2, −2), (1, 0, 0), (1, 3, 1), (1, 1, 3)
547524	(0, 2, 1), (−2, −3, −5), (2, 1, 1), (0, 0, 1)
547525	(0, 0, 1), (0, 1, 0), (2, 1, 1), (−2, −5, −7)
545317	(−3, 4, −6), (0, 1, 0), (1, 0, 0), (1, −2, 4), (3, −5, 6)
545932	(0, −1, −1), (1, −1, −3), (−2, 1, 5), (1, 0, 0), (1, 2, −2)
546013	(3, −5, 6), (1, −2, 4), (1, 0, 0), (−1, 1, −2), (−1, 3, −2)
546062	(0, 1, 3), (−2, 1, −1), (0, 1, 0), (1, 0, 0), (−1, −2, −2)
546070	(0, −2, −3), (0, 2, 1), (−2, −3, −5), (2, 1, 1), (0, 0, 1)
546205	(1, 2, −2), (−1, 0, 2), (1, 0, 0), (−2, 1, 5), (1, −1, −3)
546219	(1, 1, 1), (−3, −2, −2), (1, 0, 0), (1, 3, 1), (−1, −1, 1)
546663	(2, −3, −1), (1, 0, 0), (0, 1, 0), (0, 0, 1), (−2, −3, −3)
546862	(1, 0, 0), (0, 1, 0), (−2, 1, 5), (1, −1, −3), (1, 2, −2)
546863	(−1, −1, 1), (1, 3, 1), (0, 0, 1), (1, 0, 0), (−3, −2, −2)
547240	(−1, 1, −2), (0, 1, 0), (1, 0, 0), (1, −2, 4), (3, −5, 6)
547246	(0, −2, −3), (−2, −3, −5), (2, 1, 1), (0, 1, 0), (0, 0, 1)
532384	(1, −1, −3), (−2, 1, 5), (1, 0, 0), (1, −1, −2), (0, −1, −1), (1, 2, −2)
532606	(0, −1, 2), (−1, −1, 0), (0, 1, 0), (1, 0, 0), (2, 2, −3), (−2, 0, −3)
533513	(−1, 1, 2), (1, 0, 0), (0, 1, 0), (1, 1, 2), (−1, −2, −4), (−2, −3, −4)
534667	(1, 0, 3), (−1, −1, −1), (0, 1, 0), (1, 0, 0), (−1, −1, 0), (5, 2, 3)
534669	(1, 3, 0), (5, 3, 2), (−1, −1, −1), (0, 0, 1), (1, 0, 0), (−1, −1, 0)
534866	(−1, −1, −3), (1, 0, 0), (0, 1, 0), (1, 1, 1), (−1, −1, 0), (−3, −5, −3)
535952	(3, −5, 6), (1, −2, 4), (1, 0, 0), (0, 1, 0), (−1, 1, −2), (−1, 2, −2)
536013	(0, 1, 1), (0, 0, 1), (0, 1, 0), (2, 1, 1), (−2, −3, −5), (0, −2, −3)
536498	(1, 2, −2), (1, −1, −2), (1, 0, 0), (0, 1, 0), (−2, 1, 5), (1, −1, −3)
537834	(0, 0, 1), (1, 0, 0), (0, 1, 0), (−2, 1, 5), (1, −1, −3), (1, 2, −2)
538356	(−2, −3, −3), (−1, −3, −1), (0, 0, 1), (0, 1, 0), (1, 0, 0), (−1, −1, −3)
539063	(−1, 1, −1), (1, 1, 3), (−3, −2, −2), (1, 0, 0), (0, 1, 0), (1, 1, 2)
539304	(1, 0, 1), (−3, −1, −2), (1, 1, 2), (−2, −1, 0), (1, 0, 0), (1, 2, 0)
539313	(1, −1, −2), (1, 1, −1), (−1, 2, 2), (1, −1, −3), (−2, 1, 5), (1, 0, 0)
540602	(0, 0, 1), (1, 0, 0), (−2, 1, 5), (1, −1, −3), (−1, 2, 2), (1, 1, −1)
540663	(1, 0, 0), (0, 1, 0), (1, 1, 2), (−3, −1, −2), (1, 1, 1), (−3, −2, 0)
474457	(−1, 2, −3), (1, 0, 2), (0, 0, 1), (0, 1, 0), (1, 0, 0), (−1, −1, 0), (−3, −2, −3)
481575	(3, 2, 4), (−1, −1, −2), (−3, −1, −2), (−2, −1, 0), (0, 1, 0), (1, 0, 0), (0, 0, −1)
483109	(3, 0, 2), (1, −2, −2), (0, 0, −1), (−1, −1, 0), (1, 1, 1), (0, 1, 0), (−1, 0, 0)
490478	(1, −1, −2), (1, 1, −1), (−1, 2, 2), (1, −1, −3), (−2, 1, 5), (1, 0, 0), (−1, 0, 2)
490481	(−3, −2, 0), (−5, −3, −2), (1, 0, 0), (0, 1, 0), (1, 1, 2), (−1, −1, −1), (2, 1, 1)
490485	(−1, −1, 0), (1, 2, 0), (1, 0, 0), (−2, −1, 0), (1, 1, 2), (−3, −1, −2), (1, 0, 1)
490511	(1, 0, 0), (0, 1, 0), (−2, −1, 0), (1, 1, 2), (2, 1, 1), (1, 0, 1), (−5, −2, −4)
495687	(0, 0, −1), (1, 1, −1), (−1, 2, 2), (1, −1, −3), (−2, 1, 5), (1, 0, 0), (0, 0, 1)
499287	(1, 1, 1), (−1, −1, −3), (1, 0, 0), (0, 1, 0), (0, 0, 1), (−1, −3, −1), (−2, −3, −3)
499291	(−1, −1, −1), (−1, −1, −3), (1, 0, 0), (0, 1, 0), (0, 0, 1), (−1, −3, −1), (−2, −3, −3)
499470	(1, 0, 0), (0, 1, 0), (−2, −1, 0), (1, 1, 2), (0, 0, 1), (−5, −2, −4), (2, 1, 1)
501298	(3, −6, 8), (−1, 1, −2), (1, −2, 3), (0, 1, 0), (1, 0, 0), (0, 1, −1), (3, −5, 6)
501330	(1, 0, 0), (0, 1, 0), (−2, −1, 0), (1, 1, 2), (1, 1, 1), (0, 0, 1), (−5, −2, −4)
354912	(3, 1, 2), (1, 0, 0), (0, 1, 0), (−2, −1, 0), (1, 1, 2), (2, 1, 1), (1, 0, 1), (−5, −2, −4)
372528	(2, 1, 1), (−1, −1, −1), (1, 1, 2), (0, 1, 0), (1, 0, 0), (−5, −3, −2), (−3, −2, 0), (1, 1, 0)
372973	(−5, −2, −4), (1, 0, 1), (2, 1, 1), (1, 1, 2), (−2, −1, 0), (0, 1, 0), (1, 0, 0), (2, 1, 2)
388701	(1, 1, 1), (−2, −3, −3), (−1, −3, −1), (0, 0, 1), (0, 1, 0), (1, 0, 0), (−1, −1, −3), (−1, −1, −1)

TABLE A.6. **49 Canonical Fano 3-topes with Fine Interior of Dimension 3.** Table contains: vertices vert(Δ) of Δ.

ID	$\mathrm{vert}(\Delta^{\mathrm{FI}})$
547444	$0, (-1/2, -1, -3/2), (0, -1/3, -2/3), (0, 1/3, -1/3)$
547465	$0, (-1, -1/2, -1/2), (0, 3/4, 1/4), (0, 1/4, 3/4)$
547524	$0, (0, 1/2, 0), (1/3, 1/3, 0), (-1/3, -1/3, -1)$
547525	$0, (0, 0, -1/2), (1/3, 0, -1/3), (-1/3, -1, -5/3)$
545317	$0, (1, -3/2, 2), (2/3, -2/3, 1), (1/2, -1/2, 1), (2/3, -1, 5/3)$
545932	$0, (-1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)$
546013	$0, (1, -3/2, 2), (0, 1/2, 0), (1/2, -1/4, 1/2), (1/2, -3/4, 3/2)$
546062	$0, (-1/2, -1/2, -1/2), (-2/3, 0, -1/3), (-1/3, 0, 1/3)$
546070	$0, (0, 1/2, 0), (1/2, 1/4, 0), (0, -1/2, -1), (-1/2, -3/4, -3/2)$
546205	$0, (-1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)$
546219	$0, (-1, -1/2, -1/2), (-1/3, 1/3, 0), (-2/3, -1/3, 0)$
546663	$0, (0, -1/2, 0), (1/3, -1, -1/3), (-1/3, -1, -2/3)$
546862	$0, (-1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)$
546863	$0, (-1, -1/2, -1/2), (-1/3, 1/3, 0), (-2/3, -1/3, 0)$
547240	$0, (1, -3/2, 2), (2/3, -2/3, 1), (1/2, -1/2, 1), (2/3, -1, 5/3)$
547246	$0, (0, 0, -1/2), (1/3, 0, -1/3), (0, -1/2, -1), (-1/3, -2/3, -4/3)$
532384	$0, (-1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)$
532606	$0, (0, 1/2, -1/2), (1/3, 2/3, -1), (-1/3, 1/3, -1)$
533513	$0, (-1/2, -1/2, -1), (-1/2, 0, 0), (-1/3, 0, -1/3), (-2/3, -2/3, -1)$
534667	$0, (1/2, 1/2, 1/2), (4/3, 2/3, 1), (2/3, 1/3, 1)$
534669	$0, (1/2, 1/2, 1/2), (4/3, 1, 2/3), (2/3, 1, 1/3)$
534866	$0, (0, -1/2, -1/2), (-1/3, -2/3, -1), (-2/3, -4/3, -1)$
535952	$0, (1, -3/2, 2), (2/3, -2/3, 1), (1/2, -1/2, 1), (2/3, -1, 5/3)$
536013	$0, (0, 0, -1/2), (1/3, 0, -1/3), (0, -1/2, -1), (-1/3, -2/3, -4/3)$
536498	$0, (-1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)$
537834	$0, (-1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)$
538356	$0, (0, -1/2, -1/2), (-1/3, -2/3, -1), (-1/3, -1, -2/3), (-1/2, -1, -1)$
539063	$0, (-1, -1/2, -1/2), (-2/3, 0, -1/3), (-1/3, 0, 1/3)$
539304	$0, (0, 1/2, 0), (-1/2, 0, 0), (0, 1/3, 1/3), (-2/3, 0, -1/3)$
539313	$0, (-1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 1/2, 1/2), (-1/3, 2/3, 1)$
540602	$0, (-1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 1/2, 1/2), (-1/3, 2/3, 1)$
540663	$0, (-1/2, 0, 0), (-1, -1/2, 0), (-1/3, 0, 1/3), (-1, -1/3, -1/3)$
474457	$0, (0, 0, -1/2), (-1/3, 1/3, -1), (-2/3, -1/3, -1)$
481575	$0, (-1/2, 0, 0), (1/2, 1/2, 1), (0, 1/3, 1/3), (-1/3, 0, 1/3)$
483109	$0, (0, -1/2, 0), (2/3, -1/3, 1/3), (1/3, -2/3, -1/3)$
490478	$0, (-1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 1/2, 1/2), (-1/3, 2/3, 1)$
490481	$0, (-1/2, 0, 0), (-1, -1/2, 0), (-1/3, 0, 1/3), (-4/3, -2/3, -1/3)$
490485	$0, (0, 1/2, 0), (-1/2, 0, 0), (0, 1/3, 1/3), (-2/3, 0, -1/3)$
490511	$0, (-3/2, -1/2, -1), (-1/2, 0, 0), (-2/3, 0, -1/3), (-1, -1/3, -1/3)$
495687	$0, (-1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 1/2, 1/2), (-1/3, 2/3, 1)$
499287	$0, (0, -1/2, -1/2), (-1/3, -2/3, -1), (-1/3, -1, -2/3), (-1/2, -1, -1)$
499291	$0, (0, -1/2, -1/2), (-1/3, -2/3, -1), (-1/3, -1, -2/3), (-1/2, -1, -1)$
499470	$0, (-3/2, -1/2, -1), (-1/2, 0, 0), (-2/3, 0, -1/3), (-1, -1/3, -1/3)$
501298	$0, (1/2, -1/2, 1), (2/3, -2/3, 1), (1, -3/2, 2), (1, -5/3, 7/3)$
501330	$0, (-3/2, -1/2, -1), (-1/2, 0, 0), (-2/3, 0, -1/3), (-1, -1/3, -1/3)$
354912	$0, (-3/2, -1/2, -1), (-1/2, 0, 0), (-2/3, 0, -1/3), (-1, -1/3, -1/3)$
372528	$0, (-1/2, 0, 0), (-1, -1/2, 0), (-1/3, 0, 1/3), (-4/3, -2/3, -1/3)$
372973	$0, (-3/2, -1/2, -1), (-1/2, 0, 0), (-2/3, 0, -1/3), (-1, -1/3, -1/3)$
388701	$0, (0, -1/2, -1/2), (-1/3, -2/3, -1), (-1/3, -1, -2/3), (-1/2, -1, -1)$

Table A.7. **49 Canonical Fano 3-topes with Fine Interior of Dimension 3.** Table contains: vertices $\mathrm{vert}(\Delta^{\mathrm{FI}})$ of the Fine interior Δ^{FI}.

ID	$\mathrm{supp}(\Delta^{\mathrm{FI}})$	$\mathrm{vert}(\Delta^{\mathrm{can}})$	$\lvert\pi_1(\mathcal{S}_\Delta)\rvert$
547444	$(-2,-1,2),(-1,0,0),(-1,2,-1),(1,1,-1),(3,0,-1),$ $(5,-1,-1)$	$\mathrm{vert}(\Delta)$	1
547465	$(-1,-1,3),(-1,0,0),(-1,0,1),(-1,0,2),(-1,1,0),$ $(-1,1,1),(-1,2,0),(-1,3,-1),(2,-1,-1)$	$\mathrm{vert}(\Delta)$	2
547524	$(-1,-2,2),(-1,1,0),(-1,2,-1),(0,0,-1),(0,1,-1),$ $(0,2,-1),(1,0,-1),(1,1,-1),(2,0,-1),(3,0,-1)$	$\mathrm{vert}(\Delta),(0,-1,-1)$	1
547525	$(-1,-2,2),(-1,2,-1),(0,-1,0),(0,0,-1),(0,1,-1),$ $(1,-1,-1),(1,-1,0),(1,0,-1),(1,1,-1),(2,-1,-1),$ $(2,-1,0),(2,0,-1),(3,-1,-1),(3,-1,0),(3,0,-1),$ $(4,-1,-1),(4,0,-1),(5,-1,-1),(6,-1,-1)$	$\mathrm{vert}(\Delta),(1,1,1),$ $(-1,-2,-3)$	1
545317	$(-2,-2,-1),(-1,-1,0),(-1,2,2),(1,-1,-1),(1,2,1),$ $(3,2,0)$	$\mathrm{vert}(\Delta)$	1
545932	$(-2,-1,-1),(-1,0,0),(-1,2,-1),(0,1,0),(1,1,0),$ $(2,0,1),(3,0,1),(5,-1,2)$	$\mathrm{vert}(\Delta),(1,-1,-2),$ $(1,0,-3)$	1
546013	$(-2,-2,-1),(-1,0,1),(-1,2,2),(0,1,1),(1,0,0),$ $(1,2,1),(2,1,0),(3,0,-1),(3,2,0)$	$\mathrm{vert}(\Delta)$	2
546062	$(-1,-1,0),(-1,-1,1),(-1,-1,2),(-1,0,0),(-1,0,1),$ $(-1,1,0),(-1,2,-1),(0,-1,0),(2,1,-1)$	$\mathrm{vert}(\Delta)$	1
546070	$(-1,-2,2),(-1,2,-1),(0,0,-1),(0,1,-1),(0,2,-1),$ $(1,0,-1),(1,1,-1),(2,0,-1),(3,0,-1)$	$\mathrm{vert}(\Delta)$	2
546205	$(-2,-1,-1),(-1,0,0),(-1,2,-1),(-1,3,-1),(0,1,0),$ $(1,1,0),(1,2,0),(2,0,1),(3,0,1),(3,1,1),$ $(5,-1,2),(5,0,2),(7,-1,3)$	$\mathrm{vert}(\Delta)$	1
546219	$(-1,-1,3),(-1,0,0),(-1,0,1),(-1,0,2),(-1,1,-1),$ $(-1,1,0),(-1,1,1),(-1,2,0),(0,0,-1),(2,-1,-1)$	$\mathrm{vert}(\Delta)$	1
546663	$(-1,-1,-1),(-1,-1,0),(-1,-1,1),(-1,-1,2),(-1,0,-1),$ $(0,-1,-1),(0,-1,0),(0,-1,1),(0,0,-1),(1,-1,-1),$ $(1,-1,0),(1,0,-1),(1,2,-2),(2,-1,-1),(2,-1,0),$ $(2,0,-1),(3,-1,-1)$	$\mathrm{vert}(\Delta),(-1,-1,-1)$	1
546862	$(-2,-1,-1),(-1,0,0),(-1,2,-1),(-1,3,-1),(0,1,0),$ $(0,4,-1),(1,1,0),(1,2,0),(2,0,1),(2,3,0),$ $(3,0,1),(3,1,1),(4,2,1),(5,-1,2),(5,0,2),$ $(6,1,2),(7,-1,3),(8,0,3),(10,-1,4)$	$\mathrm{vert}(\Delta),(0,0,1)$	1
546863	$(-1,-1,3),(-1,0,0),(-1,0,1),(-1,0,2),(-1,1,-1),$ $(-1,1,0),(-1,1,1),(-1,2,0),(0,0,-1),(2,-1,-1)$	$\mathrm{vert}(\Delta)\setminus\{(0,0,1)\},$ $(1,1,1)$	1
547240	$(-2,-2,-1),(-1,-1,0),(-1,0,1),(-1,2,2),(0,-1,0),$ $(0,1,1),(1,-1,-1),(1,0,0),(1,2,1),(2,-1,-1),$ $(2,1,0),(3,0,-1),(3,2,0)$	$\mathrm{vert}(\Delta),(0,1,-1),$ $(0,0,1)$	1
547246	$(-1,-2,2),(-1,2,-1),(0,-1,0),(0,0,-1),(0,1,-1),$ $(0,2,-1),(1,-1,-1),(1,-1,0),(1,0,-1),(1,1,-1),$ $(2,-1,-1),(2,-1,0),(2,0,-1),(3,-1,-1),(3,0,-1),$ $(4,-1,-1)$	$\mathrm{vert}(\Delta),(1,1,1),$ $(-1,-1,-2)$	1
532384	$(-2,-1,-1),(-1,0,0),(-1,2,-1),(0,1,0),(1,1,0),$ $(2,0,1),(3,0,1),(5,-1,2)$	$\mathrm{vert}(\Delta),(1,0,-3)$	1

Table A.8. **49 Canonical Fano 3-topes with Fine Interior of Dimension 3.** Table contains: support $\mathrm{supp}(\Delta^{\mathrm{FI}})$ of the Fine interior Δ^{FI}, vertices $\mathrm{vert}(\Delta^{\mathrm{can}})$ of the canonical hull Δ^{can}, and order of fundamental group $\lvert\pi_1(\mathcal{S}_\Delta)\rvert$ of the minimal model $\mathcal{S}_\Delta$.

Table A.8

ID	$\mathrm{supp}(\Delta^{\mathrm{FI}})$	$\mathrm{vert}(\Delta^{\mathrm{can}})$	$\lvert\pi_1(\mathcal{S}_\Delta)\rvert$
532606	$(-1,-1,-1),(-1,1,0),(-1,2,1),(0,-1,-1),(0,1,0),$ $(1,-2,0),(1,-1,-1),(2,-1,-1)$	$\mathrm{vert}(\Delta),(0,-1,1)$	1
533513	$(-1,-1,1),(-1,0,0),(-1,1,0),(-1,2,-1),(0,-1,0),$ $(0,1,-1),(0,3,-2),(2,-2,1)$	$\mathrm{vert}(\Delta),(1,1,1),$ $(0,-1,-1)$	1
534667	$(-1,-1,2),(-1,-1,3),(-1,0,2),(-1,1,1),(-1,2,0),$ $(0,-1,1),(0,-1,2),(0,0,1),(0,1,0),(1,-2,-1),$ $(1,-1,0),(1,-1,1),(1,0,0),(2,-1,-1),(2,-1,0)$	$\mathrm{vert}(\Delta)$	1
534669	$(-1,0,2),(-1,1,1),(-1,2,-1),(-1,2,0),(0,0,1),$ $(0,1,-1),(0,1,0),(1,-1,-2),(1,0,-1),(1,0,0),$ $(2,-1,-1),(2,-1,0)$	$\mathrm{vert}(\Delta)$	1
534866	$(-2,1,1),(-1,-1,1),(-1,0,0),(-1,1,-1),(-1,2,-2),$ $(0,-1,0),(0,0,-1),(0,1,-2),(1,-1,-1),(1,-1,0),$ $(1,0,-2),(1,0,-1),(2,-1,-2),(2,-1,-1),(2,-1,0)$	$\mathrm{vert}(\Delta)$	1
535952	$(-2,-2,-1),(-1,-1,0),(-1,0,1),(-1,2,2),(0,1,1),$ $(1,-1,-1),(1,0,0),(1,2,1),(2,1,0),(3,0,-1),$ $(3,2,0)$	$\mathrm{vert}(\Delta)$	1
536013	$(-1,-2,2),(-1,2,-1),(0,-1,0),(0,0,-1),(0,1,-1),$ $(0,2,-1),(1,-1,0),(1,0,-1),(1,1,-1),(2,-1,0),$ $(2,0,-1),(3,0,-1)$	$\mathrm{vert}(\Delta)$	1
536498	$(-2,-1,-1),(-1,0,0),(-1,2,-1),(0,1,0),(1,1,0),$ $(1,2,0),(2,0,1),(2,3,0),(3,0,1),(3,1,1),$ $(4,2,1),(5,-1,2),(5,0,2),(6,1,2),(7,-1,3),$ $(8,0,3),(10,-1,4)$	$\mathrm{vert}(\Delta)$	1
537834	$(-2,-1,-1),(-1,0,0),(-1,2,-1),(-1,3,-1),(0,1,0),$ $(0,4,-1),(1,1,0),(1,2,0),(2,0,1),(2,3,0),$ $(3,0,1),(3,1,1),(4,2,1),(5,-1,2),(5,0,2),$ $(6,1,2),(7,-1,3),(8,0,3),(10,-1,4)$	$\mathrm{vert}(\Delta)$	1
538356	$(-2,1,1),(-1,-1,-1),(-1,-1,0),(-1,-1,1),(-1,0,-1),$ $(-1,0,0),(-1,1,-1),(0,-1,-1),(0,-1,0),(0,0,-1),$ $(1,-1,-1),(1,-1,0),(1,0,-1),(2,-1,-1),(2,-1,0),$ $(2,0,-1),(3,-1,-1)$	$\mathrm{vert}(\Delta),(-1,-1,-1)$	1
539063	$(-1,-1,1),(-1,0,0),(-1,0,1),(-1,0,2),(-1,1,0),$ $(-1,1,1),(-1,2,0),(-1,3,-1),(0,-1,0),(2,-1,-1)$	$\mathrm{vert}(\Delta)\setminus\{(0,1,0),(1,1,2)\},$ $(1,1,1)$	1
539304	$(-1,0,0),(-1,0,1),(-1,0,2),(-1,1,0),(-1,1,1),$ $(-1,2,0),(-1,2,1),(-1,3,0),(0,1,-1),(0,1,0),$ $(2,-2,-1)$	$\mathrm{vert}(\Delta),(-2,-1,-1)$	1
539313	$(-2,-1,-1),(-1,0,0),(-1,2,-1),(0,1,0),(1,-1,1),$ $(1,1,0),(1,2,0),(2,0,1),(2,3,0),(3,0,1),$ $(3,1,1),(4,2,1),(5,0,2),(6,1,2)$	$\mathrm{vert}(\Delta),(-1,1,2)$	1
540602	$(-2,-1,-1),(-1,0,0),(-1,2,-1),(-1,3,-1),(0,1,0),$ $(0,4,-1),(1,-1,1),(1,1,0),(1,2,0),(2,0,1),$ $(2,3,0),(3,0,1),(3,1,1),(4,2,1),(5,0,2),$ $(6,1,2)$	$\mathrm{vert}(\Delta),(-1,1,2)$	1
540663	$(-1,-1,1),(-1,-1,2),(-1,0,0),(-1,0,1),(-1,0,2),$ $(-1,1,0),(-1,1,1),(-1,2,-1),(-1,2,0),(-1,2,1),$ $(0,-1,0),(0,-1,1),(2,-2,-1)$	$\mathrm{vert}(\Delta),(-1,0,-1)$	1
474457	$(-2,1,2),(-1,-1,0),(-1,0,0),(-1,1,0),(-1,2,0),$ $(1,-1,-1),(1,0,-1),(2,-1,-1)$	$\mathrm{vert}(\Delta)$	1
481575	$(-1,-1,1),(-1,0,1),(-1,1,0),(-1,2,0),(-1,3,-1),$ $(0,-1,1),(0,1,0),(2,-2,-1)$	$\mathrm{vert}(\Delta),(-1,-1,-1)$	1

Table A.8

ID	$\mathrm{supp}(\Delta^{\mathrm{FI}})$	$\mathrm{vert}(\Delta^{\mathrm{can}})$	$\lvert\pi_1(\mathcal{S}_\Delta)\rvert$
483109	$(-1,-1,1),(0,-1,0),(0,-1,1),(0,2,-1),(1,-1,-1),$ $(1,-1,0),(1,-1,1),(1,0,-2),(1,0,-1),(1,0,0),$ $(1,0,1)$	$\mathrm{vert}(\Delta)$	1
490478	(-2, -1, -1),(-1, 0, 0),(-1, 2, -1),(0, 1, 0),(1, -1, 1), $(1,1,0),(1,2,0),(2,0,1),(3,0,1),(3,1,1),$ $(5,0,2)$	$\mathrm{vert}(\Delta)$	1
490481	$(-1,-1,2),(-1,-1,3),(-1,0,1),(-1,0,2),(-1,1,0),$ $(-1,1,1),(-1,2,-1),(-1,2,0),(0,-1,0),(0,-1,1),$ $(0,-1,2),(2,-2,-1)$	$\mathrm{vert}(\Delta)$	1
490485	$(-1,0,0),(-1,0,1),(-1,0,2),(-1,1,0),(-1,1,1),$ $(-1,2,0),(-1,2,1),(0,1,-1),(0,1,0),(2,-2,-1)$	$\mathrm{vert}(\Delta)$	1
490511	$(-1,-1,2),(-1,0,1),(-1,1,0),(-1,1,1),(-1,2,0),$ $(-1,3,0),(0,-1,0),(0,1,-1),(2,-2,-1)$	$\mathrm{vert}(\Delta),(2,1,2)$	1
495687	$(-2,-1,-1),(-1,0,0),(-1,2,-1),(-1,3,-1),(0,1,0),$ $(0,4,-1),(1,-1,1),(1,1,0),(1,2,0),(2,0,1),$ $(2,3,0),(3,0,1),(3,1,1),(4,2,1)$	$\mathrm{vert}(\Delta)$	1
499287	$(-2,1,1),(-1,-1,1),(-1,0,0),(-1,1,-1),(0,-1,0),$ $(0,0,-1),(1,-1,-1),(1,-1,0),(1,0,-1),(2,-1,-1),$ $(2,-1,0),(2,0,-1),(3,-1,-1)$	$\mathrm{vert}(\Delta),(-1,-1,-1)$	1
499291	$(-2,1,1),(-1,-1,-1),(-1,-1,0),(-1,-1,1),(-1,0,-1),$ $(-1,0,0),(-1,1,-1),(0,-1,-1),(0,-1,0),(0,0,-1),$ $(1,-1,-1),(1,-1,0),(1,0,-1),(2,-1,-1),(2,-1,0),$ $(2,0,-1),(3,-1,-1)$	$\mathrm{vert}(\Delta)$	1
499470	$(-1,-1,2),(-1,0,1),(-1,1,0),(-1,1,1),(-1,2,-1),$ $(-1,2,0),(-1,3,-1),(-1,3,0),(0,-1,0),(0,1,-1),$ $(2,-2,-1)$	$\mathrm{vert}(\Delta)$	1
501298	$(-2,-2,-1),(-1,-1,0),(-1,0,1),(-1,2,2),(0,-1,0),$ $(0,1,1),(1,-1,-1),(1,0,0),(1,2,1),(2,-1,-1),$ $(2,1,0),(3,-1,-2),(3,0,-1),(4,-1,-2),(5,0,-2),$ $(6,-1,-3)$	$\mathrm{vert}(\Delta)$	1
501330	$(-1,-1,1),(-1,-1,2),(-1,0,0),(-1,0,1),(-1,1,0),$ $(-1,1,1),(-1,2,-1),(-1,2,0),(-1,3,-1),(-1,3,0),$ $(0,-1,0),(0,1,-1),(2,-2,-1)$	$\mathrm{vert}(\Delta)$	1
354912	$(-1,-1,2),(-1,0,1),(-1,1,1),(-1,2,0),(-1,3,0),$ $(0,-1,0),(0,1,-1),(2,-2,-1)$	$\mathrm{vert}(\Delta)$	1
372528	$(-1,0,1),(-1,0,2),(-1,1,0),(-1,1,1),(-1,2,-1),$ $(-1,2,0),(0,-1,0),(0,-1,1),(0,-1,2),(2,-2,-1)$	$\mathrm{vert}(\Delta)$	1
372973	$(-1,-1,2),(-1,0,1),(-1,1,0),(-1,1,1),(-1,2,0),$ $(-1,3,0),(0,-1,0),(0,1,-1),(2,-2,-1)$	$\mathrm{vert}(\Delta)$	1
388701	$(-2,1,1),(-1,-1,1),(-1,0,0),(-1,1,-1),(0,-1,0),$ $(0,0,-1),(1,-1,-1),(1,-1,0),(1,0,-1),(2,-1,-1),$ $(2,-1,0),(2,0,-1),(3,-1,-1)$	$\mathrm{vert}(\Delta)$	1

Appendix B. Hollow 3-topes with Non-empty Fine Interior

A lattice polytope $\Delta \subseteq M_{\mathbb{Q}}$ is called *hollow* if it has no interior lattice points in its relative interior, *i.e.*, $\Delta^\circ \cap M = \varnothing$. By a Theorem 1.3 in [Tre19], any 3-dimensional hollow lattice polytope can be projected to the unimodular 1-simplex, to the double unimodular 2-simplex, or is an exceptional hollow 3-tope, whereas up to unimodular transformation there exist only a finite number of these. This theorem implies that a hollow 3-tope with non-empty Fine interior has to be exceptional because the unimodular 1-simplex and the double unimodular 2-simplex have empty Fine interior. Treutlein has found 9 maximal exceptional hollow polytopes, which was not an complete list. Averkov et al. [AWW11, AKW17] have found the complete list consisting of 12 maximal exceptional hollow 3-topes Δ_i $(1 \leq i \leq 12)$ (Table B.1, Figure B.1). Computations show that exactly 9 of 12 maximal exceptional hollow 3-topes Δ_i have non-empty Fine interior Δ_i^{FI} (Table B.1). Moreover, no one of these 9 polytopes contains a proper lattice 3-subpolytope with non-empty Fine interior. Thus, there exist exactly 9 hollow 3-topes Δ_i with non-empty Fine interior Δ_i^{FI}.

i	$\mathrm{vert}(\Delta_i)$	$\mathrm{w}(\Delta_i)$	$\dim(\Delta_i^{\mathrm{FI}})$	$\mathrm{vert}(\Delta_i^{\mathrm{FI}})$	$\lvert\pi_1(\mathcal{S}_\Delta)\rvert$
1	$(0,0,0),(6,0,0),(3,3,0),(4,0,2)$	2	-1	$\varnothing$	1
2	$(0,0,0),(4,0,0),(0,4,0),(2,0,2)$	2	-1	$\varnothing$	1
3	$(0,0,0),(3,0,0),(0,3,0),(3,0,3)$	3	-1	$\varnothing$	1
4	$(0,0,0),(4,0,0),(2,4,0),(3,0,2)$	2	0	$1/2\cdot(5,1,2)$	2
5	$(0,0,0),(2,2,0),(1,1,2),(3,-1,2)$	2	0	$1/2\cdot(3,1,2)$	2
6	$(0,0,0),(2,2,0),(4,0,0),(2,-2,0),$ $(3,1,2)$	2	0	$1/2\cdot(5,1,2)$	2
7	$(0,0,0),(1,1,0),(2,-2,0),(3,-1,0),$ $(1,-1,2),(2,0,2)$	2	0	$1/2\cdot(3,-1,2)$	2
8	$(0,0,0),(1,1,0),(1,-1,0),(2,0,0),$ $(1,-1,2),(2,0,2),(2,-2,2),(3,-1,2)$	2	0	$1/2\cdot(3,-1,2)$	2
9	$(0,0,0),(3,0,0),(1,3,0),(2,0,3)$	3	1	$(4/3,1,1),(5/3,1,1)$	3
10	$(0,0,0),(1,2,0),(1,-1,0),(3,0,0),$ $(2,1,3)$	3	1	$(4/3,2/3,1),(5/3,1/3,1)$	3
11	$(0,0,0),(1,1,0),(3,0,0),(2,-1,0),$ $(4,1,3),(2,2,3)$	3	1	$(5/3,2/3,1),(7/3,1/3,1)$	3
12	$(-1,0,0),(0,1,-2),(1,2,1),(2,-2,-1)$	3	3	$(1/5,1/5,-2/5),(2/5,2/5,-4/5),$ $(3/5,3/5,-1/5),(4/5,-1/5,-3/5)$	5

Table B.1. **12 Maximal Hollow 3-topes.** Table contains: index i of the maximal hollow 3-tope Δ_i, vertices $\mathrm{vert}(\Delta_i)$ of Δ_i, lattice width $\mathrm{w}(\Delta_i)$[1] of Δ_i, dimension $\dim(\Delta_i^{\mathrm{FI}})$ of Fine interior Δ_i^{FI}, vertices $\mathrm{vert}(\Delta_i^{\mathrm{FI}})$ of Δ_i^{FI}, and order of fundamental group $\lvert\pi_1(\mathcal{S}_\Delta)\rvert$ of the minimal model $\mathcal{S}_{\Delta_i}$.

It is remarkable that all minimal surfaces $\mathcal{S}_{\Delta_i}$ corresponding to these 9 hollow 3-topes Δ_i have non-trivial fundamental group $\pi_1(\mathcal{S}_\Delta)$ of order 2, 3, or 5 (Table B.1). There exist exactly 5 hollow 3-topes Δ_i with 0-dimensional Fine interior $\Delta_i^{\mathrm{FI}} = \{R\}$, where $R \in \frac{1}{2}M \setminus M$ is a rational point (Table B.1). The normal fans Σ^{Δ_i} of these 5 hollow polytopes Δ_i define 5 toric Fano threefolds $X_{\Sigma^{\Delta_i}}$ with at worst canonical singularities (Table B.2). These Fano threefolds can be obtained as quotients of Gorenstein toric Fano threefolds $X_{\Sigma_{\Delta''}}$ in the following 5 ways:

(i) $\mathbb{P}(1,1,2,4)$ with a μ_2-action given by $(x_0,x_1,x_2,x_3) \mapsto (x_0,-x_1,-x_2,-x_3)$;
(ii) $\mathbb{P}^3$ with a μ_4-action given by $(x_0,x_1,x_2,x_3) \mapsto (x_0,ix_1,-x_2,-ix_3)$;
(iii) $\{x_1x_2 - x_3x_4 = 0\} \subseteq \mathbb{P}(2,1,1,1,1)$ with a μ_2-action given by $(x_i)_{0\leq i\leq 4} \mapsto (-x_0,-x_1,-x_2,x_3,x_4)$;
(iv) $\mathbb{P}^1 \times \mathbb{P}(1,1,2)$ with a μ_2-action given by $(x_0,x_1,y_0,y_1,y_2) \mapsto (x_0,-x_1,y_0,-y_1,-y_2)$;
(v) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with a μ_2-action given by $(x_0,x_1,y_0,y_1,z_0,z_1) \mapsto (x_0,-x_1,y_0,-y_1,z_0,-z_1)$.

[1] The *lattice width* $\mathrm{w}(\Delta)$ of a lattice polytope Δ is defined as the infimum of $\max_{x\in\Delta}\langle x,u\rangle - \min_{x\in\Delta}\langle x,u\rangle$ for all non-zero integer lattice directions u.

i	$\Sigma^{\Delta_i}(1)$	ID(Δ')	$\Sigma_{\Delta'}(1)$	ID(Δ'')	$X_{\Sigma_{\Delta''}}$
4	$(2,-1,-3),(0,0,1),(0,1,0),(-2,-1,-1)$	547354	$(-2,-3,-5),(2,1,1),(0,1,0),(0,0,1)$	547363	(i)
5	$(1,-1,0),(1,1,-1),(-1,1,2),(-1,-1,-1)$	547364	$(0,0,1),(0,2,1),(2,1,0),(-2,-3,-2)$	547367	(ii)
6	$(0,0,1),(1,1,-2),(1,-1,-1),(-1,-1,0),(-1,1,-1)$	544353	$(1,0,0),(0,1,0),(-2,-1,0),(1,1,2),(-3,-1,-2)$	544357	(iii)
7	$(0,0,1),(-1,1,-1),(1,-1,-1),(1,1,0),(-1,-1,0)$	544310	$(-1,-1,-2),(1,1,2),(-2,-1,0),(0,1,0),(1,0,0)$	544342	(iv)
8	$(-1,1,1),(0,0,-1),(-1,-1,0),(1,1,0),(1,-1,-1),(0,0,1)$	520134	$(1,0,0),(0,1,0),(1,1,2),(-1,0,0),(0,-1,0),(-1,-1,-2)$	520140	(v)

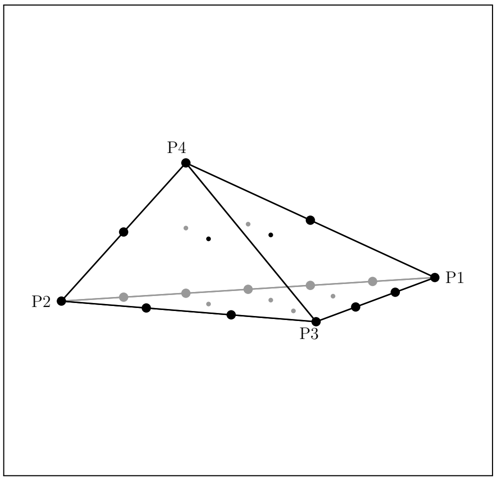
TABLE B.2. **5 Hollow 3-topes with 0-dimensional Fine Interior.** Table contains: index i of the maximal hollow 3-tope Δ_i, rays of the normal fan Σ^{Δ_i} corresponding to Δ_i, ID[2]of the canonical Fano 3-tope Δ' such that $\Sigma^{\Delta_i} \cong \Sigma_{\Delta'}$, rays of the spanning fan $\Sigma_{\Delta'}$, ID of the reflexive canonical Fano 3-tope Δ'' used to construct the Gorenstein toric Fano threefold $X_{\Sigma_{\Delta''}}$ and obtain the toric Fano threefold $X_{\Sigma_{\Delta'}}$ with at worst canonical singularities as a μ_2 quotient, and reference to the corresponding Gorenstein toric Fano threefold $X_{\Sigma_{\Delta''}}$ including the precise μ_2 action on page 23.

In addition, Table B.3 contains the support $\mathrm{supp}(\Delta_i^{\mathrm{FI}})$ of the Fine interior Δ_i^{FI} and the vertices of the canonical hull Δ_i^{can} for all 9 hollow polytopes Δ_i with non-empty Fine interior Δ_i^{FI}.

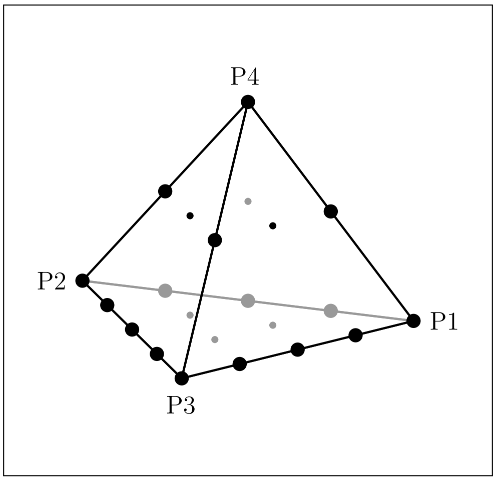
i	$\mathrm{supp}(\Delta_i^{\mathrm{FI}})$	$\mathrm{vert}(\Delta_i^{\mathrm{can}})$
4	$(-2,-1,-1),(0,-1,-2),(2,-1,-3),(0,0,1),(0,0,-1),(0,1,0)$	$\mathrm{vert}(\Delta_i)$
5	$(1,-1,0),(1,1,-1),(0,0,1),(0,0,-1),(-1,-1,-1),(-1,1,2)$	$\mathrm{vert}(\Delta_i)$
6	$(1,1,-2),(1,-1,-1),(-1,-1,0),(-1,1,-1),(0,0,1),(0,0,-1)$	$\mathrm{vert}(\Delta_i)$
7	$(1,1,0),(1,-1,-1),(-1,-1,0),(-1,1,-1),(0,0,1),(0,0,-1)$	$\mathrm{vert}(\Delta_i)$
8	$(1,1,0),(1,-1,-1),(-1,-1,0),(0,0,1),(0,0,-1),(-1,1,1)$	$\mathrm{vert}(\Delta_i)$
9	$(0,-1,-1),(0,0,1),(3,-1,-2),(0,1,0),(-3,-2,-1)$	$\mathrm{vert}(\Delta_i)$
10	$(-1,2,-1),(1,1,-1),(-1,-1,0),(2,-1,-1),(0,0,1)$	$\mathrm{vert}(\Delta_i)$
11	$(1,-1,0),(0,0,1),(-1,-2,1),(-1,1,0),(1,2,-2)$	$\mathrm{vert}(\Delta_i)$
12	$(1,1,1),(1,-1,0),(-2,-1,1),(0,1,-2)$	$\mathrm{vert}(\Delta_i)$

TABLE B.3. **9 Hollow 3-topes with Non-empty Fine Interior.** Table contains: index i of the maximal hollow 3-tope Δ_i, support $\mathrm{supp}(\Delta_i^{\mathrm{FI}})$ of Δ_i^{FI}, and vertices of the canonical hull Δ_i^{can}.

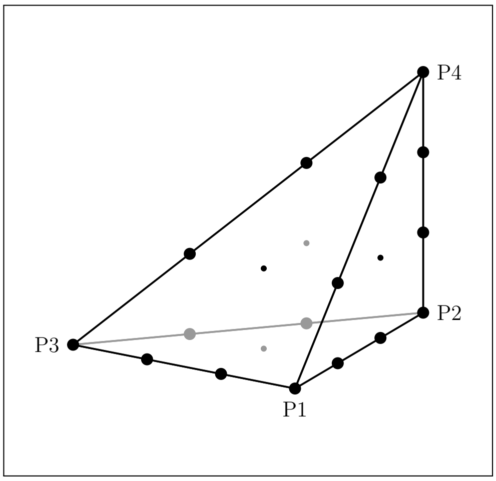
[2]ID used in the Graded Ring Database.



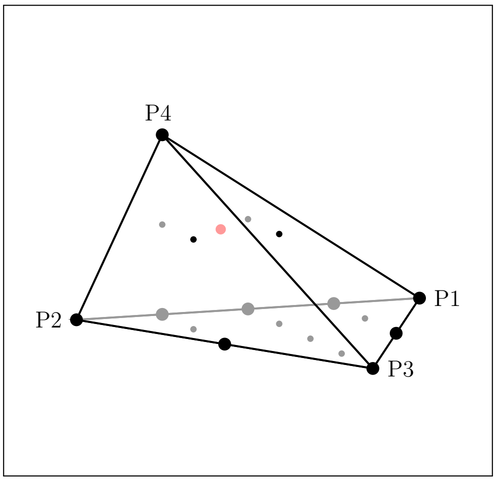
(A) Δ_1.



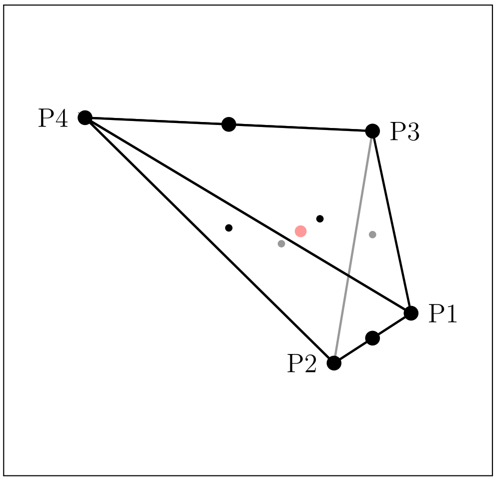
(B) Δ_2.



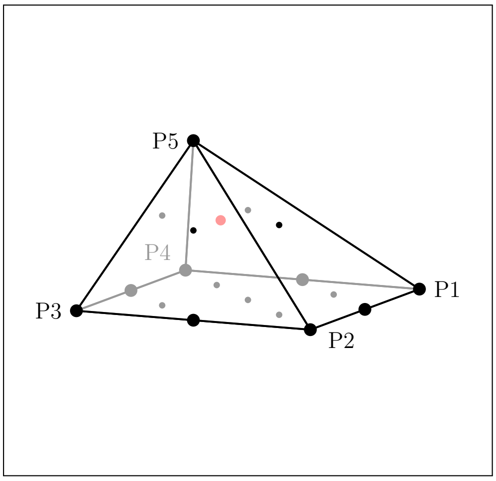
(C) Δ_3.



(D) Δ_4.

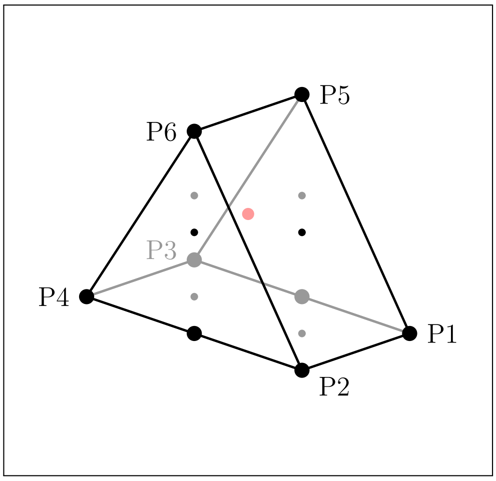


(E) Δ_5.

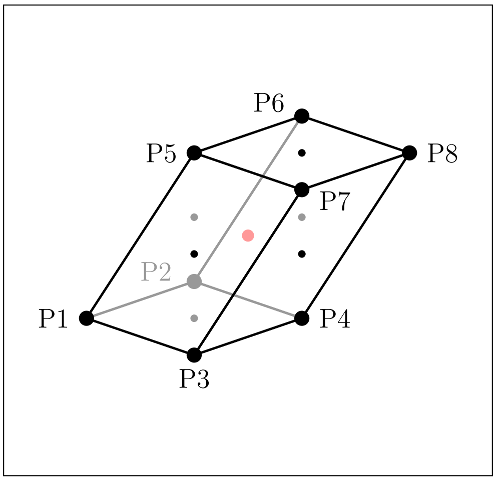


(F) Δ_6.

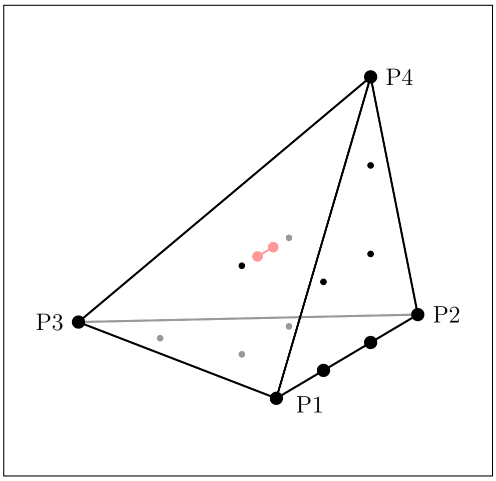
FIGURE B.1. **12 Maximal Hollow 3-topes.** Shaded faces are occulded. The Fine interior is coloured red.



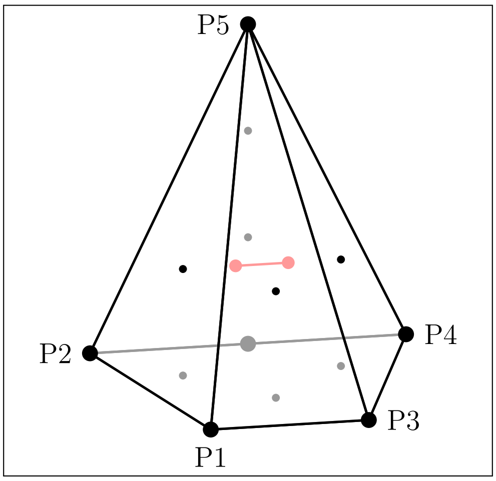
(G) Δ_7.



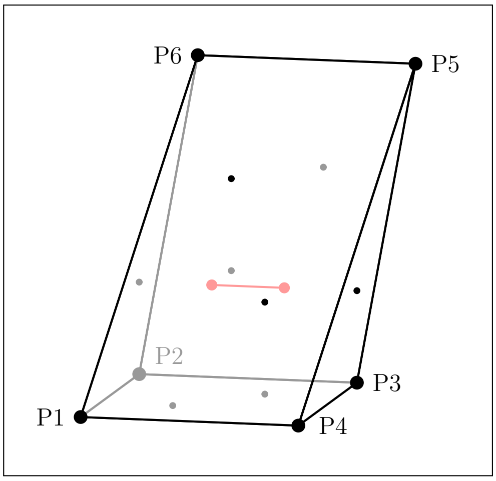
(H) Δ_8.



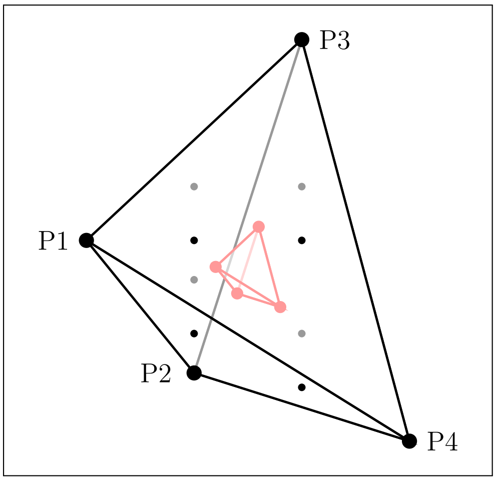
(I) Δ_9.



(J) Δ_{10}.



(K) Δ_{11}.



(L) Δ_{12}.

Figure B.1

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