MUTATIONS OF FAKE WEIGHTED PROJECTIVE SPACES

TOM COATES, SAMUEL GONSHAW, ALEXANDER KASPRZYK, AND NAVID NABIJOU

Abstract. We characterise mutations between fake weighted projective spaces, and give explicit formulas for how the weights and multiplicity change under mutation. In particular, we prove that multiplicity-preserving mutations between fake weighted projective spaces are mutations over edges of the corresponding simplices. As an application, we analyse the canonical and terminal fake weighted projective spaces of maximal degree.

1. Introduction

In this paper we analyse mutations between fake weighted projective spaces; equivalently we analyse mutations between lattice simplices. Mutations arise naturally when considering mirror symmetry for Fano manifolds. A Fano manifold X is expected to correspond under mirror symmetry to a Laurent polynomial [EHX97, BCFKvS98, HV00, BCFKvS00, Bat04, Aur07, CCG$^+$14, CCGK13]. In general there will be many different Laurent polynomials which correspond to a given Fano manifold, and it is expected that these Laurent polynomials are related via birational transformations analogous to cluster transformations [FZ02, GU10, ACGK12, GHK13]. These cluster-style transformations act on Newton polytopes via mutations; see Definition 3.2 below. A mutation between Newton polytopes can be thought of as the tropicalisation of the corresponding cluster-type transformation. Ilten [Ilt12] has shown that if two lattice polytopes P and Q are related by mutation then the corresponding toric varieties X_P and X_Q are deformation equivalent: there exists a flat family $\mathcal{X} \to \mathbb{P}^1$ such that $\mathcal{X}_0 \cong X_P$ and $\mathcal{X}_\infty \cong X_Q$. Mutations are thus expected to form the one-skeleton of the tropicalisation of the Hilbert scheme. Our understanding of this one-skeleton is rudimentary but improving [ACGK12, AK13]; in this paper we conduct the first systematic analysis in higher dimensions.

The notion of mutation raises many new and interesting combinatorial questions: for example, how can polytopes be classified up to mutation, and what properties of polytopes are mutation-invariant? Here we begin to address these questions by analysing the behaviour of lattice simplicies under mutation. In two dimensions, Akhtar and Kasprzyk determined how the weights of a fake weighted projective plane, i.e. the weights of a lattice triangle, change under mutation, and showed that mutations between fake weighted projective planes are multiplicity-preserving [AK13]. We show below that the situation in higher dimensions is different. We give an explicit formula (Theorem 4.6) for how the weights of a fake weighted projective space, i.e. the weights of a lattice simplex, change under mutation, and derive a strong necessary condition (Theorem 5.1) for a mutation to preserve multiplicity. In §§7–8 we apply our results to the study of fake weighted projective spaces of high degree with canonical and terminal singularities.

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2. Fake Weighted Projective Space

We begin by recalling some standard definitions. Throughout let $N \cong \mathbb{Z}^n$ denote a lattice of rank n with dual lattice $M := \mathrm{Hom}(N, \mathbb{Z})$. From the toric viewpoint N corresponds to the lattice of one-parameter subgroups and M corresponds to the lattice of characters. For an introduction to toric geometry see [Dan78].

Definition 2.1. A convex lattice polytope $P \subset N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ is said to be *Fano* if:

(i) P is of maximum dimension, that is $\dim P = n$;
(ii) the origin is contained in the strict interior of P, that is $\mathbf{0} \in \mathrm{int}(P)$;
(iii) the vertices $\mathrm{vert}(P)$ of P are primitive lattice points.

In addition to being compelling combinatorial objects, Fano polytopes are in bijective correspondence with toric Fano varieties: see [KN12] for an overview. The complete fan in N generated by the faces of P, which we call the *spanning fan* of P, corresponds to a toric Fano variety X_P, that is, to a (possibly singular) projective toric variety with ample anticanonical divisor $-K_X$. Two Fano polytopes P and P' give isomorphic varieties $X_P \cong X_{P'}$ if and only if there exists a change of basis of the underlying lattice N sending P to P'. Thus we regard P as being defined only up to $\mathrm{GL}_n(\mathbb{Z})$-equivalence.

Definition 2.2. Let $P := \mathrm{conv}\{v_0, \ldots, v_n\} \subset N_{\mathbb{Q}}$ be a Fano n-simplex. By Definition 2.1(ii) there exists a unique choice of $n + 1$ coprime positive integers $\lambda_0, \ldots, \lambda_n \in \mathbb{Z}_{>0}$ such that $\lambda_0 v_0 + \ldots + \lambda_n v_n = \mathbf{0}$. These are called the *(reduced) weights* of P.

Definition 2.1(iii) implies that any n of the weights are also coprime, or in other words that the weights are *well-formed*. See [IF00, §5] for details of the natural role that reduced and well-formed weights play in the study of weighted projective space.

Definition 2.3. Let $P \subset N_{\mathbb{Q}}$ be a Fano n-simplex and let $N' := v_0 \cdot \mathbb{Z} + \ldots + v_n \cdot \mathbb{Z}$ be the sublattice in N generated by the vertices of P. The rank-one $\mathbb{Q}$-factorial toric Fano variety X_P given by the spanning fan of P is $X_P = \mathbb{P}(\lambda_0, \ldots, \lambda_n)/(N/N')$, where the group N/N' acts freely in codimension one. We call X_P a *fake weighted projective space*.

Fake weighted projective spaces have been studied in [Con02, Buc08, Kas09]. One important invariant is the *multiplicity*: the index of the sublattice N' in N, denoted by $\mathrm{mult}(P) := [N : N']$. A fake weighted projective space is a weighted projective space if and only if $\mathrm{mult}(P) = 1$ [BB92, Proposition 2].

3. Mutations

We recall the definition of mutation, following [ACGK12, §3]. A primitive element $w \in M$ determines a surjective linear map $w\colon N \to \mathbb{Z}$ which extends naturally to a map $N_{\mathbb{Q}} \to \mathbb{Q}$. A point $v \in N_{\mathbb{Q}}$ is said to be at *height* $w(v)$. Given a subset $S \subset N_{\mathbb{Q}}$, if $w(v) = h$ for all $v \in S$ we say that S lies at height h and write $w(S) = h$. The hyperplane $H_{w,h}$ is defined to be the set of all points in $N_{\mathbb{Q}}$ at height h. For a convex lattice polytope $P \subset N_{\mathbb{Q}}$ we define $w_h(P) := \mathrm{conv}(H_{w,h} \cap P \cap N)$ to be the (possibly empty) convex hull of all lattice points in P

at height h. We set $h_{\min} := \min\{w(v) \mid v \in P\}$ to be the minimum height occurring amongst the the points of P, and $h_{\max}$ to be the maximum height. Since P is a lattice polytope, both $h_{\min}$ and $h_{\max}$ are integers. If $\mathbf{0} \in \mathrm{int}(P)$, and in particular if P is Fano, then $h_{\min} < 0$ and $h_{\max} > 0$.

Definition 3.1. A *factor* of $P \subset N_{\mathbb{Q}}$ with respect to a primitive height function $w \in M$ is a lattice polytope $F \subset N_{\mathbb{Q}}$ such that:

(i) $w(F) = 0$;
(ii) for every integer h with $h_{\min} \leqslant h < 0$, there exists a (possibly empty) lattice polytope $G_h \subset N_{\mathbb{Q}}$ such that $H_{w,h} \cap \mathrm{vert}(P) \subseteq G_h + (-h)F \subseteq w_h(P)$.

Definition 3.2. The *(combinatorial) mutation* of $P \subset N_{\mathbb{Q}}$ with respect to the primitive height function $w \in M$ and a factor $F \subset N_{\mathbb{Q}}$ is the convex lattice polytope

$$\mathrm{mut}_w(P,F) := \mathrm{conv}\left(\bigcup_{h=h_{\min}}^{-1} G_h \cup \bigcup_{h=0}^{h_{\max}} (w_h(P) + hF) \right) \subset N_{\mathbb{Q}}.$$

Although this is not obvious from the definition, the mutation $\mathrm{mut}_w(P,F)$ is independent of the choice of the G_h [ACGK12, Proposition 1]. Furthermore, $\mathrm{mut}_w(P,F)$ is a Fano polytope if and only if P is a Fano polytope [ACGK12, Proposition 2]. More obviously, mutations are always invertible (if $Q := \mathrm{mut}_w(P,F)$ then $P \cong \mathrm{mut}_{-w}(Q,F)$) [ACGK12, Lemma 2], and translating the factor results in an isomorphic mutation (i.e. $\mathrm{mut}_w(P,F) \cong \mathrm{mut}_w(P,F+v)$ for any $v \in N$ with $w(v) = 0$).

Mutations have a natural description as transformations of the dual polytope

$$P^\vee := \{u \in M_{\mathbb{Q}} \mid u(v) \geqslant -1 \text{ for all } v \in P\}.$$

A mutation induces a piecewise $\mathrm{GL}_n(\mathbb{Z})$-transformation $\varphi\colon u \mapsto u - u_{\min} w$ of $M_{\mathbb{Q}}$ such that $(\varphi(P^\vee))^\vee = \mathrm{mut}_w(P,F)$; here $u_{\min} := \min\{u(v) \mid v \in F\}$. This is analogous to a cluster transformation. As a consequence $|kP^\vee \cap M| = |kQ^\vee \cap M|$ for any dilation $k \in \mathbb{Z}_{\geqslant 0}$, where $Q := \mathrm{mut}_w(P,F)$ [ACGK12, Proposition 4]; hence $\mathrm{Hilb}(X_P, -K_{X_P}) = \mathrm{Hilb}\big(X_Q, -K_{X_Q}\big)$ and X_P and X_Q have the same anticanonical degree.

Example 3.3. The weighted projective spaces $\mathbb{P}(1,1,1,3)$ and $\mathbb{P}(1,1,4,6)$ have the largest degree amongst all canonical[1] *toric* Fano threefolds [Kas10] and amongst all *Gorenstein* canonical Fano threefolds [Pro05]. They are related by a mutation [ACGK12, Example 7]. The simplex associated to $\mathbb{P}(1,1,1,3)$ is $P := \mathrm{conv}\{(1,0,0),(0,1,0),(0,0,1),(-1,-1,-3)\} \subset N_{\mathbb{Q}}$. Setting $w = (-1,2,0) \in M$ gives $h_{\min} = -1$ and $h_{\max} = 2$, with $w_{-1}(P)$ equal to the edge $\mathrm{conv}\{(-1,-1,-3),(1,0,0)\}$ and $w_2(P)$ given by the vertex $(0,1,0)$. The factor

$$F := \mathrm{conv}\{(0,0,0),(2,1,3)\}$$

gives:

$$\mathrm{mut}_w(P,F) = \mathrm{conv}\{(-1,-1,-3),(0,0,1),(0,1,0),(4,3,6)\}$$

and this is the simplex associated with $\mathbb{P}(1,1,4,6)$. This mutation is illustrated in Figure 1.

[1] See §6 below for a discussion of canonical singularities.

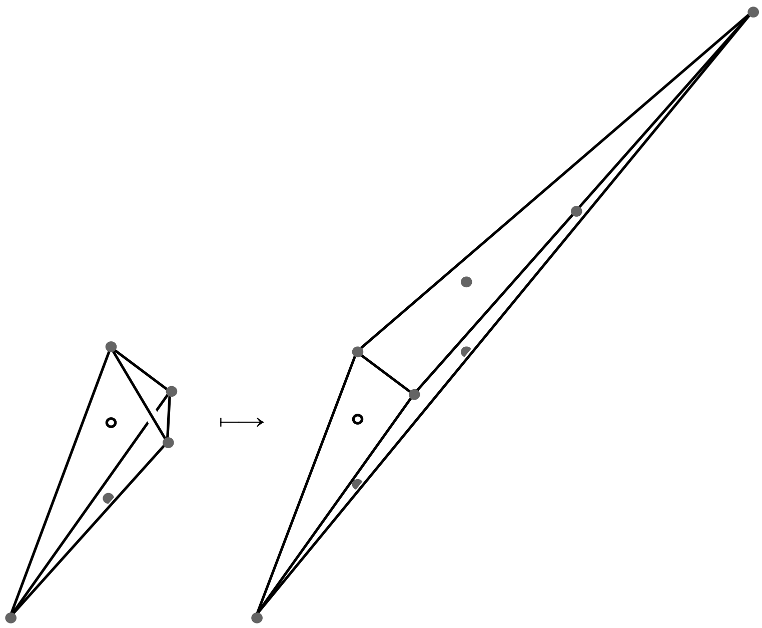

FIGURE 1. An edge mutation from the polytope corresponding to $\mathbb{P}(1,1,1,3)$, depicted on the left, to the polytope corresponding to $\mathbb{P}(1,1,4,6)$. This mutation is described in Example 3.3.

4. MUTATIONS OF n-SIMPLICES

We begin by establishing some basic properties of mutations between simplices. Throughout this section we assume that P is a Fano n-simplex, and that $w \in M$ and $F \subset N_{\mathbb{Q}}$ are (respectively) a primitive height function and a factor such that $Q := \mathrm{mut}_w(P, F)$ is a simplex. In other words, we assume that there is a mutation from the fake weighted projective space X associated with P to the fake weighted projective space Y associated with Q.

Lemma 4.1. *Let P be a Fano n-simplex, let $w \in M$ be a primitive height function, and let $F \subset N_{\mathbb{Q}}$ be a factor such that $Q := \mathrm{mut}_w(P, F)$ is a simplex. Suppose that the mutation from P to Q is non-trivial, so that $P \not\cong Q$. Then $w_{h_{\max}}(P)$ is a vertex of P and F is a translation of $\frac{1}{|h_{\min}|} w_{h_{\min}}(P)$.*

Proof. We first consider $w_{h_{\max}}(P)$. Suppose for a contradiction that $w_{h_{\max}}(P)$ is not a vertex of P. Then in particular it contains an edge E_1 of P. Let us pick an edge E_2 contained in $w_{h_{\min}}(P)$ (if such an edge does not exist then $w_{h_{\min}}(P)$ is a vertex and the mutation is trivial). Note that E_1 and E_2 cannot be parallel: if they were then the four endpoints of E_1 and E_2 would lie in a common two-dimensional affine subspace and thus would be affinely dependent; this contradicts the fact that P is a simplex. It is an immediate consequence of the definition of mutation that $w_{h_{\max}}(Q) = w_{h_{\max}}(P) + h_{\max}F$. Recall that F is a Minkowski factor of $w_{h_{\min}}(P)$. Because P is a simplex, $w_{h_{\min}}(P)$ is also a simplex and so by [She63, Result 13] F is a dilation and translation of $w_{h_{\min}}(P)$. It follows that $h_{\max}F$ is also a dilation and translation of $w_{h_{\min}}(P)$. Let E_2' denote the edge of $h_{\max}F$ corresponding to the edge E_2 of $w_{h_{\min}}(P)$. Then E_1 and E_2' are not parallel, because E_1 and E_2 are not parallel. Thus the face $E_1 + E_2'$ of $w_{h_{\max}}(Q) = w_{h_{\max}}(P) + h_{\max}F$ is a quadrilateral. Since Q is by assumption a simplex, and therefore all faces of Q are simplices, this gives a contradiction. We conclude that $w_{h_{\max}}(P)$ consists of a single vertex.

Now consider the factor F. By the definition of factor there exists a lattice polytope $G_{h_{\min}}$ such that

$$w_{h_{\min}}(P) = G_{h_{\min}} + |h_{\min}|\,F.$$

We claim that $G_{h_{\min}}$ is a point. After mutation we have $w_{h_{\min}}(Q) = G_{h_{\min}}$. As discussed above, we can mutate Q back to P by taking the height function $-w$ and the factor F. But $(-w)_{-h_{\min}}(Q) = w_{h_{\min}}(Q) = G_{h_{\min}}$ and we conclude from the first part of the proposition that $G_{h_{\min}}$ is a point: $G_{h_{\min}} = \{v\}$. It follows that $|h_{\min}|\,F = w_{h_{\min}}(P) - v$, and so F is a translation of $\frac{1}{|h_{\min}|} w_{h_{\min}}(P)$ as required. □

Definition 4.2. If $w_{h_{\min}}(P)$ has $k+1$ vertices then we say that the corresponding mutation is a *mutation over a k-face*. Cases of particular interest are $k = 1$ and $k = n-1$, which we call *mutations over edges* and *mutations over facets*, respectively.

Lemma 4.3. *Let P be a Fano n-simplex, let $w \in M$ be a primitive height function, and let $F \subset N_{\mathbb{Q}}$ be a factor such that $Q := \mathrm{mut}_w(P, F)$ is a simplex. Let $v \in \mathrm{vert}(P)$ be such that $w(v) \neq h_{\max}$ and $w(v) \neq h_{\min}$. Then $w(v) = 0$.*

Proof. Let us write $P = \mathrm{conv}\{v_0, \ldots, v_n\}$ and $w_{h_{\min}}(P) = \mathrm{conv}\{v_1, \ldots, v_k\}$. If $k = n$ then the statement holds vacuously, so let us assume that $k < n$. Without loss of generality we may assume that $\mathbf{0} \in \mathrm{vert}(F)$. In view of Lemma 4.1 we may assume further that $w_{h_{\max}}(P) = \{v_0\}$, that $w_{h_{\min}}(P) = v_1 + |h_{\min}|\,F$, and that

$$F := \mathrm{conv}\left\{\mathbf{0}, \frac{1}{|h_{\min}|}(v_2 - v_1), \ldots, \frac{1}{|h_{\min}|}(v_k - v_1)\right\}.$$

Suppose for a contradiction that there exists some $v \in \mathrm{vert}(P)$ such that $w(v) \neq h_{\max}$, $w(v) \neq h_{\min}$, and $w(v) \neq 0$. Without loss of generality we can take $v = v_n$.

Suppose first that $w(v_n) > 0$. Let $w' \in M$ be a primitive lattice point such that $w'(v_i) = 0$ for $v_i \in \{v_1, \ldots, v_{n-1}\}$. We can choose w' so that $w'(v_0) > 0$ and $w'(v_n) < 0$. Let $h'_{\max} = \sup\{w'(p) \mid p \in P\}$ and $h'_{\min} = \inf\{w'(p) \mid p \in P\}$. We see that $H_{w',h'_{\max}} \cap P = \{v_0\}$ and $H_{w',h'_{\min}} \cap P = \{v_n\}$. Note that by the definition of F, $w'(F) = 0$ and so $w'(x) = w'(x + w(x)F)$ for all $x \in P$ (with $w(x) \geqslant 0$). Then $H_{w',h'_{\max}} \cap Q = v_0 + h_{\max}F = v_0 + w(v_0)F$ and $H_{w',h'_{\min}} \cap Q = v_n + w(v_n)F$. Thus $v_0 + w(v_0)F$ and $v_n + w(v_n)F$ are two faces of Q, with vertices $v_0 + w(v_0)f$ and $v_n + w(v_n)f$ for $f \in \mathrm{vert}(F)$. Let $f_1, f_2 \in \mathrm{vert}(F)$ be any two distinct vertices of F; these exist since otherwise F is a point and the mutation is trivial. As Q is a simplex

$$F' := \mathrm{conv}\{v_0 + w(v_0)f_1, v_0 + w(v_0)f_2, v_n + w(v_n)f_1, v_n + w(v_n)f_2\}$$

is a two-dimensional face of Q. But F' has four vertices and thus is not a simplex. This gives a contradiction.

Suppose instead that $w(v_n) < 0$. By [ACGK12, Lemma 3.7] there exists a $v_Q \in \mathrm{vert}(Q)$ and $v_F \in \mathrm{vert}(F)$ such that $v_n = v_Q - w(v_Q)v_F$. Applying w to this equation yields $w(v_Q) = w(v_n) < 0$. We now have $\mathrm{mut}_{-w}(Q, F) = P$ but $-w(v_Q) > 0$ which, by the preceding argument, gives a contradiction. It follows that $w(v_n) = 0$. □

Combining the previous two results gives a necessary and sufficient combinatorial condition for the existence of a mutation between simplices.

Lemma 4.4. *Let P be a Fano n-simplex and let $w \in M$ be a primitive height function. Let the vertices of P be $\{v_0, v_1, \ldots, v_n\}$, ordered such that $w_{h_{\min}}(P) = \mathrm{conv}\{v_1, \ldots, v_k\}$. There exists a factor $F \subset N_{\mathbb{Q}}$ such that $Q := \mathrm{mut}_w(P, F)$ is a simplex if and only if the following hold:*

(i) *$w_{h_{\max}}(P)$ is a vertex;*
(ii) *$h_{\min} \mid v_i - v_1$ for $i \in \{1, \ldots, k\}$;*
(iii) *if v is a vertex of P such that $w(v) \neq h_{\max}$ and $w(v) \neq h_{\min}$, then $w(v) = 0$.*

Proof. The "only if" direction is Lemmas 4.1 and 4.3. On the other hand, it is clear that if conditions (i)–(iii) are satisfied, then defining

$$F := \operatorname{conv}\left\{\mathbf{0}, \frac{1}{|h_{\min}|}(v_2 - v_1), \ldots, \frac{1}{|h_{\min}|}(v_k - v_1)\right\}$$

gives a factor with respect to w. Let us label the vertices of $P = \operatorname{conv}\{v_0, \ldots, v_n\}$ so that:

$$w(v_0) = h_{\max}, \qquad w(v_1) = \ldots = w(v_k) = h_{\min}, \qquad w(v_{k+1}) = \ldots = w(v_n) = 0.$$

From [ACGK12, Lemma 3.7] we have that $\operatorname{vert}(Q) \subseteq \{v_0', v_1', \ldots, v_n'\}$, where:

$$v_i' = \begin{cases} v_i & \text{if } i = 0, i = 1, \text{ or } i \in \{k+1, \ldots, n\} \\ v_0 + \frac{h_{\max}}{|h_{\min}|}(v_i - v_1) & \text{if } i \in \{2, \ldots, k\}. \end{cases}$$

Furthermore by [ACGK12, Proposition 3.11] we have that Q is Fano, and hence is of maximal dimension in the n-dimensional lattice. It follows that Q must have at least $n+1$ vertices. Thus $Q = \operatorname{conv}\{v_0', v_1', \ldots, v_n'\}$, and so Q is a simplex. □

Example 4.5. Consider the 3-simplex $P = \operatorname{conv}\{(1,-1,0), (-2,-2,-1), (-2,-2,1), (0,1,0)\}$. The weighted projective space associated to P is $\mathbb{P}(1,1,4,8)$. Let $w = (1,0,0) \in M$. Lemma 4.4 implies that there is a factor F such that $Q := \operatorname{mut}_w(P, F)$ is the simplex:

$$Q = \operatorname{conv}\{(1,-1,0), (-2,-2,-1), (1,-1,1), (0,1,0)\}$$

The weighted projective space associated to Q is $\mathbb{P}(1,1,1,4)$.

The main result of this paper is:

Theorem 4.6. *Let X and Y be fake weighted projective spaces related by a mutation. Suppose that $P = \operatorname{conv}\{v_0, \ldots, v_n\} \subset N_{\mathbb{Q}}$ is the simplex corresponding to X, that $Q \subset N_{\mathbb{Q}}$ is the simplex corresponding to Y, and that the weights of X are $\lambda_0, \lambda_1, \ldots, \lambda_n$. Let $w \in M$ be the primitive height function and $F \subset N_{\mathbb{Q}}$ be the factor such that $Q = \operatorname{mut}_w(P, F)$. Then we may relabel the vertices of P such that $w_{h_{\max}}(P) = v_0$, $w_{h_{\min}}(P) = \operatorname{conv}\{v_1, \ldots, v_k\}$, and the weights of Y are:*

$$\frac{1}{d}\left(\lambda_0\lambda_1, (\lambda_1 + \ldots + \lambda_k)^2, \lambda_0\lambda_2, \ldots, \lambda_0\lambda_k, \lambda_{k+1}(\lambda_1 + \ldots + \lambda_k), \ldots, \lambda_n(\lambda_1 + \ldots + \lambda_k)\right)$$

where d is a positive integer satisfying:

$$d \cdot \frac{\operatorname{mult}(X)}{\operatorname{mult}(Y)} = \frac{\lambda_0^{k-1}}{(\lambda_1 + \ldots + \lambda_k)^{k-2}}$$

Proof. By Lemma 4.4 we have $Q = \operatorname{conv}\{v_0', v_1', \ldots, v_n'\}$ where:

$$v_i' = \begin{cases} v_i & \text{if } i = 0, i = 1, \text{ or } i \in \{k+1, \ldots, n\} \\ v_0 + \frac{h_{\max}}{|h_{\min}|}(v_i - v_1) & \text{if } i \in \{2, \ldots, k\}. \end{cases} \tag{4.1}$$

Since P is Fano we may assume that the weights $\lambda_0, \lambda_1, \ldots, \lambda_n$ are well-formed. We normalise by setting $h := \sum_{i=0}^n \lambda_i$ and $\lambda_i' := \frac{1}{h}\lambda_i$. Then $\sum_{i=0}^n \lambda_i' v_i = \mathbf{0}$, $\sum_{i=0}^n \lambda_i' = 1$, and $\lambda_i' \geqslant 0$ for all i. The sequence $\lambda_0', \lambda_1', \ldots, \lambda_n'$ is unique with these properties; these are the *normalised barycentric co-ordinates* for P.

Since Q is Fano there exist $\mu'_0, \mu'_1, \ldots, \mu'_n$ such that $\sum_{i=0}^n \mu'_i v'_i = \mathbf{0}$, $\sum_{i=0}^n \mu'_i = 1$, and $\mu'_i \geqslant 0$ for all i. From $\sum_{i=0}^n \mu'_i v'_i = \mathbf{0}$ and (4.1) we have that:

$$\left(\mu'_0 + \sum_{i=2}^k \mu'_i\right) v_0 + \left(\mu'_1 - \frac{h_{\max}}{|h_{\min}|} \sum_{i=2}^k \mu'_i\right) v_1 + \sum_{i=2}^k \mu'_i \frac{h_{\max}}{|h_{\min}|} v_i + \sum_{i=k+1}^n \mu'_i v_i = \mathbf{0}.$$

Let θ_i denote the coefficient of v_i in the expression above. We claim that the θ_i are normalised barycentric co-ordinates. It is clear that $\sum_{i=0}^n \theta_i = \sum_{i=0}^n \mu'_i = 1$, and that $\theta_0, \theta_2, \theta_3, \ldots, \theta_n \geqslant 0$. It remains to check that $\theta_1 \geqslant 0$. Suppose for a contradiction that $\theta_1 < 0$. Then we have:

$$-\theta_1 v_1 = \theta_0 v_0 + \theta_2 v_2 + \ldots + \theta_n v_n \in \operatorname{cone}\{v_0, v_2, \ldots, v_n\}$$

and $-\theta_1 > 0$, so a point on the ray from $\mathbf{0}$ through v_1 lies in the cone over $v_0, v_2, \ldots, v_n$. This contradicts the fact that $\mathbf{0}$ lies in the strict interior of P.

Hence the θ_i are normalised barycentric co-ordinates for P, and so by uniqueness we have $\theta_i = \lambda'_i$ for all i. Solving these equations for μ'_i yields:

$$\mu'_i = \begin{cases} \lambda'_0 - \frac{|h_{\min}|}{h_{\max}} \sum_{i=2}^k \lambda'_i & \text{if } i = 0 \\ \lambda'_1 + \sum_{i=2}^k \lambda'_i & \text{if } i = 1 \\ \frac{|h_{\min}|}{h_{\max}} \lambda'_i & \text{if } i \in \{2, \ldots, k\} \\ \lambda'_i & \text{if } i \in \{k+1, \ldots, n\}. \end{cases}$$

Applying w to both sides of the equation $\sum_{i=1}^n \lambda_i v_i = \mathbf{0}$, we find that

$$\frac{|h_{\min}|}{h_{\max}} = \frac{\lambda_0}{\lambda_1 + \ldots + \lambda_k} \tag{4.2}$$

and thus:

$$\mu'_i = \begin{cases} \lambda'_0 - \frac{\lambda_0}{\lambda_1 + \ldots + \lambda_k} \sum_{i=2}^k \lambda'_i & \text{if } i = 0 \\ \lambda'_1 + \sum_{i=2}^k \lambda'_i & \text{if } i = 1 \\ \frac{\lambda_0}{\lambda_1 + \ldots + \lambda_k} \lambda'_i & \text{if } i \in \{2, \ldots, k\} \\ \lambda'_i & \text{if } i \in \{k+1, \ldots, n\}. \end{cases}$$

Now we can form integer weights by defining $\mu_i = h(\lambda_1 + \ldots + \lambda_k)\mu'_i$; this gives:

$$\mu_i = \begin{cases} \lambda_0 \lambda_1 & \text{if } i = 0 \\ (\lambda_1 + \ldots + \lambda_k)^2 & \text{if } i = 1 \\ \lambda_0 \lambda_i & \text{if } i \in \{2, \ldots, k\} \\ \lambda_i(\lambda_1 + \ldots + \lambda_k) & \text{if } i \in \{k+1, \ldots, n\}. \end{cases}$$

These weights are integers, but they may not be well-formed. However since Q is Fano, we know that the weights are well-formed if and only if they are reduced. Therefore it remains to divide through by their greatest common divisor, which we denote by d. Thus the weights of Q are:

$$\frac{1}{d}\left(\lambda_0\lambda_1, (\lambda_1 + \ldots + \lambda_k)^2, \lambda_0\lambda_2, \ldots, \lambda_0\lambda_k, \lambda_{k+1}(\lambda_1 + \ldots + \lambda_k), \ldots, \lambda_n(\lambda_1 + \ldots + \lambda_k)\right).$$

Consider now the degrees of X and Y:

$$(-K_X)^n = \frac{(\lambda_0 + \ldots + \lambda_n)^n}{\lambda_0 \ldots \lambda_n \operatorname{mult}(X)}$$

$$(-K_Y)^n = \frac{\frac{1}{d^n}\left(\lambda_0\lambda_1 + \ldots + \lambda_n(\lambda_1 + \ldots + \lambda_k)\right)^n}{\frac{1}{d^{n+1}}\left((\lambda_0\lambda_1) \ldots \lambda_n(\lambda_1 + \ldots + \lambda_k)\right) \operatorname{mult}(Y)}$$

Since degree is preserved by mutation, we conclude that

$$d \cdot \frac{\operatorname{mult}(X)}{\operatorname{mult}(Y)} = \frac{\lambda_0^{k-1}}{(\lambda_1 + \ldots + \lambda_k)^{k-2}} \tag{4.3}$$

as claimed. □

Remark 4.7. When $\operatorname{mult}(X) = \operatorname{mult}(Y)$, and in particular if X and Y are weighted projective spaces, we have:

$$d = \frac{\lambda_0^{k-1}}{(\lambda_1 + \ldots + \lambda_k)^{k-2}}$$

This gives an explicit expression for the weights after mutation in terms of the weights before mutation.

Remark 4.8. In the case of a mutation over a facet we see that the new weights are:

$$\frac{1}{d}\left(\lambda_0\lambda_1, (\lambda_1 + \ldots + \lambda_k)^2, \lambda_0\lambda_2, \ldots, \lambda_0\lambda_n\right)$$

Note that λ_0 divides d, because $\lambda_0 \mid \lambda_0\lambda_i$ for $i = 1, \ldots, n$ and the weights are well-formed. On the other hand, after dividing through by λ_0 we obtain well-formed weights, and so in fact $d = \lambda_0$. In this case, therefore, we obtain an explicit formula for how the multiplicity changes:

$$\frac{\operatorname{mult}(X)}{\operatorname{mult}(Y)} = \left(\frac{\lambda_0}{\lambda_1 + \ldots + \lambda_n}\right)^{n-2}$$

5. Multiplicity-preserving Mutations

The following result places a strong restriction on which mutations of fake weighted projective spaces can preserve multiplicity.

Theorem 5.1. *Any non-trivial multiplicity-preserving mutation between fake weighted projective spaces X and Y is a mutation over an edge.*

Proof. Let $P = \operatorname{conv}\{v_0, v_1, \ldots, v_n\}$ be the simplex associated to X, and let $\lambda_0, \lambda_1, \ldots, \lambda_n$ be the corresponding weights. Let $Q = \operatorname{conv}\{v'_0, v'_1, \ldots, v'_n\}$ be the simplex associated to Y, and let $\lambda'_0, \lambda'_1, \ldots, \lambda'_n$ be the corresponding weights. Suppose for a contradiction that P and Q are related by a non-trivial mutation over a k-face, for some $k > 2$. By Remark 4.7, after reordering weights if necessary, we have:

$$(\mu_0, \ldots, \mu_n) = \frac{1}{d}\left((\lambda_1 + \ldots + \lambda_k)^2, \lambda_0\lambda_1, \lambda_0\lambda_2, \ldots, \lambda_0\lambda_k, \lambda_{k+1}(\lambda_1 + \ldots + \lambda_k), \ldots, \lambda_n(\lambda_1 + \ldots + \lambda_k)\right)$$

where:

$$d = \frac{\lambda_0^{k-1}}{(\lambda_1 + \ldots + \lambda_k)^{k-2}}$$

We recall from (4.2) that

$$\frac{h_{\max}}{|h_{\min}|} = \frac{\lambda_1 + \ldots + \lambda_k}{\lambda_0}$$

and write $h_{\max}/|h_{\min}| = A/B$ with A and B coprime integers. So, for $i \in \{1, \ldots, k\}$, we have:

$$\mu_i = \frac{\lambda_0\lambda_i}{d} = \lambda_i \left(\frac{\lambda_1 + \ldots + \lambda_k}{\lambda_0}\right)^{k-2} = \lambda_i \left(\frac{A}{B}\right)^{k-2}$$

Since A and B are coprime and $k > 2$, we have that $B \mid \lambda_i$. Similarly for $i \in \{k+1, \ldots, n\}$ we have:

$$\mu_i = \frac{\lambda_i(\lambda_1 + \ldots + \lambda_k)}{d} = \lambda_i \left(\frac{\lambda_1 + \ldots + \lambda_k}{\lambda_0}\right)^{k-1} = \lambda_i \left(\frac{A}{B}\right)^{k-1}$$

and so in this case too $B \mid \lambda_i$. However $\lambda_1, \lambda_2, \ldots, \lambda_n$ are coprime, because the weights of X are well-formed, and therefore $B = 1$.

Since $h_{\max}/|h_{\min}| = A/B = A$ is an integer, we have that $\lambda_0 \mid \lambda_1 + \ldots + \lambda_k$. It follows that $\lambda_0 \mid d\mu_i$ for all i, and since the weights μ_i are reduced we conclude that $\lambda_0 \mid d$. Then

$$\frac{d}{\lambda_0} = \left(\frac{\lambda_0}{\lambda_1 + \ldots + \lambda_k}\right)^{k-2}$$

is an integer. Taking the $(k-2)$th root (recall that $k > 2$) we see that $\frac{\lambda_0}{\lambda_1 + \ldots + \lambda_k}$ is an integer, and hence that $\lambda_1 + \ldots + \lambda_k \mid \lambda_0$. Thus $\lambda_1 + \ldots + \lambda_k = \lambda_0$. Substituting this into our expression for the μ_i shows that the mutation is trivial, which is a contradiction. □

Corollary 5.2. *Suppose that X is a weighted projective space that admits a non-trivial mutation to another weighted projective space. Let λ_0 be the weight corresponding to the vertex $w_{h_{\max}}(P)$, and let λ_1, λ_2 be the weights corresponding to the vertices of the edge $w_{h_{\min}}(P)$. Then:*

(i) $\lambda_0 \mid (\lambda_1 + \lambda_2)^2$
(ii) $\gcd\{\lambda_1, \lambda_2\} \mid \lambda_0$

Proof. The mutation is an edge mutation, and so Theorem 4.6 implies both that $d = \lambda_0$ and that $d \mid (\lambda_1 + \lambda_2)^2$. This proves (i). Looking again at Theorem 4.6 and using well-formedness of weights, we see that $\gcd\{\lambda_1, \lambda_2\} \mid d$. This proves (ii). □

Remark 5.3. Let X be a fake weighted projective plane. Akhtar and Kasprzyk characterise mutations from X to other fake weighted projective planes in terms of solutions to an associated Diophantine equation [AK13, Proposition 3.12]. Their argument relies on the fact that, for lattice triangles, the square-free parts of the weights are preserved (up to reordering) under mutation. This phenomenon does not persist in higher dimensions:

(i) Example 3.3 above shows that, in general, neither the square-free parts of the weights nor the kth-power-free parts of the weights nor the square parts of the weights nor the kth-power parts of the weights are preserved.
(ii) Example 4.5 above shows that, in general, neither the nth-power-free parts nor the nth-power parts of the weights are preserved.

6. Canonical and Terminal Singularities

Terminal and canonical singularities were introduced by Miles Reid; they play a fundamental role in birational geometry [Rei80, Rei83, Rei87]. Terminal singularities form the smallest class of singularities that must be allowed if one wishes to construct minimal models in dimensions three or more. Canonical singularities can be regarded as the limit of terminal singularities; they arise naturally as the singularities occurring on canonical models of varieties of general type. From the toric viewpoint, terminal and canonical singularities have a particularly elegant combinatorial description. A toric singularity corresponds to a strictly convex rational polyhedral cone $\sigma \subset N_{\mathbb{Q}}$ [Dan78]. The cone σ is *terminal* if and only if:

(i) the lattice points $\rho_1, \ldots, \rho_m$ corresponding to the primitive generators of the rays of σ are contained in an affine hyperplane $H_u := \{v \in N_{\mathbb{Q}} \mid u(v) = 1\}$ for some $u \in M_{\mathbb{Q}}$;

(ii) with the exception of the origin $\mathbf{0}$ and the generators ρ_i of the rays, no other lattice points of N are contained in the part of σ on or under H_u, i.e.

$$N \cap \sigma \cap \{v \in N_{\mathbb{Q}} \mid u(v) \leqslant 1\} = \{\mathbf{0}, \rho_1, \ldots, \rho_m\}$$

The cone σ is *canonical* if and only if (i) holds and

(ii′) the origin $\mathbf{0}$ is the only lattice point contained in the part of σ under H_u, i.e.

$$N \cap \sigma \cap \{v \in N_{\mathbb{Q}} \mid u(v) < 1\} = \{\mathbf{0}\}$$

When the hyperplane H_u in condition (i) corresponds to a lattice point $u \in M$, the singularity is called *Gorenstein*.

7. Weighted Projective Spaces With Canonical Singularities

We now apply our results to the study of weighted projective spaces with canonical singularities, giving some geometric context, in terms of mutations, for a recent combinatorial result that characterises the fake weighted projective spaces of maximal degree with at worst canonical singularities [AKN13]. We say that a (fake) weighted projective space with at worst canonical singularities is a *canonical* (fake) weighted projective space.

Remark 7.1. If $\mathbb{P}(\lambda_0, \ldots, \lambda_n)/G$ is a canonical fake weighted projective space then $\mathbb{P}(\lambda_0, \ldots, \lambda_n)$ has canonical singularities too. Thus a canonical fake weighted projective space of maximal degree is necessarily a weighted projective space.

Definition 7.2. The *Sylvester numbers* $y_0, y_1, y_2, \ldots$ are defined by:

$$y_n = \begin{cases} 2 & \text{if } n = 0 \\ 1 + \prod_{i=0}^{n-1} y_i & \text{otherwise.} \end{cases}$$

We set $t_n := y_n - 1$.

Lemma 7.3 ([Syl80]).

(i) *The Sylvester numbers are pairwise coprime;*

(ii) *If $n \geqslant 1$ and $i < n$ then t_n/y_i is an integer. Furthermore if p is a prime dividing y_i then $p \mid t_n/y_j$ for $j < n$, $j \neq i$, and $p \nmid t_n/y_i$;*

(iii) *We have:*

$$\sum_{i=0}^{n-1} \frac{1}{y_i} = \frac{t_n - 1}{t_n}.$$

Define:

$$X_n := \mathbb{P}\left(1, 1, \tfrac{2t_{n-1}}{y_{n-2}}, \ldots, \tfrac{2t_{n-1}}{y_0}\right)$$

X_n is an n-dimensional canonical weighted projective space. Averkov, Krümpelmann, and Nill [AKN13] have proved that, if $n \geqslant 4$, then X_n is the unique n-dimensional canonical weighted projective space of maximum degree; this generalizes the corresponding result for Gorenstein weighted projective spaces, which is due to Nill [Nil07]. In dimension three there are precisely two canonical weighted projective spaces of maximum degree, and these are connected by a mutation: see Example 3.3. We next determine all weighted projective spaces $X_n^{(m,a)}$ that might be connected to X_n by a sequence of mutations through weighted projective spaces, and show that none of the $X_n^{(m,a)}$ are canonical.

Proposition 7.4. *Define integers* $\lambda_i^{(m,a)}$*, where* $0 \leqslant i \leqslant n$ *and* $0 \leqslant a \leqslant n-2$*, by:*

$$\lambda_i^{(0,a)} = \begin{cases} 1 & \text{if } i = 0 \\ \frac{2t_{n-1}}{y_a} & \text{if } i = 1 \\ 1 & \text{if } i = 2 \\ \frac{2t_{n-1}}{y_{k_i}} & \text{otherwise} \end{cases}$$

where $\{k_3, \ldots, k_n\}$ *is* $\{0, 1, \ldots, \widehat{a}, \ldots, n-2\}$ *with* a *omitted, and:*

$$\lambda_i^{(m,a)} = \begin{cases} \lambda_2^{(m-1,a)} & \text{if } i = 0 \\ \lambda_1^{(m-1,a)} & \text{if } i = 1 \\ \frac{\left(\lambda_1^{(m-1,a)} + \lambda_2^{(m-1,a)}\right)^2}{\lambda_0^{(m-1,a)}} & \text{if } i = 2 \\ \frac{\lambda_i^{(m-1,a)}\left(\lambda_1^{(m-1,a)} + \lambda_2^{(m-1,a)}\right)}{\lambda_0^{(m-1,a)}} & \text{otherwise} \end{cases}$$

for $m \geqslant 1$*. Set:*

$$X_n^{(m,a)} := \mathbb{P}\left(\lambda_0^{(m,a)}, \lambda_1^{(m,a)}, \ldots, \lambda_n^{(m,a)}\right)$$

Then:

(i) $X_n^{(0,a)} = X_n$ *for all* a*;*

(ii) *if* $m > 0$ *and if* X *is a weighted projective space with a non-trivial mutation to* $X_n^{(m,a)}$ *then either* $X = X_n^{(m-1,a)}$ *or* $X = X_n^{(m+1,a)}$*;*

(iii) $X_n^{(m,a)}$ *is not canonical for any* $m \geqslant 1$*.*

In other words, the graph of mutations between weighted projective spaces starting at X_n *is a subtree of the following graph:*

$$\begin{array}{cccc} & & X_n & \\ X_n^{(1,0)} & X_n^{(1,1)} & \cdots & X_n^{(1,n-2)} \\ | & | & | & | \\ X_n^{(2,0)} & X_n^{(2,1)} & \cdots & X_n^{(2,n-2)} \\ | & | & | & | \\ X_n^{(3,0)} & X_n^{(3,1)} & \cdots & X_n^{(3,n-2)} \\ | & | & | & | \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

and the only canonical weighted projective space in this graph is X_n*.*

Proof. Statement (i) is trivial. For (iii), recall that a weighted projective space $\mathbb{P}(\mu_0, \ldots, \mu_n)$ is canonical if and only if

$$\sum_{i=0}^{n} \left\{ \frac{\mu_i \kappa}{h} \right\} \in \{1, \ldots, n-1\}$$

for every $\kappa \in \{2, \ldots, h-2\}$, where $\{x\}$ denotes the fractional part of x and h is the sum of the μ_i [Kas13, Proposition 2.5]. This fails for $m \geqslant 1$ and $\kappa = h^{(m,a)} - \lambda_1^{(m,a)} - \lambda_2^{(m-1,a)}$, where $h^{(m,a)}$ is the sum of the weights of $X_n^{(m,a)}$: see Lemma A.14.

It remains to prove (ii). Suppose first that there is a non-trivial mutation from X_n to a weighted projective space X. By Theorem 5.1, this must be an edge mutation. Let P be the simplex associated to X_n, let μ_0 denote the weight associated to the vertex $w_{h_{\max}}(P)$, and let

μ_1, μ_2 be the weights associated to the vertices of $w_{h_{\min}}(P)$. We consider the possible values of μ_0, μ_1, μ_2.

Case 1. $\mu_0 \neq 1$. Then $\mu_0 = 2t_{n-1}/y_{a_0}$ for some $a_0 \in \{0, \ldots, n-2\}$.

Case 1.1. $\mu_1 = 1$, $\mu_2 = 1$. Since $n \geqslant 4$, there exist a_1, $a_2 \in \{0, 1, \ldots, n-2\}$ distinct such that $y_{a_1} \neq y_{a_0}$ and $y_{a_2} \neq y_{a_0}$. At least one of y_{a_1} or y_{a_2} is not equal to 2, and hence is a divisor of μ_0 not equal to 2 or 4. But $(\mu_1 + \mu_2)^2 = 4$, and this contradicts Corollary 5.2(i).

Case 1.2. $\mu_1 \neq 1$, $\mu_2 = 1$. Then $\mu_1 = 2t_{n-1}/y_{a_1}$ for some $a_1 \neq a_0$. Choose a_2 not equal to a_0 or a_1, and let p be a prime dividing y_{a_2}. Lemma 7.3(ii) imples that p divides both $2t_{n-1}/y_{a_0}$ and $2t_{n-1}/y_{a_1}$. So p does not divide $(2t_{n-1}/y_{a_1} + 1)^2$, and this contradicts Corollary 5.2(i).

Case 1.3. $\mu_1 \neq 1$, $\mu_2 \neq 1$. Then $\mu_1 = 2t_{n-1}/y_{a_1}$ and $\mu_2 = 2t_{n-1}/y_{a_2}$ for a_1, a_2 distinct and not equal to a_0. Suppose first that $y_{a_0} \neq 2$, and let p be a prime dividing y_{a_0}. Lemma 7.3(ii) implies that $p \mid \gcd\{\mu_1, \mu_2\}$ and $p \nmid \mu_0$, contradicting Corollary 5.2(ii). On the other hand if $y_{a_0} = 2$ then the same argument with $p = 2$ shows that $4 \mid \gcd\{\mu_1, \mu_2\}$ and $4 \nmid \mu_0$, again contradicting Corollary 5.2(ii)

Case 2. $\mu_0 = 1$.

Case 2.1. $\mu_1 \neq 1$, $\mu_2 = 1$. Then $\mu_1 = 2t_{n-1}/y_a$ for some $a \in \{0, \ldots, n-2\}$, and Theorem 4.6 implies that $X = X_n^{(1,a)}$.

Case 2.2. $\mu_1 \neq 1$, $\mu_2 \neq 1$. Then $\mu_1 = 2t_{n-1}/y_{a_1}$,$\mu_2 = 2t_{n-1}/y_{a_2}$ for a_1, a_2 distinct. Choose a_0 not equal to a_1 or a_2, and let p be a prime dividing y_{a_0}. Lemma 7.3(ii) implies that $p \mid \gcd\{\mu_1, \mu_2\}$ and $p \nmid \mu_0$, contradicting Corollary 5.2(ii).

This completes the proof in the case where $m = 0$.

Suppose now that $m > 1$, and that there is a non-trivial mutation from $X_n^{(m,a)}$ to a weighted projective space X. Theorem 5.1 implies that this is an edge mutation. Let P be the simplex corresponding to $X_n^{(m,a)}$, let μ_0 denote the weight associated to the vertex $w_{h_{\max}}(P)$, and let μ_1, μ_2 denote the weights associated to the vertices of $w_{h_{\min}}(P)$. To declutter the notation, write λ_i for the weight $\lambda_i^{(m,a)}$ of $X_n^{(m,a)}$. We consider the possible values of μ_0, μ_1, μ_2 in turn.

Example Case. $\mu_0 = \lambda_i$ with $i \geqslant 3$, $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$. By Lemma A.3 we have that μ_0 and μ_2 share a common factor that does not divide μ_1, and so does not divide $\mu_1 + \mu_2$. Thus $\mu_0 \nmid (\mu_1 + \mu_2)$, contradicting Corollary 5.2(i).

The remaining cases are entirely analogous. We arrive at a contradiction in all but two cases, as summarized in Table 1 below; here λ_i, λ_j, λ_k denote distinct elements of $\{3, \ldots, n\}$. The case $\mu_0 = \lambda_0$, $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$ yields $X_n^{(m+1,a)}$, and the case $\mu_0 = \lambda_2$, $\mu_1 = \lambda_0$, $\mu_2 = \lambda_1$ yields $X_n^{(m-1,a)}$. This completes the proof in the case where $m > 1$.

Suppose now that $m = 1$. We can argue exactly as for $m > 1$, except in those cases where Lemma A.5 is used. We consider these three cases separately. As before, write λ_i for $\lambda_i^{(m,a)}$.

Case 3. $\mu_0 = \lambda_0$, $\mu_1 = \lambda_1$, $\mu_2 = \lambda_i$ with $i \geqslant 3$. Then $\mu_0 = 1$, $\mu_1 = 2t_{n-1}/y_a$, and $\mu_2 = \frac{2t_{n-1}}{y_{a_i}}\left(\frac{2t_{n-1}}{y_a} + 1\right)$. Choose a_j not equal to a or a_i, and let p be a prime dividing y_{a_j}. Then p divides μ_1 and μ_2 but does not divide μ_0, contradicting Corollary 5.2(ii).

μ_0	μ_1	μ_2	Contradicts	Using	μ_0	μ_1	μ_2	Contradicts	Using
λ_0	λ_1	λ_2	–	–	λ_2	λ_1	λ_i	Corollary 5.2(i)	Lemma A.4
λ_0	λ_1	λ_i	Corollary 5.2(i)	Lemma A.5	λ_2	λ_i	λ_j	Corollary 5.2(ii)	Lemma A.6
λ_0	λ_2	λ_i	Corollary 5.2(i)	Lemma A.5	λ_i	λ_0	λ_1	Corollary 5.2(i)	Lemma A.4
λ_0	λ_i	λ_j	Corollary 5.2(ii)	Lemma A.6	λ_i	λ_0	λ_2	Corollary 5.2(i)	Lemma A.4
λ_1	λ_0	λ_2	Corollary 5.2(i)	Lemma A.8	λ_i	λ_0	λ_j	Corollary 5.2(i)	Lemma A.4
λ_1	λ_0	λ_i	Corollary 5.2(i)	Lemma A.7	λ_i	λ_1	λ_2	Corollary 5.2(i)	Lemma A.4
λ_1	λ_2	λ_i	Corollary 5.2(i)	Lemma A.7	λ_i	λ_1	λ_j	Corollary 5.2(i)	Lemma A.4
λ_1	λ_i	λ_j	Corollary 5.2(i)	Lemma A.7	λ_i	λ_2	λ_j	Corollary 5.2(i)	Lemma A.5
λ_2	λ_0	λ_1	–	–	λ_i	λ_j	λ_k	Corollary 5.2(i)	Lemma A.7
λ_2	λ_0	λ_i	Corollary 5.2(i)	Lemma A.4					

TABLE 1. A summary of the argument in the case where $m > 1$.

Case 4. $\mu_0 = \lambda_0$, $\mu_1 = \lambda_2$, $\mu_2 = \lambda_i$ with $i \geqslant 3$. Then $\mu_0 = 1$, $\mu_1 = (2t_{n-1}/y_a + 1)^2$, and $\mu_2 = \frac{2t_{n-1}}{y_{a_i}}\left(\frac{2t_{n-1}}{y_a} + 1\right)$. Let p be a prime dividing $2t_{n-1}/y_a + 1$. Then p divides μ_1 and μ_2 but does not divide μ_0, contradicting Corollary 5.2(ii).

Case 5. $\mu_0 = \lambda_i$ with $i \geqslant 3$, $\mu_1 = \lambda_2$, $\mu_2 = \lambda_j$ with $j \geqslant 3$. Then $\mu_0 = \frac{2t_{n-1}}{y_{a_i}}\left(\frac{2t_{n-1}}{y_a} + 1\right)$, $\mu_1 = (2t_{n-1}/y_a + 1)^2$, and $\mu_2 = \frac{2t_{n-1}}{y_{a_j}}\left(\frac{2t_{n-1}}{y_a} + 1\right)$. Let p be a prime dividing y_a. Then p divides μ_0 and μ_2 (Lemma 7.3) but does not divide μ_1 (Lemma A.1). This contradicts Corollary 5.2(i).

This completes the proof in the case where $m = 1$. □

8. Weighted Projective Spaces With Terminal Singularities

A (fake) weighted projective space with at worst terminal singularities is called a *terminal* (fake) weighted projective space. As before, a terminal fake weighted projective space of maximal degree is necessarily a weighted projective space. We now give a conjectural classification of terminal weighted projective spaces of maximal degree, and analyse them in terms of mutations.

In dimension 3, the unique terminal weighted projective space of maximum degree is $\mathbb{P}^3$. In dimension 4, Kasprzyk has shown [Kas13, Lemma 3.5] that the unique terminal weighted projective space of maximum degree is:

$$\mathbb{P}(1,1,6,14,21) = \mathbb{P}\left(1, 1, \tfrac{t_3}{y_2}, \tfrac{t_3}{y_1}, \tfrac{t_3}{y_0}\right)$$

The classification problem for terminal weighted projective spaces of maximum degree in dimensions 5 and higher is open. The space

$$X_n = \mathbb{P}\left(1, 1, \tfrac{t_{n-1}}{y_{n-2}}, \dots, \tfrac{t_{n-1}}{y_0}\right)$$

is terminal [Kas13, Lemma 3.7] and it seems reasonable to conjecture that, for $n \geqslant 4$, X_n is the unique terminal weighted projective space of maximum degree. Note that the methods of [AKN13] do not apply to this problem, as they are unable to distinguish between canonical and terminal singularities (they are insensitive to lattice points on the boundary of a simplex).

Proposition 8.1. *Define integers* $\lambda_i^{(m,a)}$*, where* $0 \leqslant i \leqslant n$ *and* $0 \leqslant a \leqslant n-2$*, by:*

$$\lambda_i^{(0,a)} = \begin{cases} 1 & \text{if } i = 0 \\ \frac{t_{n-1}}{y_a} & \text{if } i = 1 \\ 1 & \text{if } i = 2 \\ \frac{t_{n-1}}{y_{k_i}} & \text{otherwise} \end{cases}$$

where $\{k_3, \ldots, k_n\}$ *is* $\{0, 1, \ldots, \widehat{a}, \ldots, n-2\}$ *with* a *omitted, and:*

$$\lambda_i^{(m,a)} = \begin{cases} \lambda_2^{(m-1,a)} & \text{if } i = 0 \\ \lambda_1^{(m-1,a)} & \text{if } i = 1 \\ \frac{\left(\lambda_1^{(m-1,a)} + \lambda_2^{(m-1,a)}\right)^2}{\lambda_0^{(m-1,a)}} & \text{if } i = 2 \\ \frac{\lambda_i^{(m-1,a)}\left(\lambda_1^{(m-1,a)} + \lambda_2^{(m-1,a)}\right)}{\lambda_0^{(m-1,a)}} & \text{otherwise} \end{cases}$$

for $m \geqslant 1$*. Suppose that* $n \geqslant 5$*. Set:*

$$X_n^{(m,a)} := \mathbb{P}\left(\lambda_0^{(m,a)}, \lambda_1^{(m,a)}, \ldots, \lambda_n^{(m,a)}\right)$$

Then:

(i) $X_n^{(0,a)} = X_n$ *for all* a*;*

(ii) *if* $m > 0$ *and if* X *is a weighted projective space with a non-trivial mutation to* $X_n^{(m,a)}$ *then either* $X = X_n^{(m-1,a)}$ *or* $X = X_n^{(m+1,a)}$*;*

(iii) $X_n^{(m,a)}$ *is not terminal for any* $m \geqslant 1$*.*

In other words, the graph of mutations between weighted projective spaces starting at X_n *is a subtree of the following graph:*

$$\begin{array}{cccc} & & X_n & \\ X_n^{(1,0)} & X_n^{(1,1)} & \cdots & X_n^{(1,n-2)} \\ | & | & | & | \\ X_n^{(2,0)} & X_n^{(2,1)} & \cdots & X_n^{(2,n-2)} \\ | & | & | & | \\ X_n^{(3,0)} & X_n^{(3,1)} & \cdots & X_n^{(3,n-2)} \\ | & | & | & | \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

and the only terminal weighted projective space in this graph is X_n*.*

Proof. Statement (i) is trivial. For (iii), recall that a weighted projective space $\mathbb{P}(\mu_0, \ldots, \mu_n)$ is terminal if and only if

$$\sum_{i=0}^{n} \left\{ \frac{\mu_i \kappa}{h} \right\} \in \{2, \ldots, n-1\} \tag{8.1}$$

for every $\kappa \in \{2, \ldots, h-2\}$, where $\{x\}$ denotes the fractional part of x and h is the sum of the μ_i [Kas13, Proposition 2.3]. This fails for $m \geqslant 1$ and $\kappa = h^{(m,a)} - h^{(m-1,a)}$, where $h^{(m,a)}$ is the sum of the weights of $X_n^{(m,a)}$: see Lemma A.16.

It remains to prove (ii). This is analogous to the proof of Proposition 7.4(ii). The analogs of Lemmas A.2–A.5 and Lemmas A.7, A.8 hold for these weights $\lambda_i^{(m,a)}$ too, with almost identical proofs, and Lemma A.15 functions as a replacement for Lemma A.6. Suppose first that $m > 1$ and that there is a non-trivial mutation from $X_n^{(m,a)}$ to a weighted projective space X. By Theorem 5.1, this must be an edge mutation. Let P be the simplex associated to $X_n^{(m,a)}$, let μ_0 denote the weight associated to the vertex $w_{h_{\max}}(P)$, and let μ_1, μ_2 be the weights associated to the vertices of $w_{h_{\min}}(P)$. To declutter the notation, we again write λ_i for the weight $\lambda_i^{(m,a)}$ of $X_n^{(m,a)}$. We consider the possible values of μ_0, μ_1, μ_2 in turn, arriving at a contradiction in all but two cases. This is summarized in Table 2 below, where λ_i, λ_j, λ_k denote distinct elements of $\{3, \ldots, n\}$. The case $\mu_0 = \lambda_0$, $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$ yields $X_n^{(m+1,a)}$, and the case $\mu_0 = \lambda_2$, $\mu_1 = \lambda_0$, $\mu_2 = \lambda_1$ yields $X_n^{(m-1,a)}$.

μ_0	μ_1	μ_2	Contradicts	Using	μ_0	μ_1	μ_2	Contradicts	Using
λ_0	λ_1	λ_2	–	–	λ_2	λ_1	λ_i	Corollary 5.2(i)	Lemma A.4
λ_0	λ_1	λ_i	Corollary 5.2(i)	Lemma A.5	λ_2	λ_i	λ_j	Corollary 5.2(ii)	Lemma A.15
λ_0	λ_2	λ_i	Corollary 5.2(i)	Lemma A.5	λ_i	λ_0	λ_1	Corollary 5.2(i)	Lemma A.4
λ_0	λ_i	λ_j	Corollary 5.2(ii)	Lemma A.15	λ_i	λ_0	λ_2	Corollary 5.2(i)	Lemma A.4
λ_1	λ_0	λ_2	Corollary 5.2(i)	Lemma A.8	λ_i	λ_0	λ_j	Corollary 5.2(i)	Lemma A.4
λ_1	λ_0	λ_i	Corollary 5.2(i)	Lemma A.7	λ_i	λ_1	λ_2	Corollary 5.2(i)	Lemma A.4
λ_1	λ_2	λ_i	Corollary 5.2(i)	Lemma A.7	λ_i	λ_1	λ_j	Corollary 5.2(i)	Lemma A.4
λ_1	λ_i	λ_j	Corollary 5.2(i)	Lemma A.7	λ_i	λ_2	λ_j	Corollary 5.2(i)	Lemma A.5
λ_2	λ_0	λ_1	–	–	λ_i	λ_j	λ_k	Corollary 5.2(i)	Lemma A.7
λ_2	λ_0	λ_i	Corollary 5.2(i)	Lemma A.4					

TABLE 2. A summary of the argument in the case where $m > 1$.

Suppose now that $m = 1$. We argue exactly as for $m > 1$, except in those cases where Lemma A.5 is used. We consider these three cases separately, once again writing λ_i for $\lambda_i^{(m,a)}$. When $\mu_0 = \lambda_0$, $\mu_1 = \lambda_1$, $\mu_2 = \lambda_i$ with $i \geqslant 3$, we argue as in Proposition 7.4 case 3. When $\mu_0 = \lambda_0$, $\mu_1 = \lambda_2$, $\mu_2 = \lambda_i$ with $i \geqslant 3$, we argue as in Proposition 7.4 case 4. When $\mu_0 = \lambda_i$ with $i \geqslant 3$, $\mu_1 = \lambda_2$, $\mu_2 = \lambda_j$ with $j \geqslant 3$, we have $\mu_0 = \frac{t_{n-1}}{y_{a_i}}\left(\frac{t_{n-1}}{y_a} + 1\right)$, $\mu_1 = \left(\frac{t_{n-1}}{y_a} + 1\right)^2$ and $\mu_2 = \frac{t_{n-1}}{y_{a_j}}\left(\frac{t_{n-1}}{y_a} + 1\right)$. Since $n \geqslant 5$ we can find $a_k \in \{0, 1, \ldots, n-2\} \backslash \{a, a_i, a_j\}$. Let p be a prime dividing y_{a_k}. Then p divides μ_0 and μ_2 but p does not divide μ_1, contradicting Corollary 5.2(i). □

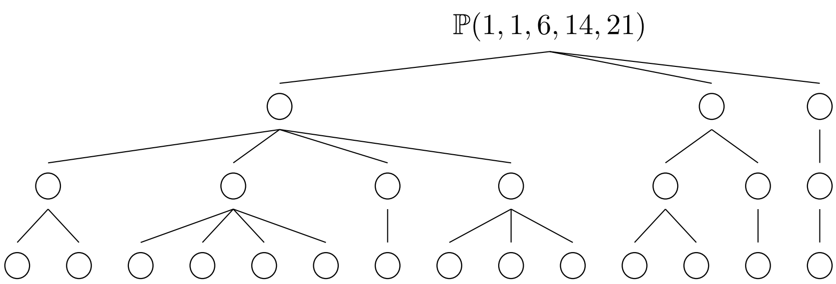

FIGURE 2. The graph of mutations between four-dimensional weighted projective spaces to a depth of 3

Remark 8.2. The requirement in Proposition 8.1 that $n \geqslant 5$ is necessary. Figure 2 shows the graph of mutations between four-dimensional weighted projective spaces starting from $X_4 =$

$\mathbb{P}(1,1,6,14,21)$ to a depth of three, where $\bigcirc$ denotes some weighted projective space. The rightmost branch of the tree above corresponds to a mutation with $\lambda_0 = 1, \lambda_1 = 21$ and $\lambda_2 = 1$. It can be shown using the methods of §7 that this branch continues as a chain. We do not know what happens in the other branches at greater depth.

Appendix A. The Sylvester Numbers and Related Sequences

Recall the definition of Sylvester numbers from Definition 7.2. Recall also that $t_n := y_n - 1$.

Lemma A.1. *Let $n \geqslant 1$ and let $i \in \{0, \ldots, n-1\}$. Then $y_i \mid t_n/y_i + 1$ and $y_i \nmid 2t_n/y_i + 1$.*

Proof. We proceed by induction on n. The base case $n = 1$ holds trivially. Note that:

$$t_n/y_i + 1 = (t_{n-1}/y_i + 1)y_{n-1} - (y_{n-1} - 1)$$

If $i \leqslant n-2$ then $y_i \mid y_{n-1} - 1$ and, by the induction hypothesis, $y_i \mid t_{n-1}/y_i + 1$. Thus $y_i \mid t_n/y_i + 1$. If $i = n-1$ then $t_n/y_i + 1 = y_{n-1}$, which is certainly divisible by y_i. This completes the induction step, proving that $y_i \mid t_n/y_i + 1$. It follows that $y_i \mid 2t_n/y_i + 2$, and so $y_i \nmid 2t_n/y_i + 1$. □

Lemma A.2. *Let $\lambda_i^{(m,a)}$ be as in Proposition 7.4. For $i \in \{0, \ldots, n\}$ the sequence $\left(\lambda_i^{(m,a)}\right)_{m \geqslant 0}$ is increasing.*

Proof. This is a straightforward induction on m. □

Lemma A.3. *Let $\lambda_i^{(m,a)}$ be as in Proposition 7.4. Fix $m \geqslant 0$. Then $\lambda_0^{(m,a)}$, $\lambda_1^{(m,a)}$, and $\lambda_2^{(m,a)}$ are pairwise coprime.*

Proof. We begin by showing that $\lambda_1^{(m,a)}$ and $\lambda_2^{(m,a)}$ are coprime, proceeding by induction. The base case $m = 0$ is trivial. Suppose that $\lambda_1^{(m-1,a)}$ and $\lambda_2^{(m-1,a)}$ are coprime, and suppose that there exists a prime p dividing both $\lambda_1^{(m,a)}$ and $\lambda_2^{(m,a)}$. Then:

$$p \mid \lambda_1^{(m-1,a)} \qquad \text{and} \qquad p \mid \frac{(\lambda_1^{(m-1,a)} + \lambda_2^{(m-1,a)})^2}{\lambda_0^{(m-1,a)}}$$

Thus $p \mid \lambda_2^{(m-1,a)}$, contradicting the hypothesis that $\lambda_1^{(m-1,a)}$ and $\lambda_2^{(m-1,a)}$ are coprime. This completes the induction step, showing that $\lambda_1^{(m,a)}$ and $\lambda_2^{(m,a)}$ are coprime for all m. It follows immediately that $\lambda_0^{(m,a)} = \lambda_2^{(m-1,a)}$ and $\lambda_1^{(m,a)} = \lambda_1^{(m-1,a)}$ are also coprime for all m.

Suppose now that $p \mid \lambda_0^{(m,a)} = \lambda_2^{(m-1,a)}$. Then $p \nmid \lambda_1^{(m,a)}$, as we have just seen, and so p does not divide the numerator of

$$\lambda_2^{(m,a)} = \frac{\left(\lambda_1^{(m,a)} + \lambda_2^{(m-1,a)}\right)^2}{\lambda_0^{(m-1,a)}}$$

Thus $p \nmid \lambda_2^{(m,a)}$, so $\lambda_0^{(m,a)}$ and $\lambda_2^{(m,a)}$ are coprime. □

Lemma A.4. *Let $\lambda_i^{(m,a)}$ be as in Proposition 7.4, let $m \geqslant 1$, and let $a \in \{0, 1, \ldots, n-2\}$. There exists a prime p such that $p \nmid \lambda_0^{(m,a)}$, $p \nmid \lambda_1^{(m,a)}$, and $p \mid \lambda_i^{(m,a)}$ for all $i \geqslant 2$.*

Proof. Lemma A.2 implies that, for $m \geqslant 2$, $\lambda_0^{(m-1,a)} = \lambda_2^{(m-2,a)} \leqslant \lambda_2^{(m-1,a)}$. Thus $\frac{\lambda_1^{(m,a)} + \lambda_2^{(m-1,a)}}{\lambda_0^{(m-1,a)}} > 1$ for all $m \geqslant 1$, and so there is some prime p such that $p \mid \lambda_1^{(m-1,a)} + \lambda_2^{(m-1,a)}$ but $p \nmid \lambda_0^{(m-1,a)}$. Thus $p \mid \lambda_i^{(m)}$ for all $i \in \{2, \ldots, n\}$. Lemma A.3 now implies that $p \nmid \lambda_0^{(m,a)}$ and $p \nmid \lambda_1^{(m,a)}$. □

Lemma A.5. *Let $\lambda_i^{(m,a)}$ be as in Proposition 7.4, let $m \geqslant 2$, and let $a \in \{0, 1, \ldots, n-2\}$. There exists a prime p such that $p \nmid \lambda_1^{(m,a)}$, $p \nmid \lambda_2^{(m,a)}$, and $p \mid \lambda_i^{(m,a)}$ for $i = 0$ and all $i \geqslant 3$.*

Proof. By Lemma A.4 there exists a prime p such that p divides $\lambda_i^{(m-1,a)}$ for $i \in \{2,\dots,n\}$ but p does not divide $\lambda_0^{(m-1,a)}$ or $\lambda_1^{(m-1,a)}$. Thus $p \mid \lambda_i^{(m,a)}$ for all $i \in \{3,\dots,n\}$ and, as $p \mid \lambda_2^{(m-1,a)} = \lambda_0^{(m,a)}$, we have by Lemma A.3 that p does not divide $\lambda_1^{(m,a)}$ or $\lambda_2^{(m,a)}$. ☐

Lemma A.6. *Let $\lambda_i^{(m,a)}$ be as in Proposition 7.4, let $m \geqslant 0$, and let $a \in \{0,1,\dots,n-2\}$. Then $2 \nmid \lambda_0^{(m,a)}$, $2 \nmid \lambda_2^{(m,a)}$, and $2 \mid \lambda_i^{(m,a)}$ for $i = 1$ and all $i \geqslant 3$.*

Proof. This is a straightforward induction on m. ☐

Lemma A.7. *Let $\lambda_i^{(m,a)}$ be as in Proposition 7.4, let $a \in \{0,1,\dots,n-2\}$, and let $i \in \{3,\dots,n\}$. Then there exists $k > 1$ such that for all $m \geqslant 0$, $k \nmid \lambda_j^{(m,a)}$ for $j \in \{0,2,i\}$ and $k \mid \lambda_j^{(m,a)}$ for $j \in \{0,\dots,n\}\backslash\{0,2,i\}$.*

Proof. We proceed by induction on m. For $m = 0$ we have $\lambda_j^{(m,a)} = 2t_{n-1}/y_{a_j}$ for some a_j. If $y_{a_i} \neq 2$ then let k be a prime dividing y_{a_i}; otherwise, let $k = 4$. In either case k has the desired properties and the claim holds.

Suppose now that there exists $k > 1$ such that $k \nmid \lambda_j^{(m-1,a)}$ for $j \in \{0,2,i\}$ and $k \mid \lambda_j^{(m-1,a)}$ for $j \in \{0,\dots,n\}\backslash\{0,2,i\}$. Lemma A.3 implies that k and $\lambda_0^{(m-1)}$ are coprime. Thus as

$$\lambda_j^{(m,a)} = \frac{\lambda_j^{(m-1,a)}(\lambda_1^{(m-1,a)}+\lambda_2^{(m-1,a)})}{\lambda_0^{(m-1,a)}} \qquad \text{for } j \in \{3,\dots,n\}$$

we see that $k \mid \lambda_j^{(m,a)}$ for $j \in \{3,\dots,n\}\backslash\{i\}$. Since $k \mid \lambda_1^{(m,a)} = \lambda_1^{(m-1,a)}$, by Lemma A.3 again we have that $k \nmid \lambda_0^{(m,a)}$ and $k \nmid \lambda_2^{(m,a)}$. Finally as k does not divide $\lambda_2^{(m-1,a)}$ but does divide $\lambda_1^{(m-1,a)}$, it does not divide $\lambda_1^{(m-1,a)} + \lambda_2^{(m-1,a)}$; since k also does not divide $\lambda_i^{(m-1,a)}$, by the recursion formula it cannot divide $\lambda_i^{(m,a)}$ either. ☐

Lemma A.8. *Let $\lambda_i^{(m,a)}$ be as in Proposition 7.4. There exists a prime p such that, for $m \geqslant 0$, $p \mid \lambda_1^{(m,a)}$ and $p \nmid \lambda_0^{(m,a)} + \lambda_2^{(m,a)}$.*

Proof. For the case $m = 0$, take p to be an odd prime dividing $\lambda_1 = 2t_{n-1}/y_a$. Suppose now that there exists a prime p such that $p \mid \lambda_1^{(m-1,a)}$ and $p \nmid \lambda_0^{(m-1,a)} + \lambda_2^{(m-1,a)}$. Then $p \nmid \lambda_1^{(m,a)} = \lambda_1^{(m-1,a)}$ and and $p \mid \lambda_1^{(m-1,a)}(\lambda_1^{(m-1,a)} + 2\lambda_2^{(m-1,a)})$. Now:

$$\lambda_0^{(m,a)} + \lambda_2^{(m,a)} = \frac{\lambda_2^{(m-1,a)}\left(\lambda_0^{(m-1,a)}+\lambda_2^{(m-1,a)}\right)+\lambda_1^{(m-1,a)}\left(\lambda_1^{(m-1,a)}+2\lambda_2^{(m-1,a)}\right)}{\lambda_0^{(m-1,a)}}$$

and by Lemma A.3, $p \nmid \lambda_2^{(m-1,a)}$. Thus $p \nmid \lambda_0^{(m)} + \lambda_2^{(m)}$. The result follows by induction on m. ☐

Lemma A.9. *Let $\lambda_i^{(m,a)}$ be as in Proposition 7.4, let $b \in \{0,1,\dots,n-2\}$ be such that $b \neq a$, and let $m \geqslant 0$. Then $y_b \mid \lambda_2^{(m,a)} - 1$*

Proof. The cases $m = 0$ and $m = 1$ are straightforward. Suppose now that $y_b \mid \lambda_2^{(m-2,a)} - 1$ and $y_b \mid \lambda_2^{(m-1,a)} - 1$. We have that:

$$\lambda_2^{(m,a)} - 1 = \frac{\lambda_1^{(m-1,a)}\left(\lambda_1^{(m-1,a)}+2\lambda_2^{(m-1,a)}\right)+\left(\lambda_2^{(m-1,a)}-1\right)\left(\lambda_2^{(m-1,a)}+\lambda_0^{(m-1,a)}\right)-\left(\lambda_2^{(m-2,a)}-1\right)\lambda_2^{(m-1,a)}}{\lambda_0^{(m-1,a)}}$$

Since $y_b \mid 2t_{n-1}/y_a = \lambda_1^{(m,a)}$, we have by Lemma A.3 that y_b and $\lambda_0^{(m-1,a)}$ are coprime. It thus suffices to show that y_b divides the numerator of the above expression. But this holds by assumption. The Lemma follows by induction on m. ☐

Definition A.10. Let $X_n^{(m,a)}$ be as in §7 and let $h^{(m,a)}$ denote the sum of the weights of $X_n^{(m,a)}$.

Lemma A.11.

$$h^{(m,a)} = \begin{cases} 2t_{n-1} & \text{if } m = 0 \\ \frac{\left(\lambda_1^{(m-1,a)}+\lambda_2^{(m-1,a)}\right)h^{(m-1,a)}}{\lambda_0^{(m-1,a)}} & \text{otherwise} \end{cases}$$

and, for any $i \in \{3,\dots,n\}$:

$$h^{(m,a)} = y_{k_i}\lambda_i^{(m,a)}$$

Proof. This is a straightforward calculation. □

Lemma A.12. *Let $\lambda_i^{(m,a)}$ be as in Proposition 7.4, let $a \in \{0,1,\dots,n-2\}$, and let $m \geqslant 1$. Then:*

$$\frac{\lambda_0^{(m,a)}\lambda_1^{(m,a)}}{h^{(m,a)}} < \frac{1}{2t_{n-1}}$$

Proof. By Lemma A.11 it suffices to prove that $\frac{2t_{n-1}\lambda_1^{(m,a)}}{y_{k_i}} < \frac{\lambda_i^{(m,a)}}{\lambda_0^{(m,a)}}$. This evidently holds for $m = 1$, and

$$\frac{\lambda_i^{(m,a)}}{\lambda_0^{(m,a)}} = \frac{\lambda_i^{(m-1,a)}}{\lambda_0^{(m-1,a)}}\frac{(\lambda_1^{(m-1,a)}+\lambda_2^{(m-1,a)})}{\lambda_2^{(m-1,a)}} > \frac{\lambda_i^{(m-1,a)}}{\lambda_0^{(m-1,a)}}$$

The result follows by induction on m. □

Lemma A.13. *Let $\lambda_i^{(m,a)}$ be as in Proposition 7.4, let $a \in \{0,1,\dots,n-2\}$, let $m \geqslant 0$, and let $i \in \{3,\dots,n\}$. Then:*

$$y_{k_i} \mid \lambda_2^{(m,a)} - \lambda_0^{(m,a)}$$

Proof. We proceed by induction on m. The base case $m = 0$ is trivial. Suppose now that $y_{k_i} \mid \lambda_2^{(m-1,a)} - \lambda_0^{(m-1,a)}$. Then, since $\lambda_2^{(m,a)} - \lambda_0^{(m,a)} = (\lambda_2^{(m,a)} - \lambda_0^{(m-1,a)}) - (\lambda_2^{(m-1,a)} - \lambda_0^{(m-1,a)})$, it suffices to show that y_{k_i} divides $\lambda_2^{(m,a)} - \lambda_0^{(m-1,a)}$. But:

$$\lambda_2^{(m,a)} - \lambda_0^{(m-1,a)} = \frac{\lambda_1^{(m-1,a)}(\lambda_1^{(m-1,a)} + 2\lambda_2^{(m-1,a)}) + (\lambda_2^{(m-1,a)} - \lambda_0^{(m-1,a)})(\lambda_2^{(m-1,a)} + \lambda_0^{(m-1,a)})}{\lambda_0^{(m-1,a)}}$$

Now y_{k_i} divides $\lambda_1^{(m,a)} = \frac{2t_{n-1}}{y_a}$ and $y_{k_i} \mid (\lambda_2^{(m-1,a)} - \lambda_0^{(m-1,a)})$, so y_{k_i} divides the numerator here. Lemma A.3 implies that y_{k_i} is coprime to the denominator. Thus y_{k_i} divides $\lambda_2^{(m,a)} - \lambda_0^{(m-1,a)}$. □

Lemma A.14. *Let $\lambda_i^{(m,a)}$ be as in Proposition 7.4, let $a \in \{0,1,\dots,n-2\}$, let $m \geqslant 1$, and let $\kappa^{(m,a)} = h^{(m,a)} - (\lambda_1^{(m-1,a)} + \lambda_2^{(m-1,a)})$. Then $\kappa^{(m,a)} \in \{2,\dots,h^{(m,a)} - 2\}$, and:*

$$\sum_{i=0}^{n} \left\{ \frac{\lambda_i^{(m,a)}\kappa^{(m,a)}}{h^{(m,a)}} \right\} > n-1$$

where $\{x\}$ denotes the fractional part of x.

Proof. The first statement follows immediately from Lemma A.2. We claim that:

$$\left\{ \frac{\kappa^{(m,a)}\lambda_0^{(m,a)}}{h^{(m,a)}} \right\} = \begin{cases} 1 - \frac{1}{2t_{n-1}} & \text{if } m \text{ is odd} \\ 1 - \frac{1}{2t_{n-1}} - \frac{1}{y_a} & \text{if } m \text{ is even} \end{cases} \tag{A.1}$$

Note that:

$$\begin{aligned}
\frac{\kappa^{(m,a)}\lambda_0^{(m,a)}}{h^{(m,a)}} &= 1 - \frac{(\lambda_1^{(m-1,a)}+\lambda_2^{(m-1,a)})\lambda_0^{(m,a)}}{y_{k_i}\lambda_i^{(m,a)}} && \text{by Lemma A.11} \\
&= 1 - \frac{\lambda_0^{(m-1,a)}\lambda_0^{(m,a)}}{y_{k_i}\lambda_i^{(m-1,a)}} \\
&= 1 - \frac{\lambda_0^{(m-2,a)}\lambda_0^{(m-1,a)}\lambda_2^{(m-1,a)}}{y_{k_i}\lambda_i^{(m-2,a)}(\lambda_1^{(m-2,a)}+\lambda_2^{(m-2,a)})} \\
&= 1 - \frac{\lambda_2^{(m-2,a)}(\lambda_1^{(m-2,a)}+\lambda_2^{(m-2,a)})}{y_{k_i}\lambda_i^{(m-2,a)}} \\
&= 1 - \frac{\lambda_0^{(m-2,a)}(\lambda_1^{(m-3,a)}+\lambda_2^{(m-3,a)})}{y_{k_i}\lambda_i^{(m-2,a)}} \\
&\quad - \frac{\lambda_2^{(m-2,a)}(\lambda_1^{(m-2,a)}+\lambda_2^{(m-2,a)}) - \lambda_0^{(m-2,a)}(\lambda_1^{(m-2,a)}+\lambda_0^{(m-2,a)})}{y_{k_i}\lambda_i^{(m-2,a)}} \qquad \text{(A.2)}
\end{aligned}$$

We claim that the last term (A.2) here is an integer. It is equal to:

$$\frac{(\lambda_2^{(m-2,a)}-\lambda_0^{(m-2,a)})(\lambda_0^{(m-2,a)}+\lambda_1^{(m-2,a)}+\lambda_2^{(m-2,a)})}{h^{(m-2,a)}}$$

Now:

$$\begin{aligned}
\frac{\lambda_0^{(m-2,a)}+\lambda_1^{(m-2,a)}+\lambda_2^{(m-2,a)}}{h^{(m-2,a)}} &= \frac{h^{(m-2,a)}-\sum_{i=3}^n\lambda_i^{(m-2,a)}}{h^{(m-2,a)}} \\
&= 1 - \sum_{i=3}^n \frac{\lambda_i^{(m-2,a)}}{h^{(m-2,a)}} \\
&= 1 - \sum_{i=3}^n \frac{1}{y_{k_i}} && \text{by Lemma A.11} \\
&= 1 - \sum_{i=0}^{n-2} \frac{1}{y_i} + \frac{1}{y_a} \\
&= \frac{1}{t_{n-1}} + \frac{1}{y_a} && \text{by Lemma 7.3}
\end{aligned}$$

Hence (A.2) is equal to:

$$\frac{(\lambda_2^{(m-2,a)}-\lambda_0^{(m-2,a)})(t_{n-1}/y_a+1)}{t_{n-1}}$$

Recall that $t_{n-1}=\prod_{i=0}^{n-2}y_i$. We will show that each y_i divides the numerator of this expression. By Lemma A.1 we have that $y_a \mid \frac{t_{n-1}}{y_a}+1$, and by Lemma A.13 we have that, for all $i\neq a$, $y_i \mid \lambda_2^{(m-2,a)}-\lambda_0^{(m-2,a)}$. Hence (A.2) is an integer. Thus:

$$\left\{\frac{\kappa^{(m,a)}\lambda_0^{(m,a)}}{h^{(m,a)}}\right\} = \left\{1-\frac{\lambda_0^{(m-2,a)}(\lambda_1^{(m-3,a)}+\lambda_2^{(m-3,a)})}{y_{k_i}\lambda_i^{(m-2,a)}}\right\} = \left\{\frac{\kappa^{(m-2,a)}\lambda_0^{(m-2,a)}}{h^{(m-2,a)}}\right\}$$

Since (A.1) holds for $m=1$ and $m=2$, by induction it holds for all m.

Lemma A.11 implies that

$$\frac{\kappa^{(m,a)}}{h^{(m,a)}} = 1 - \frac{\lambda_0^{(m-1,a)}}{h^{(m-1,a)}}$$

We have:

$$\begin{aligned}\left\{\frac{\kappa^{(m,a)}\lambda_1^{(m,a)}}{h^{(m,a)}}\right\} &= \left\{1-\frac{\lambda_0^{(m-1,a)}\lambda_1^{(m,a)}}{h^{(m-1,a)}}\right\} \\ &= 1-\frac{\lambda_0^{(m-1,a)}\lambda_1^{(m,a)}}{h^{(m-1,a)}} && \text{by Lemma A.12}\end{aligned}$$

and for $i \in \{3, \ldots n\}$ we have:

$$\begin{aligned}\left\{\frac{\kappa^{(m,a)}\lambda_i^{(m,a)}}{h^{(m,a)}}\right\} &= \left\{1-\frac{\lambda_i^{(m-1,a)}(\lambda_1^{(m-1,a)}+\lambda_2^{(m-1,a)})}{h^{(m-1,a)}}\right\} \\ &= \left\{1-\frac{\lambda_1^{(m-1,a)}+\lambda_2^{(m-1,a)}}{y_{k_i}}\right\} && \text{by Lemma A.11} \\ &= \left\{1-\frac{\lambda_1^{(m-1,a)}}{y_{k_i}}-\frac{\lambda_2^{(m-1,a)}-1}{y_{k_i}}-\frac{1}{y_{k_i}}\right\} \\ &= 1-\frac{1}{y_{k_i}} && \text{by Lemma A.9}\end{aligned}$$

Putting this all together, for $m = 1$ we obtain:

$$\begin{aligned}\sum_{i=0}^{n}\left\{\frac{\kappa^{(1,a)}\lambda_i^{(1,a)}}{h^{(1,a)}}\right\} &= \left\{\frac{\kappa^{(1,a)}\lambda_2^{(1,a)}}{h^{(1,a)}}\right\} + n - \sum_{i=0}^{n-2}\frac{1}{y_i} - \frac{1}{2t_{n-1}} \\ &= \left\{\frac{\kappa^{(1)}\lambda_2^{(1)}}{h^{(1)}}\right\} + n - \frac{t_{n-1}-1}{t_{n-1}} - \frac{1}{2t_{n-1}} && \text{by Lemma 7.3} \\ &> n-1\end{aligned}$$

and for $m \geqslant 2$ we obtain:

$$\begin{aligned}\sum_{i=0}^{n}\left\{\frac{\kappa^{(m,a)}\lambda_i^{(m,a)}}{h^{(m,a)}}\right\} &\geqslant \left\{\frac{\kappa^{(m,a)}\lambda_2^{(m,a)}}{h^{(m,a)}}\right\} + n - \sum_{i=0}^{n-2}\frac{1}{y_i} - \frac{1}{2t_{n-1}} - \frac{\lambda_0^{(m-1,a)}\lambda_1^{(m,a)}}{h^{(m-1,a)}} \\ &\geqslant (n-1) + \frac{1}{2t_{n-1}} - \frac{\lambda_0^{(m-1,a)}\lambda_1^{(m,a)}}{h^{(m-1,a)}} \\ &> n-1\end{aligned}$$

where at the last step we used Lemma A.12. □

Lemma A.15. *Let $\lambda_i^{(m,a)}$ be as in Proposition 8.1, let $n \geqslant 5$, let $a \in \{0, 1, \ldots, n-2\}$, and let i, $j \in \{3, \ldots, n\}$ be distinct. There exists a prime p such that, for all $m \geqslant 0$, p divides $\lambda_1^{(m,a)}$, $\lambda_i^{(m,a)}$, and $\lambda_j^{(m,a)}$ but p does not divide $\lambda_0^{(m,a)}$ or $\lambda_2^{(m,a)}$.*

Proof. We have $\lambda_i^{(0,a)} = \frac{t_{n-1}}{y_{k_i}}$, $\lambda_j^{(0,a)} = \frac{t_{n-1}}{y_{k_j}}$, and $\lambda_1^{(0,a)} = \frac{t_{n-1}}{y_a}$. Since $n \geqslant 5$ we can find $k_l \in \{0, 1, \ldots, n-2\}\backslash\{a, k_i, k_j\}$. Let p be a prime dividing y_{k_l}. Then p divides $\lambda_i^{(0,a)}$, $\lambda_j^{(0,a)}$, and $\lambda_1^{(0,a)}$ and thus, by Lemma A.3, p divides neither $\lambda_0^{(0,a)}$ nor $\lambda_2^{(0,a)}$. We now proceed by induction on m: the induction step is straightforward. □

Lemma A.16. *Let $\lambda_i^{(m,a)}$ be as in Proposition 8.1, let $a \in \{0, 1, \ldots, n-2\}$, let $m \geqslant 1$, and let $\kappa^{(m,a)} = h^{(m,a)} - h^{(m-1,a)}$. Then $\kappa^{(m,a)} \in \{2, \ldots, h^{(m,a)} - 2\}$, and:*

$$\sum_{i=0}^{n}\left\{\frac{\lambda_i^{(m,a)}\kappa^{(m,a)}}{h^{(m,a)}}\right\} < 2$$

where $\{x\}$ denotes the fractional part of x.

Proof. The first statement follows immediately from Lemma A.2. The conclusions of Lemma A.11 hold here too, and thus:

$$\frac{\kappa^{(m,a)}}{h^{(m,a)}} = 1 - \frac{\lambda_0^{(m-1,a)}}{\lambda_1^{(m-1,a)} + \lambda_2^{(m-1,a)}}$$

It follows that:

$$\left\{ \frac{\kappa^{(m,a)}\lambda_2^{(m,a)}}{h^{(m,a)}} \right\} = \left\{ -\lambda_2^{(m,a)} \frac{\lambda_0^{(m-1,a)}}{\lambda_1^{(m-1,a)} + \lambda_2^{(m-1,a)}} \right\} = 0$$

and, for $i \in \{3, \ldots, n\}$:

$$\left\{ \frac{\kappa^{(m,a)}\lambda_i^{(m,a)}}{h^{(m,a)}} \right\} = \left\{ -\lambda_i^{(m,a)} \frac{\lambda_0^{(m-1,a)}}{\lambda_1^{(m-1,a)} + \lambda_2^{(m-1,a)}} \right\} = 0$$

Thus:

$$\sum_{i=0}^{n} \left\{ \frac{\kappa^{(m,a)}\lambda_i^{(m,a)}}{h^{(m,a)}} \right\} = \left\{ \frac{\kappa^{(m,a)}\lambda_0^{(m,a)}}{h^{(m,a)}} \right\} + \left\{ \frac{\kappa^{(m,a)}\lambda_1^{(m,a)}}{h^{(m,a)}} \right\} < 2$$

□

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Department of Mathematics, Imperial College London, London, SW7 2AZ, UK

E-mail address: `t.coates@imperial.ac.uk`

E-mail address: `samuel.gonshaw10@imperial.ac.uk`

E-mail address: `a.m.kasprzyk@imperial.ac.uk`

E-mail address: `navid.nabijou09@imperial.ac.uk`