Paving Tropical Ideals

Nicholas Anderson, Felipe Rincón

Queen Mary University of London

November 16, 2021

Outline

"Tropical Geometry"

1. $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ with operations $a \oplus b := \min\{a, b\}$ and $a \otimes b := a + b$ the **tropical semiring**

"Tropical Geometry"

1. $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ with operations $a \oplus b := \min\{a, b\}$ and $a \otimes b := a + b$ the **tropical semiring**
2. $\mathbb{B} := \{0, \infty\} \subset \overline{\mathbb{R}}$ is the **boolean semiring**

"Tropical Geometry"

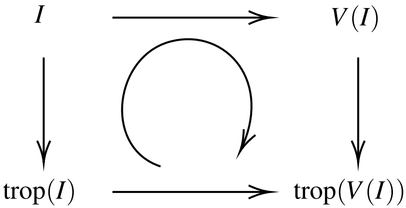
1. $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ with operations $a \oplus b := \min\{a, b\}$ and $a \otimes b := a + b$ the **tropical semiring**
2. $\mathbb{B} := \{0, \infty\} \subset \overline{\mathbb{R}}$ is the **boolean semiring**
3. Given a field K, we consider $\mathrm{val} : K \to T$ a "non-archemedian field valuation"

"Tropical Geometry"

1. $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ with operations $a \oplus b := \min\{a, b\}$ and $a \otimes b := a + b$ the **tropical semiring**
2. $\mathbb{B} := \{0, \infty\} \subset \overline{\mathbb{R}}$ is the **boolean semiring**
3. Given a field K, we consider $\mathrm{val} : K \to T$ a "non-archemedian field valuation"
4. For this talk $val(a) = 0 \iff a \neq 0_K$, $\mathrm{val}(0_K) = \infty$.

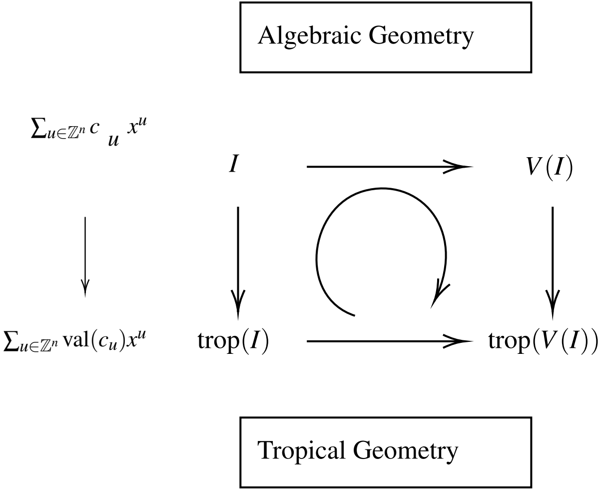
Tropical Algebra

Algebraic Geometry

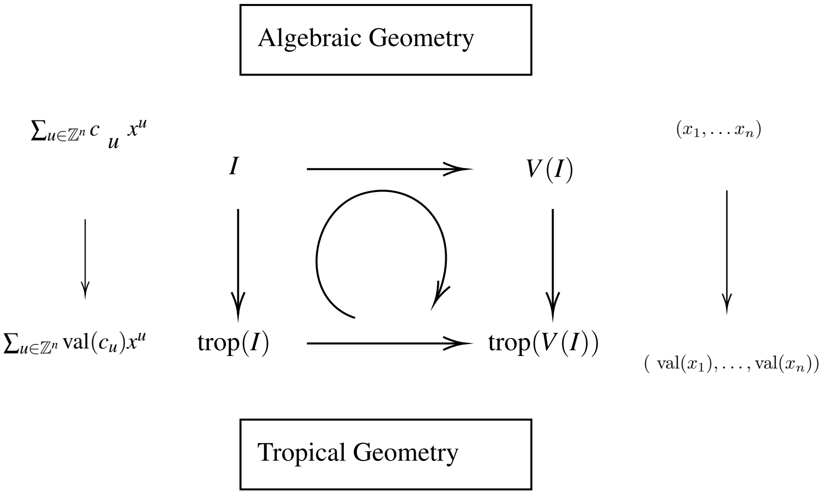


Tropical Geometry

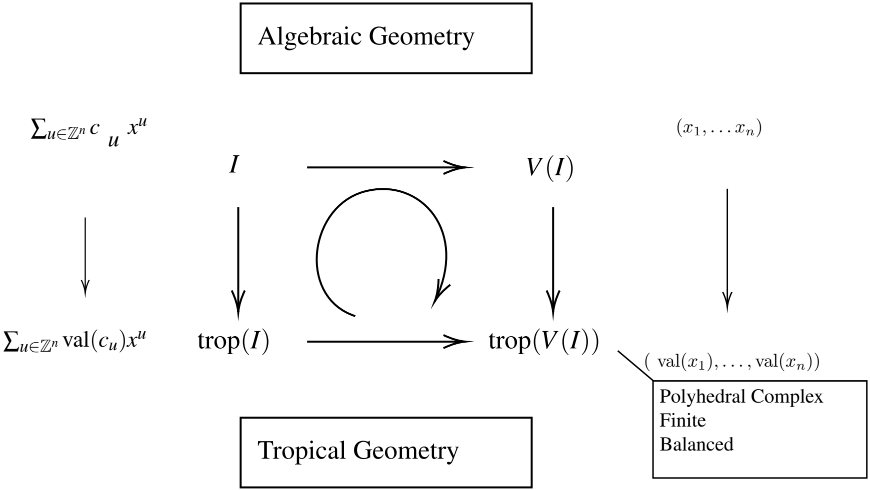
Tropical Algebra



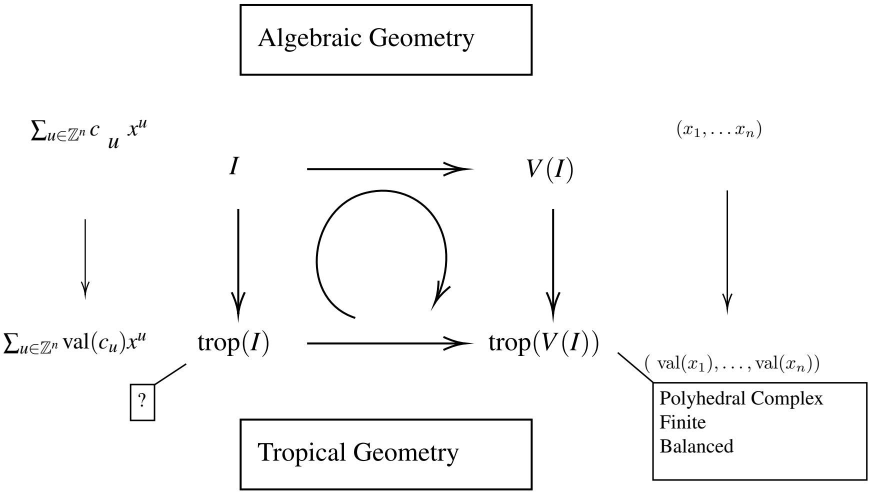
Tropical Algebra



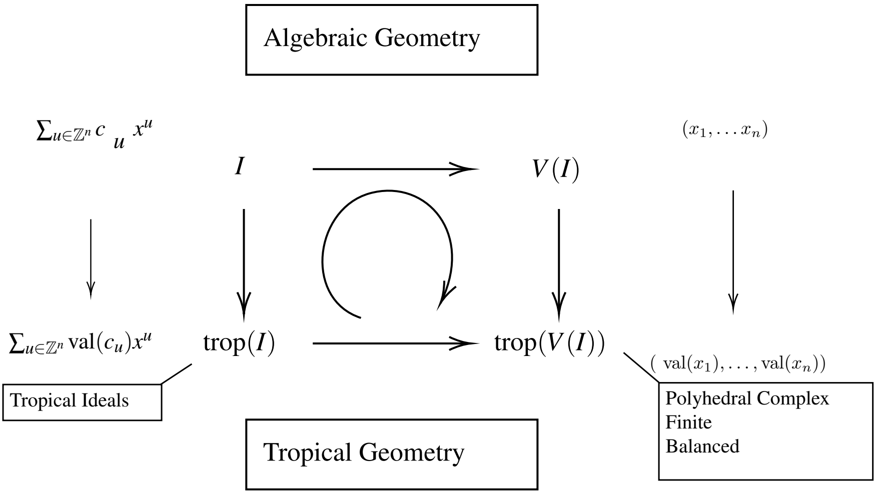
Tropical Algebra



Tropical Algebra



Tropical Algebra



Motivation

- "Non-realizable" tropical ideals are hard to construct.

Motivation

- "Non-realizable" tropical ideals are hard to construct.
- (Zajaczkowska 2018)[Zaj18]: Zero-dimensional, degree-2 homogeneous tropical ideals in $\mathbb{B}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ are in 1-1 correspondence with sublattices $\mathbb{Z}^n$

Contribution

1. Provide a simple way of constructing zero-dimensional tropical ideals of any degree.

Contribution

1. Provide a simple way of constructing zero-dimensional tropical ideals of any degree.
2. Understand the degree-2 sublattice correspondence from a combinatorial perspective, "generalizing" to higher degrees

Method/Synopsis

1. identify a subclass of tropical ideals that admits a well-defined notion of ideal generation (paving tropical ideals)

Method/Synopsis

1. identify a subclass of tropical ideals that admits a well-defined notion of ideal generation (paving tropical ideals)
2. study conditions on the generating set that controls the structure of the resulting ideal

Method/Synopsis

1. identify a subclass of tropical ideals that admits a well-defined notion of ideal generation (paving tropical ideals)
2. study conditions on the generating set that controls the structure of the resulting ideal
3. use these conditions to construct and study examples.

Matroids (Circuits)

Definition

A **matroid presented by circuits** is a set E along with a set system $\mathcal{C}$ on E such that

1. $\emptyset \notin \mathcal{C}$,
2. the elements of $\mathcal{C}$ are finite,
3. $\mathcal{C}$ is a clutter, and
4. (**Circuit Elimination Axiom**) For each pair $C_1, C_2 \in \mathcal{C}$, and each element $e \in C_1 \cap C_2$ there exists a $C_3 \in \mathcal{C}$ such that
$$C_3 \subset (C_1 \cup C_2) \setminus e$$

Matroids (Circuits)

Definition

A **matroid presented by circuits** is a set E along with a set system $\mathcal{C}$ on E such that

1. $\emptyset \notin \mathcal{C}$,
2. the elements of $\mathcal{C}$ are finite,
3. $\mathcal{C}$ is a clutter, and
4. (**Circuit Elimination Axiom**) For each pair $C_1, C_2 \in \mathcal{C}$, and each element $e \in C_1 \cap C_2$ there exists a $C_3 \in \mathcal{C}$ such that
$$C_3 \subset (C_1 \cup C_2) \setminus e$$

- Sets of E which do not contain a circuit are **independent**.

Matroids (Circuits)

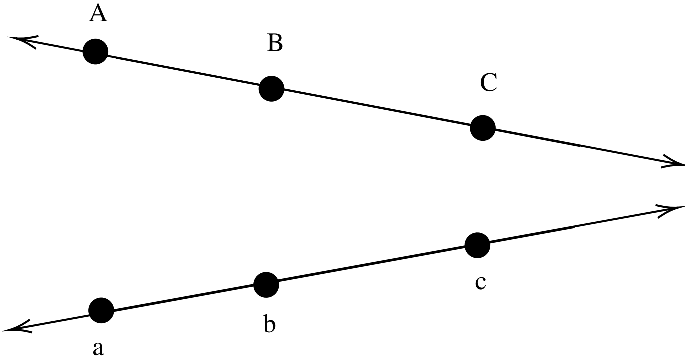
Definition

A **matroid presented by circuits** is a set E along with a set system $\mathcal{C}$ on E such that

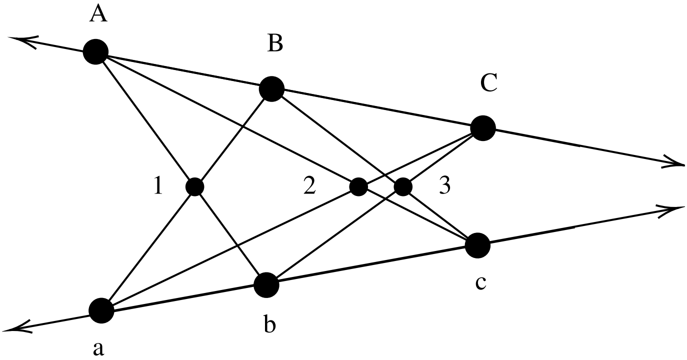
1. $\emptyset \notin \mathcal{C}$,
2. the elements of $\mathcal{C}$ are finite,
3. $\mathcal{C}$ is a clutter, and
4. (**Circuit Elimination Axiom**) For each pair $C_1, C_2 \in \mathcal{C}$, and each element $e \in C_1 \cap C_2$ there exists a $C_3 \in \mathcal{C}$ such that
$$C_3 \subset (C_1 \cup C_2) \setminus e$$

- Sets of E which do not contain a circuit are **independent**.
- The size of a maximal independent set is the **rank** of the matroid.

Example: Points in Space

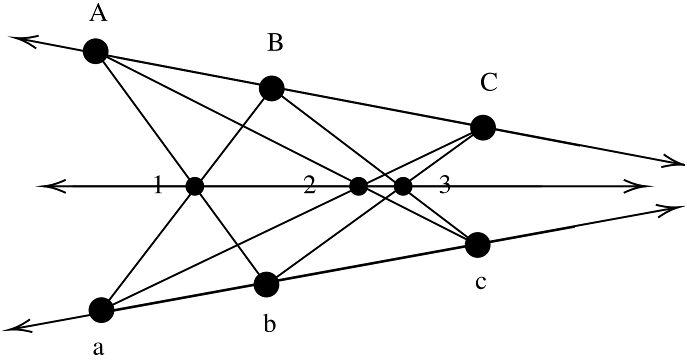


Example: Points in Space



This is a Matroid

Example: Points in Space



Pappus' Theorem: $1, 2, 3$ must lie on a line in any vector space

Example: Ideals

- The polynomials in an ideal I with **minimal support** correspond to the circuits of a matroid.

Example: Ideals

- The polynomials in an ideal I with **minimal support** correspond to the circuits of a matroid.
- $$\sum_{u \in \mathbb{Z}^n} c_u x^u \mapsto \{u \in \mathbb{Z}^n \mid c_u \neq 0\}$$

Example: Ideals

- The polynomials in an ideal I with **minimal support** correspond to the circuits of a matroid.
- $$\sum_{u \in \mathbb{Z}^n} c_u x^u \mapsto \{u \in \mathbb{Z}^n \mid c_u \neq 0\}$$
- **An ideal in $\mathbf{K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]}$ gives us a matroid on the set $\mathbb{Z}^{\mathbf{n}}$**, called it's **underlying matroid**.

Tropical Ideals

Definition

A tropical ideal is an ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ whose polynomials of minimal support form the circuits of a matroid.

Example: Tropical Ideals Aren't So Nice

- Consider $I := \langle x - y \rangle \subset \mathbb{C}[x, y]$.

Example: Tropical Ideals Aren't So Nice

- Consider $I := \langle x - y \rangle \subset \mathbb{C}[x, y]$.
- $\mathrm{trop}(I) \subset \overline{\mathbb{R}}[x_1^{\pm 1}, x_2^{\pm 1}]$, is not finitely generated as $x^d - y^d \in I$ hence $x^d \oplus y^d \in \mathrm{trop}(I)$.

Example: Tropical Ideals Aren't So Nice

- Consider $I := \langle x - y \rangle \subset \mathbb{C}[x, y]$.
- $\text{trop}(I) \subset \overline{\mathbb{R}}[x_1^{\pm 1}, x_2^{\pm 1}]$, is not finitely generated as $x^d - y^d \in I$ hence $x^d \oplus y^d \in \text{trop}(I)$.
- Since no cancellation occurs naturally in $\overline{\mathbb{R}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, no finite collection of $x^d \oplus y^d$ may be used to generate all such binomials.

Example: Tropical Ideals Aren't So Nice

- Consider $I := \langle x - y \rangle \subset \mathbb{C}[x, y]$.
- $\text{trop}(I) \subset \overline{\mathbb{R}}[x_1^{\pm 1}, x_2^{\pm 1}]$, is not finitely generated as $x^d - y^d \in I$ hence $x^d \oplus y^d \in \text{trop}(I)$.
- Since no cancellation occurs naturally in $\overline{\mathbb{R}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, no finite collection of $x^d \oplus y^d$ may be used to generate all such binomials.
- **This makes it quite hard to construct nontrivial examples of tropical ideals: what can we do to specify an ideal with an infinite generating set?**

Tropical Ideals are Nice

1. Variety is finite, balanced polyhedral complex
2. Satisfy ascending chain condition
3. Hilbert **polynomial** encodes meaningful combinatorial data
4. Weak Nulstellensatz holds.

Paving Matroids

Definition

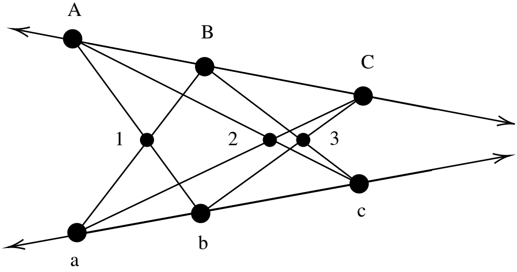
A **Paving Matroid** of rank r is a matroid (of rank r) whose circuits have size r or $r+1$

Paving Matroids

Definition

A **Paving Matroid** of rank r is a matroid (of rank r) whose circuits have size r or $r + 1$

Example:



Generalized Partitions

Definition

Given a set E, a d-partition on E is a set system $\mathcal{H}$ such that

P1) $|\mathcal{H}| \geq 2$,

P2) for all $H \in \mathcal{H}$, $|H| \geq d$, and

P3) each d-subset of E appears in a unique element of $\mathcal{H}$.

Elements of $\mathcal{H}$ are called **blocks**.

From d-Partitions to Circuits

- d-partitions exactly encode paving matroids of rank $d+1$

From d-Partitions to Circuits

- d-partitions exactly encode paving matroids of rank $d + 1$
- The circuits of size $d + 1$ are exactly the subsets of blocks of size at least $d + 1$.

From d-Partitions to Circuits

- d-partitions exactly encode paving matroids of rank $d+1$
- The circuits of size $d+1$ are exactly the subsets of blocks of size at least $d+1$.
- The circuits of size $d+2$ are implicit: take all $d+2$ subsets of E not containing a circuit of size $d+1$.

Generating d-Partitions

- We can generate d-partitions

Generating d-Partitions

- We can generate d-partitions
- Let $\mathcal{S}$ be a set system on E with elements of size at least $d+1$, and with pairwise intersections of size less than d.

Generating d-Partitions

- We can generate d-partitions
- Let $\mathcal{S}$ be a set system on E with elements of size at least $d+1$, and with pairwise intersections of size less than d.
- Just fill in the d-subsets not covered by S:

$$\mathcal{H} := S$$

Generating d-Partitions

- We can generate d-partitions
- Let $\mathcal{S}$ be a set system on E with elements of size at least $d+1$, and with pairwise intersections of size less than d.
- Just fill in the d-subsets not covered by S:

$$\mathcal{H} := S \cup \{T \subset E \quad | \quad |T| = d \text{ and } T \not\subset S \text{ for all } S \in \mathcal{S}\}$$

Recap

- Tropical ideals are ideals in $\mathbb{B}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ that are matroids.
- Matroids, and thus tropical ideals, are very complicated
- d-Partitions provide a succinct way of describing the circuits of a paving matroid.
- d-Partitions can be generated

Definition

Definition

A zero-dimensional tropical ideal is called a **paving tropical ideal** if its underlying matroid $\underline{\mathrm{Mat}}(I)$ is a paving matroid.

Structural Observation

- A (tropical) ideal I gives us a matroid on $\mathbb{Z}^n$ called its' underlying matroid $\underline{\mathrm{Mat}}(I)$ via the map

$$\sum_{u \in \mathbb{Z}^n} c_u x^u \mapsto \{u \in \mathbb{Z}^n \mid c_u \neq 0\}$$

Structural Observation

- A (tropical) ideal I gives us a matroid on $\mathbb{Z}^n$ called its' underlying matroid $\underline{\text{Mat}}(I)$ via the map

$$\sum_{u \in \mathbb{Z}^n} c_u x^u \mapsto \{u \in \mathbb{Z}^n \mid c_u \neq 0\}$$

- If $S \in \underline{\text{Mat}}(I)$, then the set $S + u := \{t + u \mid t \in S\}$ is also in $\underline{\text{Mat}}(I)$, as I is closed under multiplication by the monomial x^u.

Structural Observation

- A (tropical) ideal I gives us a matroid on $\mathbb{Z}^n$ called its' underlying matroid $\underline{\text{Mat}}(I)$ via the map

$$\sum_{u \in \mathbb{Z}^n} c_u x^u \mapsto \{u \in \mathbb{Z}^n \mid c_u \neq 0\}$$

- If $S \in \underline{\text{Mat}}(I)$, then the set $S + u := \{t + u \mid t \in S\}$ is also in $\underline{\text{Mat}}(I)$, as I is closed under multiplication by the monomial x^u. Succinctly: $\underline{\text{Mat}}(I)$ is a matroid on $\mathbb{Z}^n$ that is invariant under the action of $\mathbb{Z}^n$.

Invariance under Z^n action

Definition

We say that a d-partition $\mathcal{H}$ of $\mathbb{Z}^n$ is $\mathbb{Z}^n$-invariant if for each $\mathbf{u} \in \mathbb{Z}^n$ and $H \in \mathcal{H}$, $H + \mathbf{u} \in \mathcal{H}$.

Correspondence Theorem

Theorem (Correspondence Theorem)

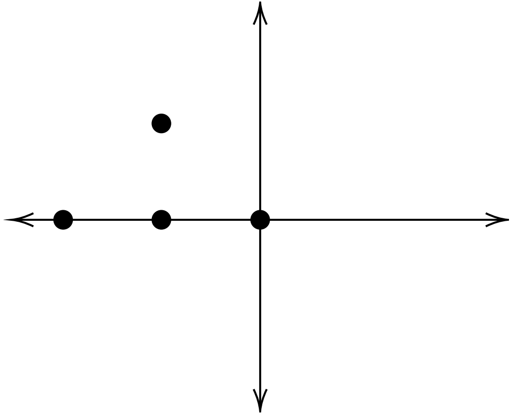
There is a natural one-to-one correspondence between tropical ideals $I \subset B[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $\mathbb{Z}^n$-invariant matroids on Z^n. In particular, there is a one-to-one correspondence between degree $d+1$ paving tropical ideals in $\mathbb{B}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $\mathbb{Z}^n$-invariant d-partitions of $\mathbb{Z}^n$.

Constructing a Non-Example

Not just any subset of $\mathbb{Z}^n$ can be a block in a paving tropical ideal; there is a geometric constraint.

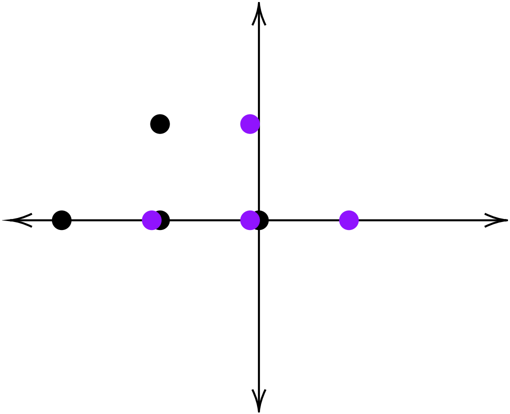
Constructing a Non-Example

Consider $S = \{(0,0), (-2,0), (-4,0), (-2,2)\}$ as a block in a 2-partition



Constructing a Non-Example

The set S+(2,0) is also a block in our paving tropical ideal

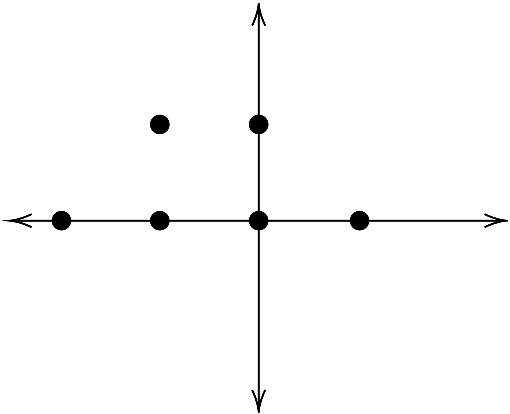


Constructing a Non-Example

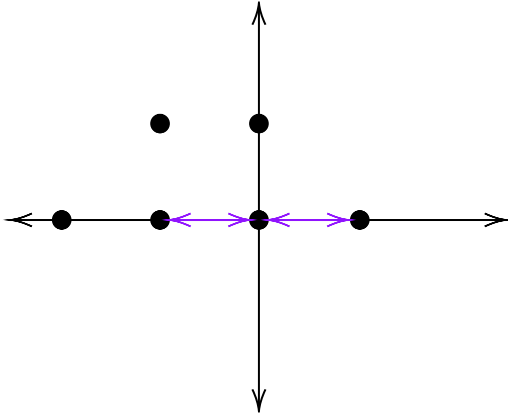
This is a problem because $S \cap S + (2,0) = \{(-2,0),(0,0)\}$, which has more than one element.

Constructing a Non-Example

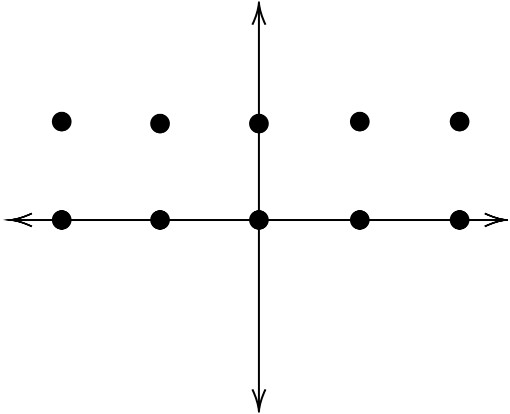
What if we try and fix this? The minimal block containing S contains S and $S + (2, 0)$



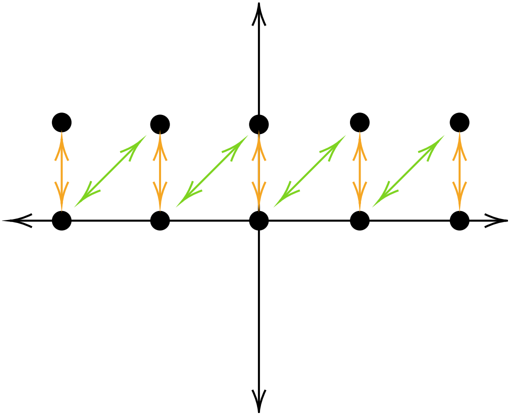
Constructing a Non-Example



Constructing a Non-Example

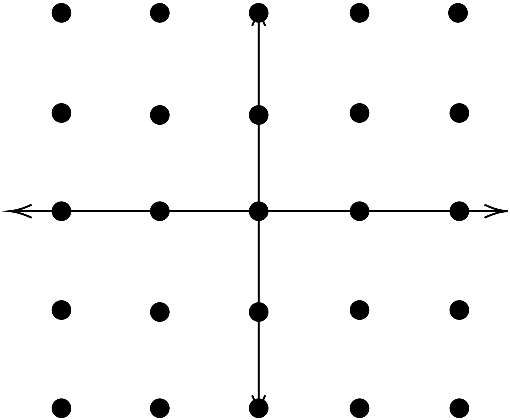


Constructing a Non-Example



Constructing a Non-Example

The minimal block containing S is $(2\mathbb{Z})^2$



d-Sparsity

Definition

A subset $S \subset \mathbb{Z}^n$ is called d-**sparse** if there is no $\mathbf{u} \in \mathbb{Z}^n \setminus \mathbf{0}$ such that $|S \cap S + \mathbf{u}| \geq d$.

Lattice Blocks

Proposition (A, Rincón, [AR21])

Suppose $\mathcal{P}$ is a $\mathbb{Z}^n$-invariant d-partition of $\mathbb{Z}^n$. Then any block $S \in \mathcal{P}$ is either d-sparse or a non-trivial affine sublattice of $\mathbb{Z}^n$, i.e. it has the form $S = \mathbf{v} + L$ for $\mathbf{v} \in \mathbb{Z}^n$ and $\{\mathbf{0}\} \subsetneq L \subsetneq \mathbb{Z}^n$ a sublattice.

Generalizing Zajaczkowska

- Setting $d = 1$ we consider partitions of $\mathbb{Z}^n$.

Generalizing Zajaczkowska

- Setting $d = 1$ we consider partitions of $\mathbb{Z}^n$.
- Every subset of Z^n of size at least 2 intersects a translate of itself in one point.

Generalizing Zajaczkowska

- Setting $d = 1$ we consider partitions of $\mathbb{Z}^n$.
- Every subset of Z^n of size at least 2 intersects a translate of itself in one point.
- Every block in a degree 2 paving tropical ideal is the translate of a unique lattice L.

Generalizing Zajaczkowska

- Setting $d = 1$ we consider partitions of $\mathbb{Z}^n$.
- Every subset of Z^n of size at least 2 intersects a translate of itself in one point.
- Every block in a degree 2 paving tropical ideal is the translate of a unique lattice L.
- This is precisely the result of Zajaczkowska [Zaj18][Theorem 4.2.4]

Generating d-Partitions

Definition

Suppose $\mathcal{A}$ is a collection of subsets of $\mathbb{Z}^n$ satisfying:

(A1) $\mathbb{Z}^n \notin \mathcal{A}$.

(A2) $|A| \geq d+1$ for all $A \in \mathcal{A}$.

(A3) If $A_1, A_2 \in \mathcal{A}$ and $\mathbf{u} \in \mathbb{Z}^n$ satisfy $|A_1 \mathcal{A}p(\mathbf{u} + A_2)| \geq d$ then $A_1 = \mathbf{u} + A_2$.

Define:

$$\mathcal{P}_d(\mathcal{A}) := (\mathbb{Z}^n + \mathcal{A}) \cup \mathcal{D},$$

where

$$\mathbb{Z}^n + \mathcal{A} := \{\mathbf{u} + A : \mathbf{u} \in \mathbb{Z}^n \text{ and } A \in \mathcal{A}\}$$

and

$$\mathcal{D} := \{S \subset \mathbb{Z}^n : |S| = d \text{ and } S \not\subset X \text{ for all } X \in \mathbb{Z}^n + \mathcal{A}\}.$$

We call $\mathcal{P}_d(\mathcal{A})$ the **$\mathbb{Z}^n$-invariant d-partition of $\mathbb{Z}^n$ generated by $\mathcal{A}$.**

The key content of the previous frame

Given a collection of subsets of size at least $d+1$, whose translations intersect in fewer than d points, we can generate a paving tropical ideal by simply considering $\mathbb{Z}^n$'s action on our set system, and then generating a d-partition as usual.

Lots of Paving Tropical Ideals

Proposition (A, Rincón)

There are uncountably many degree 3 paving tropical ideals in $\mathbb{B}[x^{\pm 1}]$.

Proof by Example

- To any $S \subset \mathcal{P}(\mathbb{N})$ of size at least 3 we associate the set $T_S := \{2^t \mid t \in S\}$

Proof by Example

- To any $S \subset \mathcal{P}(\mathbb{N})$ of size at least 3 we associate the set $T_S := \{2^t \mid t \in S\}$
- T_S is a 2-sparse set that is in $1-1$ correspondence with S.

Proof by Example

- To any $S \subset \mathcal{P}(\mathbb{N})$ of size at least 3 we associate the set $T_S := \{2^t \mid t \in S\}$
- T_S is a 2-sparse set that is in $1-1$ correspondence with S.
- The $\mathbb{Z}$ invariant 2-partition generated by $\{T_S\}$ is in 1-1 correspondence with S.

Proof by Example

- To any $S \subset \mathcal{P}(\mathbb{N})$ of size at least 3 we associate the set $T_S := \{2^t \mid t \in S\}$
- T_S is a 2-sparse set that is in $1-1$ correspondence with S.
- The $\mathbb{Z}$ invariant 2-partition generated by $\{T_S\}$ is in 1-1 correspondence with S.
- There are uncountably many such S

Short Corollary

Corollary (A, Rincón)

Most zero-dimensional tropical ideals are not representable.

Proof.

Only countably many zero-dimensional tropical ideals are representable [Sil21] □

Degree 2 Paving Tropical Ideals

- All Rank 2 matroids are realizable

Degree 2 Paving Tropical Ideals

- All Rank 2 matroids are realizable
- Degree 2 paving tropical ideals are "everywhere a rank-2 matroid"

Degree 2 Paving Tropical Ideals

- All Rank 2 matroids are realizable
- Degree 2 paving tropical ideals are "everywhere a rank-2 matroid"
- Degree 2 paving tropical ideals in one variable are realisable [Zaj18, Theorem 5.1.5].

Degree 2 Paving Tropical Ideals

- All Rank 2 matroids are realizable
- Degree 2 paving tropical ideals are "everywhere a rank-2 matroid"
- Degree 2 paving tropical ideals in one variable are realisable [Zaj18, Theorem 5.1.5].
- Question: Are all degree 2 paving tropical ideals realisable.

Degree 2 Paving Tropical Ideals

- All Rank 2 matroids are realizable
- Degree 2 paving tropical ideals are "everywhere a rank-2 matroid"
- Degree 2 paving tropical ideals in one variable are realisable [Zaj18, Theorem 5.1.5].
- Question: Are all degree 2 paving tropical ideals realisable.
- Answer: No

Characteristic Two

Lemma (Proposition 5.2.9, Zajaczkowska)

The degree 2 paving tropical ideal associated to the lattice $(2n, 2m)$ is not realisable except in characteristic 2

Characteristic Two

Lemma (Proposition 5.2.9, Zajaczkowska)

The degree 2 paving tropical ideal associated to the lattice $(2n, 2m)$ *is not realisable except in characteristic 2*

It should suffice to find a tropical ideal that is not realisable in characteristic 2.

Not Characteristic Two

Lemma

The (homogeneous) tropical ideal associated to $4\mathbb{Z}$ is not realisable in characteristic two.

Not Characteristic Two

Lemma

The (homogeneous) tropical ideal associated to $4\mathbb{Z}$ is not realisable in characteristic two.

Proof: a simple proof by contradiction

Gluing

Proposition

If I is a paving tropical ideal in $\mathbb{B}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and J is a paving tropical ideal in $\mathbb{B}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$, then The d-partition $\mathcal{H}$ of $\mathbb{Z}^{n+m}$ generated by $\mathcal{H}(\underline{Mat}(I)) \cup \mathcal{H}(\underline{Mat}(J))$ is defined and $\mathcal{H}|_{\mathbb{Z}^n} = \mathcal{H}(\underline{Mat}(I))$ and $\mathcal{H}|_{Z^m} = \mathcal{H}(\underline{Mat}(J))$.

Proof: Any translation of Z^m intersects Z^n in exactly one point and vice versa; the case where $d = 1$ is the exception, but the generators in this case form the basis of a lattice and are lineraly independent by definition.

Counterexample

The degree 2 tropical ideal associated to the lattice $(4x, 2y, 2z)$ is not realisable over any field.

References I

Nicholas Anderson and Felipe Rincón, *Paving tropical ideals*, arXiv preprint arXiv:2102.09848 (2021).

Rob Silversmith, *The matroid stratification of the Hilbert scheme of points on $\mathbb{P}^1$*, Manuscripta Math. (2021).

Magdalena Anna Zajaczkowska, *Tropical ideals with Hilbert function two*, Ph.D. thesis, University of Warwick, 2018.