Constructing and Machine Learning Calabi-Yau Five-Folds

Daniele Angella

joint with

Rashid Alawadhi, Andrea Leonardo, and Tancredi Schettini Gherardini

Università di Firenze

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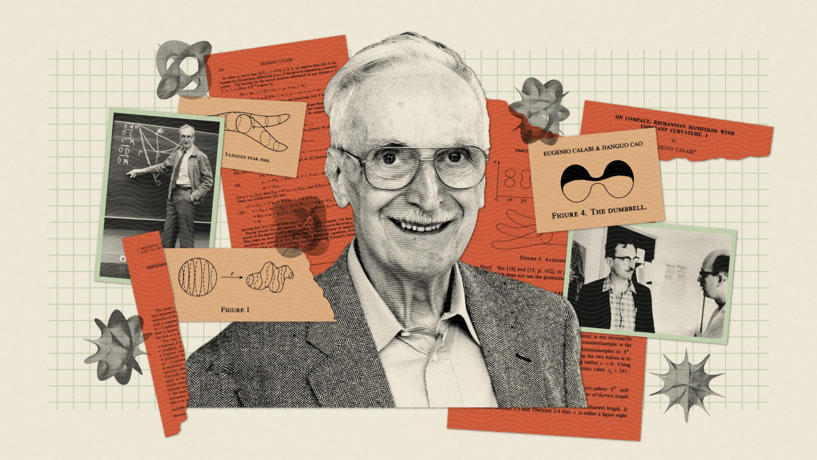
Calabi-Yau manifolds



"A lot of mathematicians like to solve problems that finish off work on a particular subject," Chen said. "Calabi was someone who liked to start a subject."

Calabi-Yau manifolds

between Mathematics. . .

Definition

A *Calabi-Yau n-fold* is a compact Kähler manifold X^n with trivial canonical bundle.

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X^n compact Kähler:

$$\begin{pmatrix} K_X \simeq \mathcal{O}_X \\ \Updownarrow \\ \exists \Omega \in H^0(X; \Omega^n_X) \text{ s.t. } \Omega(x) \neq 0 \\ \Updownarrow \\ \text{struct group } TX \text{ red to } \mathrm{SU(n)} \\ \Updownarrow \\ \exists g \text{ Kähler s.t. } \mathrm{Hol}(g) \subseteq \mathrm{SU}(n) \end{pmatrix} \Rightarrow \begin{pmatrix} c_1(X) = 0 \\ \Updownarrow \\ \exists g \text{ Kähler s.t. } \mathrm{Ric}(g) = 0 \\ \Updownarrow \\ \exists g \text{ Kähler s.t. } \mathrm{Hol}^0(g) \subseteq \mathrm{SU}(n) \\ \Updownarrow \\ \exists m \text{ s.t. } mK_X \simeq \mathcal{O}_X \\ \Updownarrow \\ \exists X' \stackrel{m:1}{\to} X \text{ with } K_{X'} \simeq \mathcal{O}_{X'} \end{pmatrix}$$

(converse if $\pi_1(X) = 0$)

Calabi-Yau manifolds

between Mathematics...

$$\text{Calabi–Yau} \quad \Rightarrow \quad \underbrace{\text{Kähler}}_{d\omega=0} + \underbrace{\text{Einstein}}_{\mathrm{Ric}(g)=\lambda g}$$

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D. Examples

0.23. Despite the simplicity of the condition $r = \lambda g$ the reader should not imagine that examples are easy to find. If you are not convinced, try to find one yourself which is not in our book. And if you succeed, please write to us immediately. Ricci flat compact manifolds are even harder to come by. The author will be happy to stand you a meal in a starred restaurant in exchange of one of these!

A. L. Besse, *Einstein Manifolds*, Springer (1987).

Calabi-Yau manifolds

...and Physics

Calabi-Yau manifolds are the most promising candidates for internal spaces in the compactification mechanism. Their geometry affects the properties of the subsequent 4-dim theory.

$$\mathbb{R}^{1,3} \times X^6$$

and Hull-Strominger are necessary and sufficient for spacetime supersymmetry:

- (X, g) complex Hermitian with Plank-scale, (V, h) Hermitian vector bundle, Ω holomorphic 3-form never vanishing,
- $\partial\overline{\partial}g = \sqrt{-1}\,\mathrm{tr}\, F_h \wedge F_h - \sqrt{-1}\,\mathrm{tr}\, R_g \wedge R_g$,
- $d^* g = \sqrt{-1}\,(\partial - \overline{\partial})\,\lg \|\Omega\|$,
- F Yang-Mills.

P. Candelas, G. T. Horowitz, A. Strominger, E. Witten, Vacuum configurations for superstrings, *Nucl. Phys. B* **258** (1985) 46–74.

1 Introduction

2 Construction

3 Cohomology

4 Statistics

5 Machine Learning

CY manifolds in dim 1 and 2

dim $= 1$

Calabi-Yau onefold is the torus.
It is a complex elliptic curve, in particular, algebraic:

$$y^2 z = x^3 + axz^2 + bz^3$$

(where $4a^3 + 27b^2 \neq 0$).

dim $= 2$

Simply-connected Calabi-Yau twofolds are K3 surfaces.
They are quartic surfaces in $\mathbb{P}^3$:

$$x^4 + y^4 + z^4 + w^4 = 0.$$

CY manifolds in dim 3

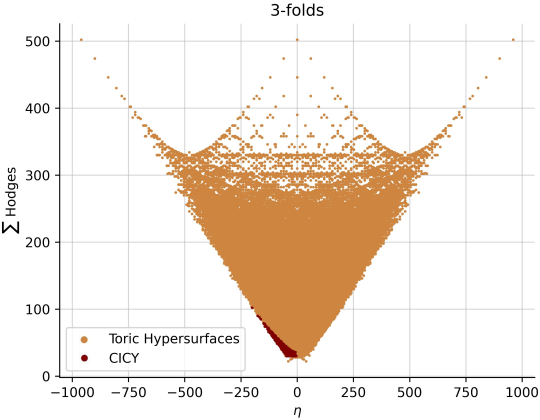
dim = 3

- Yau conjectures finitely many topological types of CY in each dim
- at least 2590 distinct diffeom classes of CICY
- over 473 million toric embeddings of CY
- Reid's fantasy: there is only one CY threefold, all CY threefolds are connected through conifold transitions

M. Reid, The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible, *Math. Ann.* **278** (1987), 329–334.

S. T. Yau, A survey of Calabi-Yau manifolds, *Adv. Lect. Math. (ALM)*, **18**, International Press, 2011, 521–563.

CY manifolds in dim 3



A. Ashmore, Y.-H. He, Calabi-Yau three-folds: Poincaré polynomials and fractals, in *Strings, gauge fields, and the geometry behind*, 173–186, World Scientific, 2013

CY hypersurfaces in $\mathbb{P}^n$

$$X^{n-1} = \{x \in \mathbb{P}^n : p(x) = 0\}$$

where $p(x) \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous polynomial of degree d

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- $0 \to T_X \to T_{\mathbb{P}^n}|_X \to N_{X|\mathbb{P}^n} \to 0 \rightsquigarrow K_X = K_{\mathbb{P}^n}|_X \otimes N_{X|\mathbb{P}^n}$

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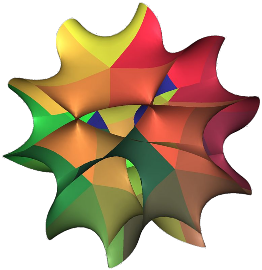
Proposition

$$X \text{ Calabi-Yau} \quad \Leftrightarrow \quad d = n+1$$

Fermat quintic threefold

$$X^3 = \{[x_0 : \cdots : x_4] \in \mathbb{P}^4 \ : \ x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\}$$



- $c(T_X) = 1 + 10H^2 - 40H^3$, where H is hyperplane div class $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$
- $\chi(X) = -40\int_X H^3 = -200$ by Bézout
- $h^{1,1}(X) = 1,\ h^{2,1}(X) = 101$

CICY

Complete Intersection Calabi-Yau

$$X^n := \{p_1 = \cdots = p_K = 0\} \subseteq \mathcal{A}$$

where

- $\mathcal{A} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ ambient space
- $p_1, \ldots, p_K$ polyn depending on m sets of homogeneous coord
- q_α^r the homog degree of p_α in the coord of the rth proj space

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such that:

- X smooth
- $\dim X = \dim \mathcal{A} - K = \sum^m n_r - K$
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such that:

- X smooth
- $\dim X = \dim \mathcal{A} - K = \sum^m n_r - K$
- $K_X = \mathcal{O}_X$, i.e. for any r, $\sum^K q_\alpha^r = n_r + 1$

P. Candelas, A. Dale, C. Lütken, R. Schimmrigk, Complete intersection Calabi-Yau manifolds, *Nucl. Phys. B* **298** (1988), 493.

CICY : configuration matrix

Cohomological invariants of X only depends on q^r_α

$$X \quad \rightsquigarrow \quad \left(\begin{array}{c|ccc} n_1 & q^1_1 & \cdots & q^1_K \\ \vdots & \vdots & \ddots & \vdots \\ n_m & q^m_1 & \cdots & q^m_K \end{array}\right)$$

where

- $K = \sum^m n_r - n$
- for any r, $\sum^K q^r_\alpha = n_r + 1$

CICY : configuration matrix

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where

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- for any r, $\sum^K q_\alpha^r = n_r + 1$

Remark

There are redundancies!

CICY : finiteness

X^5

CICY : finiteness

$$X^5 \quad \hookrightarrow \quad \mathcal{A} = (\mathbb{P}^1)^f \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_F}$$

- assume $\sum_{r=1}^{f+F} q_\alpha^r \geq 2$ for any α, since $\left(\begin{array}{c|c} 1 & 1 \end{array} \right) = \text{pt}$
- then $f + F + 5 \geq K$
- then $F \leq 10$
- assume bilin constraint involving $\mathbb{P}^1$ also involves $\mathbb{P}^{n_j}$, since $\left(\begin{array}{c|c} 1 & 1 \\ 1 & 1 \end{array} \right) = \mathbb{P}^1$
- assume $\sum_{r=1}^{f} q_\alpha^r > 2$ for any α, otherwise product
- . . . then $f \leq 15$
- then $K = f + \sum^{f+F} n_r - 5 \leq 30$

maximum size for a configuration matrix for CICY fivefold is

$$25 \times 30$$

CICY : configuration matrix' redundancies

- permutations of rows and columns
  (lexicographic order, Gray-Haupt-Lukas algorithm, brute force)

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- ineffective splittings

$$\left(\begin{array}{c|cc} n & \sum_{a=1}^{n+1} u_a & q \end{array} \right) \quad \rightsquigarrow \quad \left(\begin{array}{c|cccccc} n & 1 & 1 & \cdots & 1 & 0 \\ n & u_1 & u_2 & \cdots & u_{n+1} & q \end{array} \right)$$

CICY : configuration matrix' redundancies

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- accidental identities

$$\left(\begin{array}{c|cc} 2 & 2 & a \\ n & 0 & q \end{array}\right) = \left(\begin{array}{c|c} 1 & 2a \\ n & q \end{array}\right), \quad \left(\begin{array}{c|cc} 1 & 1 & a \\ 1 & 1 & b \\ n & 0 & q \end{array}\right) = \left(\begin{array}{c|c} 1 & a+b \\ n & q \end{array}\right), \quad \left(\begin{array}{c|cc} 3 & 2 & a \\ n & 0 & q \end{array}\right) = \left(\begin{array}{c|c} 1 & a \\ 1 & a \\ n & q \end{array}\right),$$

$$\left(\begin{array}{c|cc} 1 & 2 & 0 \\ 2 & 1 & a \\ n & 0 & q \end{array}\right) = \left(\begin{array}{c|c} 1 & a \\ 1 & a \\ n & q \end{array}\right), \quad \left(\begin{array}{c|ccc} 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & a \\ n & 0 & 0 & q \end{array}\right) = \left(\begin{array}{c|cc} 1 & 2 & 0 \\ 1 & 2 & a \\ n & 0 & q \end{array}\right), \quad \ldots$$

CICY : summary

$n = 3$ 7 890 spaces, 265 distinct Hodge diamonds

$n = 4$ 921 497 spaces, 4 417 distinct Hodge diamonds

$n = 5$ expected $\sim 10^8$ spaces, we constructed 27 068 configuration matrices of dimensions up to 4×4, up to permutations, 2 375 distinct Hodge diamonds

The list of complete intersection Calabi-Yau three-folds, http://www-thphys.physics.ox.ac.uk/projects/CalabiYau/cicylist/.

Complete intersection Calabi-Yau four-folds, http://www-thphys.physics.ox.ac.uk/projects/CalabiYau/Cicy4folds/index.html.

R. Alawadhi, D. A., A. Leonardo, T. Schettini Gherardini, Constructing and Machine Learning Calabi-Yau Five-folds, *Fortschr. Phys.* (arXiv:2310.15966).

Complete intersection Calabi-Yau five-folds, https://www.dropbox.com/scl/fo/z7ii5idt6qxu36e0b8azq/h?rlkey=0qfhx3tykytduobpld510gsfy&dl=0

M. Kreuzer, H. Skarke, Calabi-Yau data, http://hep.itp.tuwien.ac.at/~kreuzer/CY/.

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CY fivefolds : Hodge numbers

$$
\begin{array}{ccccccccccc}
 & & & & & h^{0,0}(X) & & & & & \\
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- $\chi = \sum_k (-1)^k b_k = 2h^{1,1} - 4h^{1,2} + 4h^{1,3} + 2h^{2,2} - 2h^{1,4} - 2h^{2,3}$ (Euler)

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- $11h^{1,1} - 10h^{1,2} - h^{2,2} + h^{2,3} + 10h^{1,3} - 11h^{1,4} = 0$ (Atiyah-Singer)

CY fivefolds : Hodge numbers

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & h^{1,1} & & 0 & & & \\
 & & 0 & & h^{2,1} & & h^{2,1} & & 0 & & \\
 & 0 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 0 & \\
1 & & h^{4,1} & & h^{3,2} & & h^{3,2} & & h^{4,1} & & 1 \\
 & 0 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 0 & \\
 & & 0 & & h^{2,1} & & h^{2,1} & & 0 & & \\
 & & & 0 & & h^{1,1} & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

where $11h^{1,1} - 10h^{2,1} - h^{2,2} + h^{3,1} + 10h^{3,1} - 11h^{4,1} = 0$

$$
\begin{aligned}
b_0 &= 1 \\
b_1 &= 0 \\
b_2 &= h^{1,1} \\
b_3 &= 2h^{1,2} \\
b_4 &= 2h^{3,1} + h^{2,2} \\
b_5 &= 2 + 2h^{1,4} + 2h^{3,2} \\
b_6 &= b_4 \\
b_7 &= b_3 \\
b_8 &= b_2 \\
b_9 &= b_1 = 0 \\
b_{10} &= b_0 = 1
\end{aligned}
$$

$$\chi = \sum_k (-1)^k b_k = 2h^{1,1} - 4h^{1,2} + 4h^{1,3} + 2h^{2,2} - 2h^{1,4} - 2h^{2,3}$$

Example CICY threefold in $\mathbb{P}^5$: Hodge numbers

$$X^3 = \left(\begin{array}{c|cc} 5 & 2 & 4 \end{array} \right)$$

Example CICY threefold in $\mathbb{P}^5$: Hodge numbers

$$X^3 = \left(\; 5 \; \middle| \; 2 \quad 4 \; \right)$$

$p(x) \in H^0(\mathbb{P}^5; \mathcal{O}_{\mathbb{P}^5}(2))$, $q(x) \in H^0(\mathbb{P}^5; \mathcal{O}_{\mathbb{P}^5}(4))$, generic

$$X^3 \quad = \quad \{x \in \mathbb{P}^5 : p(x) = q(x) = 0\}$$

Example CICY threefold in $\mathbb{P}^5$: Hodge numbers

$$X^3 = \left(\; 5 \; \middle| \; 2 \quad 4 \; \right)$$

$p(x) \in H^0(\mathbb{P}^5; \mathcal{O}_{\mathbb{P}^5}(2))$, $q(x) \in H^0(\mathbb{P}^5; \mathcal{O}_{\mathbb{P}^5}(4))$, generic

$$\begin{aligned} X^3 &= \{x \in \mathbb{P}^5 : p(x) = q(x) = 0\} \\ &= \left\{x \in Y^4 := \left\{y \in \mathbb{P}^5 : q(x) = 0\right\} : p(x) = 0\right\} \end{aligned}$$

Example CICY threefold in $\mathbb{P}^5$: Hodge numbers

$$X^3 = \left(5 \,\middle|\, 2 \quad 4 \right)$$

$p(x) \in H^0(\mathbb{P}^5; \mathcal{O}_{\mathbb{P}^5}(2))$, $q(x) \in H^0(\mathbb{P}^5; \mathcal{O}_{\mathbb{P}^5}(4))$, generic

$$\begin{aligned} X^3 &= \{x \in \mathbb{P}^5 : p(x) = q(x) = 0\} \\ &= \left\{x \in Y^4 := \left\{y \in \mathbb{P}^5 : q(x) = 0\right\} : p(x) = 0\right\} \end{aligned}$$

$$\begin{array}{ccccccc} & & & h^{0,0}(X) & & & \\ & & h^{1,0}(X) & & h^{0,1}(X) & & \\ & h^{2,0}(X) & & h^{1,1}(X) & & h^{0,2}(X) & \\ h^{3,0}(X) & & h^{2,1}(X) & & h^{1,2}(X) & & h^{0,3}(X) \\ & h^{3,1}(X) & & h^{2,2}(X) & & h^{1,3}(X) & \\ & & h^{3,2}(X) & & h^{2,3}(X) & & \\ & & & h^{3,3}(X) & & & \end{array}$$

Example CICY threefold in $\mathbb{P}^5$: Hodge numbers

$$X^3 = \left(\; 5 \,\middle|\, 2 \quad 4 \; \right)$$

$p(x) \in H^0(\mathbb{P}^5; \mathcal{O}_{\mathbb{P}^5}(2))$, $q(x) \in H^0(\mathbb{P}^5; \mathcal{O}_{\mathbb{P}^5}(4))$, generic

$$\begin{aligned} X^3 &= \{x \in \mathbb{P}^5 : p(x) = q(x) = 0\} \\ &= \{x \in Y^4 := \{y \in \mathbb{P}^5 : q(x) = 0\} : p(x) = 0\} \end{aligned}$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1}(X) & & 0 & \\ 1 & & h^{2,1}(X) & & h^{2,1}(X) & & 1 \\ & 0 & & h^{1,1}(X) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

CY threefolds in $\mathbb{P}^5$: Hodge numbers

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1}(X) & & 0 & \\
1 & & h^{2,1}(X) & & h^{2,1}(X) & & 1 \\
 & 0 & & h^{1,1}(X) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Remark

- $H^{1,1}(X)$ *Kähler class parameters*
- $H^{2,1}(X) = H^{1,2}(X) = H^2(X;\Omega^1_X) = H^1(X; T_X)$ *cplx struct param*

P. Green, T. Hübsch, Polynomial deformations and cohomology of Calabi-Yau manifolds, *Comm. Math. Phys.* 113 (1987), no. 3, 505–528.

T. Hübsch, *Calabi-Yau manifolds. A bestiary for physicists*, World Scientific, 1992.

Example CICY threefold in $\mathbb{P}^5$: cohomology of $\mathcal{O}_X$

$X = \left(\begin{array}{c|cc} 5 & 2 & 4 \end{array} \right)$

By Dolbeault-Grothendieck: $h^{p,q}(X) = H^q(X; \mathcal{O}_X)$, then

$$h(\mathcal{O}_X) = \left(h^{0,0}(X), h^{0,1}(X), h^{0,2}(X), h^{0,3}(X), \cdots\right)$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of $\mathcal{O}_X$

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$$0 \to \mathcal{O}_{\mathbb{P}^5}(-2) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_Y \to 0$$

$$0 \to \mathcal{O}_Y(-4) \to \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of $\mathcal{O}_X$

$X = \left(\begin{array}{c|cc} 5 & 2 & 4 \end{array} \right)$

By Dolbeault-Grothendieck: $h^{p,q}(X) = H^q(X; \mathcal{O}_X)$, then

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$$0 \to \mathcal{O}_{\mathbb{P}^5}(-2) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_Y \to 0$$
$$0 \to \mathcal{O}_Y(-4) \to \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

Theorem

$$\dim H^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{d} \quad \textit{if} \quad d \geq 0$$

$$\dim H^n(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(d)) = \binom{-d-1}{-n-d-1} \quad \textit{if} \quad d \leq -n-1$$

$$\dim H^i(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(d)) = 0 \quad \textit{otherwise}$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of $\mathcal{O}_X$

$$X = \left(\; 5 \; \middle| \; 2 \quad 4 \; \right)$$

Apply long exact seq in cohomology to $0 \to \mathcal{O}_{\mathbb{P}^5}(-2) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_Y \to 0$:

$$h(\mathcal{O}_{\mathbb{P}^5}) = (1, 0, 0, 0, 0, 0)$$

$$h(\mathcal{O}_{\mathbb{P}^5}(-2)) = (0, 0, 0, 0, 0, 0)$$

$$\rightsquigarrow h(\mathcal{O}_Y) = (1, 0, 0, 0, 0)$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of $\mathcal{O}_X$

$X = \left(\begin{array}{c|cc} 5 & 2 & 4 \end{array} \right)$

Apply long exact seq in cohomology to $0 \to \mathcal{O}_{\mathbb{P}^5}(-2) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_Y \to 0$:

$$h(\mathcal{O}_{\mathbb{P}^5}) = (1,0,0,0,0,0)$$

$$h(\mathcal{O}_{\mathbb{P}^5}(-2)) = (0,0,0,0,0,0)$$

$$\rightsquigarrow h(\mathcal{O}_Y) = (1,0,0,0,0)$$

Apply long exact seq in cohomology to $0 \to \mathcal{O}_{\mathbb{P}^5}(-6) \to \mathcal{O}_{\mathbb{P}^5}(-4) \to \mathcal{O}_Y(-4) \to 0$:

$$h(\mathcal{O}_{\mathbb{P}^5})(-4) = (0,0,0,0,0,0)$$

$$h(\mathcal{O}_{\mathbb{P}^5}(-6)) = (0,0,0,0,0,1)$$

$$\rightsquigarrow h(\mathcal{O}_Y(-4)) = (0,0,0,0,1)$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of $\mathcal{O}_X$

$$X = \left(\begin{array}{c|cc} 5 & 2 & 4 \end{array} \right)$$

Apply long exact seq in cohomology to $0 \to \mathcal{O}_{\mathbb{P}^5}(-2) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_Y \to 0$:

$$h(\mathcal{O}_{\mathbb{P}^5}) = (1, 0, 0, 0, 0, 0)$$

$$h(\mathcal{O}_{\mathbb{P}^5}(-2)) = (0, 0, 0, 0, 0, 0)$$

$$\rightsquigarrow h(\mathcal{O}_Y) = (1, 0, 0, 0, 0)$$

Apply long exact seq in cohomology to $0 \to \mathcal{O}_{\mathbb{P}^5}(-6) \to \mathcal{O}_{\mathbb{P}^5}(-4) \to \mathcal{O}_Y(-4) \to 0$:

$$h(\mathcal{O}_{\mathbb{P}^5})(-4) = (0, 0, 0, 0, 0, 0)$$

$$h(\mathcal{O}_{\mathbb{P}^5}(-6)) = (0, 0, 0, 0, 0, 1)$$

$$\rightsquigarrow h(\mathcal{O}_Y(-4)) = (0, 0, 0, 0, 1)$$

Apply long exact seq in cohomology to $0 \to \mathcal{O}_Y(-4) \to \mathcal{O}_Y \to \mathcal{O}_X \to 0$:

$$h(\mathcal{O}_Y) = (1, 0, 0, 0, 0)$$

$$h(\mathcal{O}_Y(-4)) = (0, 0, 0, 0, 1)$$

$$\rightsquigarrow h(\mathcal{O}_X) = (1, 0, 0, 1)$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of Ω_X^1

$X = \left(\begin{array}{c|cc} 5 & 2 & 4 \end{array} \right)$

$$h(\Omega_X^1) = \left(h^{1,0}(X), h^{1,1}(X), h^{1,2}(X), h^{1,3}(X), \cdots\right)$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of Ω^1_X

$X = \left(\begin{array}{c|cc} 5 & 2 & 4 \end{array} \right)$

$$h(\Omega^1_X) = \left(h^{1,0}(X), h^{1,1}(X), h^{1,2}(X), h^{1,3}(X), \cdots\right)$$

$$0 \to T_X \to T_{\mathbb{P}^5}|_X \to N_{X|\mathbb{P}^5} \to 0$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of Ω_X^1

$$X = \left(5 \mid 2 \quad 4 \right)$$

$$h(\Omega_X^1) = \left(h^{1,0}(X), h^{1,1}(X), h^{1,2}(X), h^{1,3}(X), \cdots\right)$$

$$0 \to T_X \to T_{\mathbb{P}^5}|_X \to N_{X|\mathbb{P}^5} \to 0$$

$$0 \to \underbrace{N_{X|\mathbb{P}^5}^\vee}_{=\mathcal{O}_X(-2)\oplus\mathcal{O}_X(-4)} \to \Omega_{\mathbb{P}^5}^1|_X \to \Omega_X^1 \to 0$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of Ω^1_X

$X = \left(5 \mid 2 \quad 4 \right)$

$$h(\Omega^1_X) = \left(h^{1,0}(X), h^{1,1}(X), h^{1,2}(X), h^{1,3}(X), \cdots\right)$$

$$0 \to T_X \to T_{\mathbb{P}^5}|_X \to N_{X|\mathbb{P}^5} \to 0$$

$$0 \to \underbrace{N^\vee_{X|\mathbb{P}^5}}_{=\mathcal{O}_X(-2)\oplus\mathcal{O}_X(-4)} \to \Omega^1_{\mathbb{P}^5}|_X \to \Omega^1_X \to 0$$

Theorem (Euler sequence)

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

$$\overset{\otimes\mathcal{O}_X}{\rightsquigarrow} 0 \to \Omega^1_{\mathbb{P}^5}|_X \to \mathcal{O}_X(-1)^{\oplus 6} \to \mathcal{O}_X \to 0$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of Ω^1_X

$X = \left(\begin{array}{c|cc} 5 & 2 & 4 \end{array} \right)$

- $h(\mathcal{O}_X(-2)) = (0,0,0,20)$,
  $h(\mathcal{O}_X(-4)) = (0,0,0,104)$

$$\rightsquigarrow h(N^\vee_{X|\mathbb{P}^5}) = h(\mathcal{O}_X(-2)) + h(\mathcal{O}_X(-4)) = (0,0,0,124)$$

- $h(\mathcal{O}_X(-1)^{\oplus 6}) = (0,0,0,6)^{\oplus 6} = (0,0,0,36)$,
  $h(\mathcal{O}_X) = (1,0,0,1)$

$$\rightsquigarrow h(\Omega^1_{\mathbb{P}^5}|_X) = (0,1,0,35)$$

- $h(N^\vee_{X|\mathbb{P}^5}) = (0,0,0,124)$,
  $h(\Omega^1_{\mathbb{P}^5}|_X) = (0,1,0,35)$

$$\rightsquigarrow h(\Omega^1_X) = (0,1,89,0)$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of Ω^1_X

$$X = \left(\; 5 \; \middle| \; 2 \quad 4 \; \right)$$

$$0 \to N^\vee_{X|\mathbb{P}^5} \to \Omega^1_{\mathbb{P}^5}|_X \to \Omega^1_X \to 0$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X; N^\vee_{X|\mathbb{P}^5}) & \longrightarrow & H^0(X; \Omega^1_{\mathbb{P}^5}|_X) & \longrightarrow & H^0(X; \Omega^1_X) & & \\
& & H^1(X; N^\vee_{X|\mathbb{P}^5}) & \longrightarrow & H^1(X; \Omega^1_{\mathbb{P}^5}|_X) & \longrightarrow & H^1(X; \Omega^1_X) & & \\
& & H^2(X; N^\vee_{X|\mathbb{P}^5}) & \longrightarrow & H^2(X; \Omega^1_{\mathbb{P}^5}|_X) & \longrightarrow & H^2(X; \Omega^1_X) & & \\
& & H^3(X; N^\vee_{X|\mathbb{P}^5}) & \longrightarrow & H^3(X; \Omega^1_{\mathbb{P}^5}|_X) & \longrightarrow & H^3(X; \Omega^1_X) & \longrightarrow & 0
\end{array}$$

Example CICY threefold in $\mathbb{P}^5$: cohomology of Ω^1_X

$X = \left(\ 5 \mid 2 \quad 4\ \right)$

$$0 \to N^\vee_{X|\mathbb{P}^5} \to \Omega^1_{\mathbb{P}^5}|_X \to \Omega^1_X \to 0$$

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow H^0(X;\Omega^1_X)$$

$$\longrightarrow 0 \longrightarrow 1 \longrightarrow H^1(X;\Omega^1_X)$$

$$\longrightarrow 0 \longrightarrow 0 \longrightarrow H^2(X;\Omega^1_X)$$

$$\longrightarrow 124 \longrightarrow 35 \longrightarrow H^3(X;\Omega^1_X) \longrightarrow 0$$

$$\rightsquigarrow h(\Omega^1_X) = (0, 1, 89, 0)$$

Example CICY threefold in $\mathbb{P}^5$: Hodge diamond

$$X^3 = \left(\begin{array}{c|cc} 5 & 2 & 4 \end{array} \right)$$

			1				$b_0 = 1$
		0		0			$b_1 = 0$
	0		1		0		$b_2 = 1$
1		89		89		1	$b_3 = 180$
	0		1		0		$b_4 = 1$
		0		0			$b_5 = 0$
			1				$b_6 = 1$

$\chi = -176$

CICY : Koszul sequence

Koszul complex

Let $X = \{\xi = 0\}$ for $\xi \in H^0(\mathbb{P}^n; \mathcal{E})$, where $\mathcal{E} := \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_k)$,

$$0 \to \wedge^k \xi^\vee \to \wedge^{k-1} \xi^\vee \to \cdots \to \wedge^2 \xi^\vee \to \xi^\vee \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

Then

$$E_0^{q,k} = H^q(\mathbb{P}^n; \wedge^k \xi^\vee) \Rightarrow H^q(X; \mathcal{O}_X)$$

Example CICY threefold in $\mathbb{P}^5$: Koszul sequence

$$X = \left(\; 5 \; \middle| \; 2 \quad 4 \; \right) \quad \rightsquigarrow \quad 0 \to \mathcal{O}_{\mathbb{P}^5}(-6) \to \mathcal{O}_{\mathbb{P}^5}(-2) \oplus \mathcal{O}_{\mathbb{P}^5}(-4) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_X \to 0$$

$E_0^{q,k} = H^q(\mathbb{P}^n; \wedge^k \xi^\vee)$	$k=2$	$k=1$	$k=0$	$H^q(X; \mathcal{O}_X)$
$q=0$	0	$0+0$	1	1
$q=1$	0	$0+0$	0	0
$q=2$	0	$0+0$	0	0
$q=3$	0	$0+0$	0	1
$q=4$	0	$0+0$	0	
$q=5$	1	$0+0$	0	

Example CICY threefold in $\mathbb{P}^5$: Koszul sequence

$$X = \left(\; 5 \;|\; 2 \quad 4 \; \right) \quad \overset{\otimes \Omega^{\mathbf{1}}_{\mathbb{P}^{\mathbf{5}}}}{\rightsquigarrow} \quad 0 \to \Omega^1_{\mathbb{P}^5}(-6) \to \Omega^1_{\mathbb{P}^5}(-2) \oplus \Omega^1_{\mathbb{P}^5}(-4) \to \Omega^1_{\mathbb{P}^5} \to \Omega^1_{\mathbb{P}^5}|_X \to 0$$

Bott formula $\rightsquigarrow$

$E_0^{q,k} = H^q(\mathbb{P}^n; \Omega^1_{\mathbb{P}^5} \otimes \wedge^k \xi^\vee)$	$k=2$		$k=1$		$k=0$	$H^q(X; \Omega^1_{\mathbb{P}^5}\|_X)$
$q=0$	0	$\longrightarrow$	$0+0$	$\longrightarrow$	0	0
$q=1$	0	$\longrightarrow$	$0+0$	$\longrightarrow$	1	1
$q=2$	0	$\longrightarrow$	$0+0$	$\longrightarrow$	0	0
$q=3$	0	$\longrightarrow$	$0+0$	$\longrightarrow$	0	35
$q=4$	0	$\longrightarrow$	$0+0$	$\longrightarrow$	0	
$q=5$	35	$\longrightarrow$	$0+0$	$\longrightarrow$	0	

CICY : Bott formula

$$\dim H^q(\mathbb{P}^n; \wedge^p T_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} \binom{k+n+1+p}{p} \cdot \binom{k+n}{n-p} & \text{if} \quad q = 0, \quad k \geq -p \\ 1 & \text{if} \quad q = n-p, \quad k = -(n+1) \\ \binom{-k-n-2}{p} \cdot \binom{-k-p-1}{n-p} & \text{if} \quad q = n, \quad k \leq -(n+p+2) \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

Example CICY threefold in $\mathbb{P}^5$: Koszul sequence

$$X = \left(\ 5 \ \middle|\ 2 \quad 4\ \right) \quad \rightsquigarrow \quad 0 \to N^{\vee}_{X|\mathbb{P}^5} \to \Omega^1_{\mathbb{P}^5}|_X \to \Omega^1_X \to 0$$

	$H^q(X;\mathcal{O}_X(-2)\oplus\mathcal{O}_X(-4))$	$H^q(X;\Omega^1_{\mathbb{P}^5}\|_X)$	$H^q(X;\Omega^1_X)$
$q=0$	0	0	$h^{1,0}(X)=0$
$q=1$	0	1	1
$q=2$	0	0	89
$q=3$	$20+104$	35	$h^{2,0}(X)=0$

Example CICY threefold in $\mathbb{P}^5$: Koszul sequence

$$X = \left(\; 5 \; \middle| \; 2 \quad 4 \; \right) \quad \overset{\otimes \Omega^{\mathbf{2}}_{\mathbb{P}^{\mathbf{5}}}}{\rightsquigarrow} \quad 0 \to \Omega^2_{\mathbb{P}^5}(-6) \to \Omega^2_{\mathbb{P}^5}(-2) \oplus \Omega^2_{\mathbb{P}^5}(-4) \to \Omega^2_{\mathbb{P}^5} \to \Omega^2_{\mathbb{P}^5}|_X \to 0$$

$E_0^{q,k} = H^q(\mathbb{P}^n; \Omega^2_{\mathbb{P}^5} \otimes \wedge^k \xi^\vee)$	$k=2$	$k=1$	$k=0$	$H^q(X; \Omega^2_{\mathbb{P}^5}\|_X)$
$q=0$	0	0 + 0	0	0
$q=1$	0	0 + 0	0	0
$q=2$	0	0 + 0	1	1
$q=3$	0	0 + 0	0	265
$q=4$	0	0 + 0	0	
$q=5$	280	0 + 15	0	

Example CICY threefold in $\mathbb{P}^5$: Koszul sequence

$X = \left(\begin{array}{c|cc} 5 & 2 & 4 \end{array} \right)$

$$0 \longrightarrow \mathrm{Sym}^2(N^\vee_{X|\mathbb{P}^5}) \longrightarrow N^\vee_{X|\mathbb{P}^5} \otimes \Omega^1_{\mathbb{P}^5}|_X \longrightarrow \Omega^2_{\mathbb{P}^5}|_X \to \Omega^2_X \longrightarrow 0$$

$$N^\vee_{X|\mathbb{P}^5} \otimes \Omega^1_{\mathbb{P}^5}|_X \dashrightarrow K \dashrightarrow \Omega^2_{\mathbb{P}^5}|_X$$

where

$$\mathrm{Sym}^2(N^\vee_{X|\mathbb{P}^5}) = \mathcal{O}_X(-4) \oplus \mathcal{O}_X(-6) \oplus \mathcal{O}_X(-8)$$

Example CICY fivefold in $\mathbb{P}^5$: Hodge diamond

$$X = \left(\begin{array}{c|ccc} 2 & 1 & 1 & 1 \\ 6 & 0 & 0 & 7 \end{array} \right)$$

$$\begin{array}{ccccccccccc|c}
 & & & & & 1 & & & & & & b_0 = 1 \\
 & & & & 0 & & 0 & & & & & b_1 = 0 \\
 & & & 0 & & 1 & & 0 & & & & b_2 = 1 \\
 & & 0 & & 0 & & 0 & & 0 & & & b_3 = 0 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & & b_4 = 1 \\
1 & & 1667 & & 18327 & & 18327 & & 1667 & & 1 & b_5 = 39990 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & & b_6 = 1 \\
 & & 0 & & 0 & & 0 & & 0 & & & b_7 = 0 \\
 & & & 0 & & 1 & & 0 & & & & b_8 = 1 \\
 & & & & 0 & & 0 & & & & & b_9 = 0 \\
 & & & & & 1 & & & & & & b_{10} = 1 \\
\hline
\end{array}$$

$$\chi = -39984$$

CICY fivefolds : summary

Problems and (possible) solutions:

- supplementary variables required
- if they decouple, solution is unique in the Hodge numbers
- otherwise, consider ineffective splitting
- limited computational resources

Results:

- 27 068 configuration matrices of dimensions up to 4×4, up to permutations
- ignoring 3 909 of them which are products
- full Hodge diamond computed for 12 433 manifolds, $\sim 53,7\%$
- 2 375 distinct Hodge diamonds

1 Introduction

2 Construction

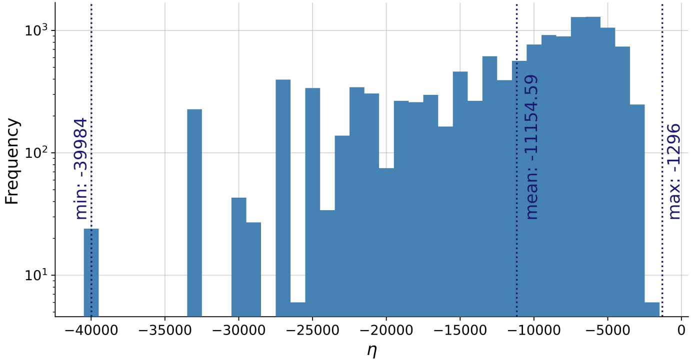
3 Cohomology

4 Statistics

5 Machine Learning

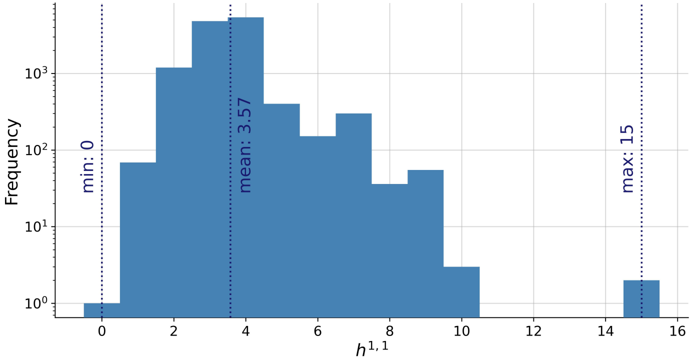
CICY fivefolds : distribution of Euler number

$$\langle\chi\rangle = -11154,59^{-1296}_{-39984}$$



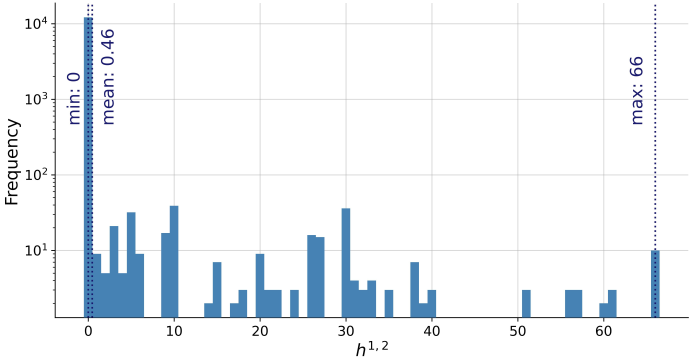
CICY fivefolds : distribution of Hodge number $h^{1,1}$

$$\langle h^{1,1} \rangle = 3,57_{0}^{15}$$



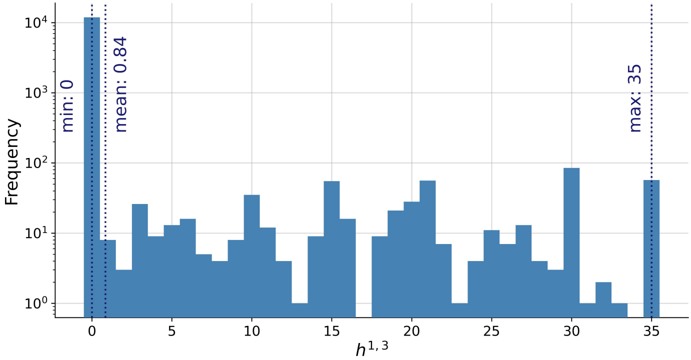
CICY fivefolds : distribution of Hodge number $h^{1,2}$

$$\langle h^{1,2} \rangle = 0,46_0^{66}$$



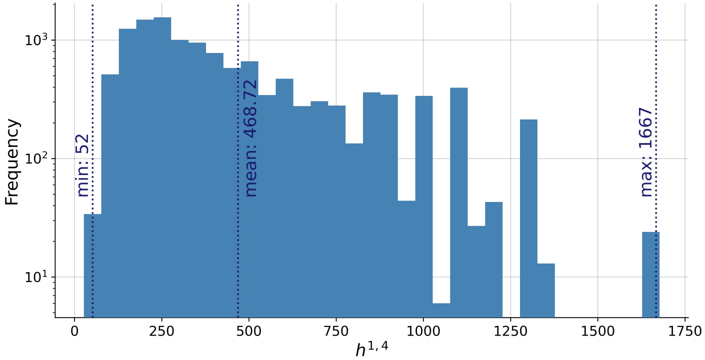
CICY fivefolds : distribution of Hodge number $h^{1,3}$

$$\langle h^{1,3} \rangle = 0,84_0^{35}$$



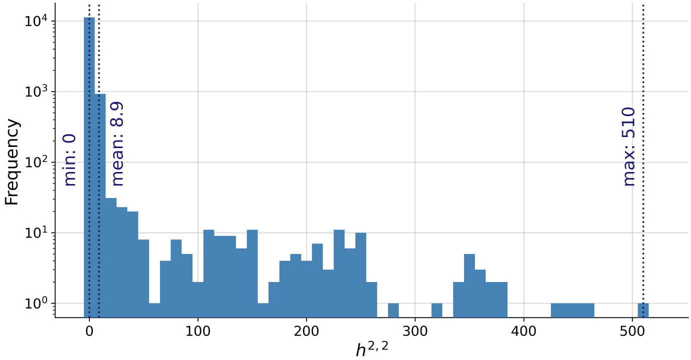
CICY fivefolds : distribution of Hodge number $h^{1,4}$

$$\langle h^{1,4} \rangle = 468,72_{52}^{1667}$$



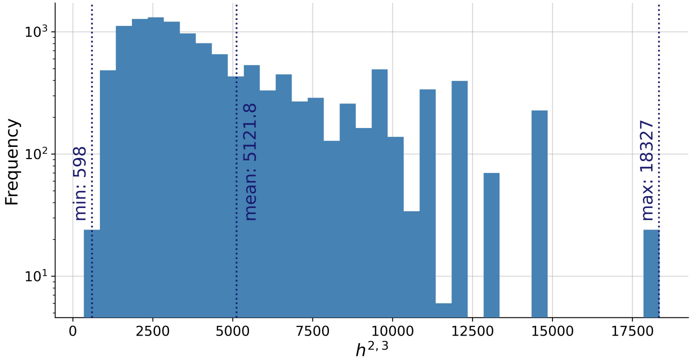
CICY fivefolds : distribution of Hodge number $h^{2,2}$

$$\langle h^{2,2} \rangle = 8,9_0^{510}$$

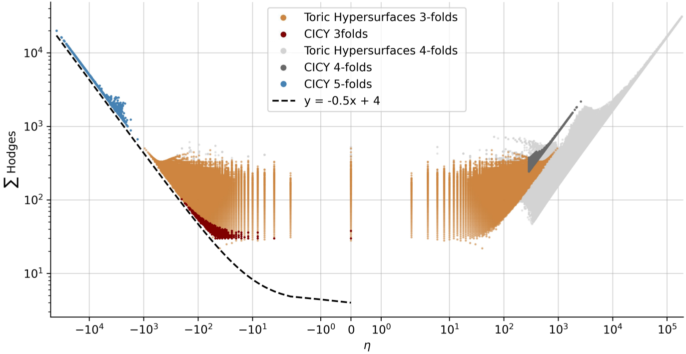


CICY fivefolds : distribution of Hodge number $h^{2,3}$

$$\langle h^{2,3} \rangle = 5121, 8_{598}^{18327}$$



CY cohomology : asymptotic behaviour



1 Introduction

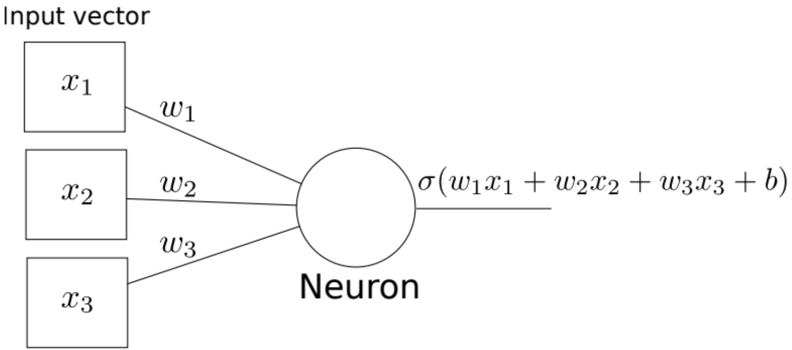
2 Construction

3 Cohomology

4 Statistics

5 Machine Learning

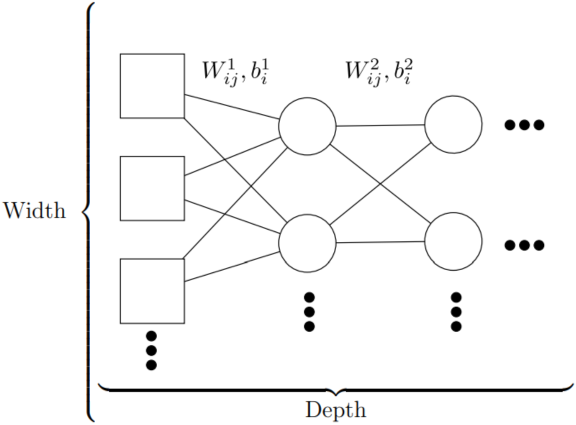
Neural network



(*e.g.* $\sigma(x) = \lg(1 + e^x)$ softplus)

Neural network

$$f_i^n = f(W_{ij}^n f_j^{n-1} + b_i^n)\,.$$



Universal approximation theorem

THEOREM 12 (Universal Approximation Theorem) *We have the following approximations by feed-forward NNs:*

Arbitrary Width *For every continuous function* $f : \mathbb{R}^d \to \mathbb{R}^D$, *every compact subset* $K \subset \mathbb{R}^d$, *and every* $\epsilon > 0$, *there exists [347] a continuous function* $f_\epsilon : \mathbb{R}^d \to \mathbb{R}^D$ *such that* $f_\epsilon = W_2(\sigma(W_1))$, *where* σ *is a fixed continuous function,* $W_{1,2}$ *affine transformations and composition appropriately defined, so that* $\sup_{x \in K} |f(x) - f_\epsilon(x)| < \epsilon$.

Arbitrary Depth *Consider a feed-forward NN with* n *input neurons,* m *output neuron and an arbitrary number of hidden layers each with* $n + m + 2$ *neurons, such that every hidden neuron has activation function* φ *and every output neuron has activation function the identity* [8] *Then [348], given any vector-valued function* f *from a compact subset* $K \subset \mathbb{R}^m$, *and any* $\epsilon > 0$, *one can find an* F, *a NN of the above type, so that* $|F(x) - f(x)| < \epsilon$ *for all* $x \in K$.

[8] Here $\varphi : \mathbb{R} \to \mathbb{R}$ is any non-affine continuous function which is continuously differentiable at least at one point and with non-zero derivative at that point.

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Machine Learning CICY : literature

CICY 3fold	dense	convolutional	convolutional	inception
$h^{1,1}$	84%	90%	94%	99%
$h^{2,1}$	–	–	37%	50%

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Machine Learning CICY fivefolds : architectures

- classifier:
  $h^{1,1}$: $N_{4*4}(512, \sigma, \delta_{0.4}, 256, \sigma, \delta_{0.3}, 256, \sigma, 16, S)$,
  other: $N_{4*4}(512, \sigma, \delta_{0.4}, 512, \sigma, \delta_{0.3}, 512, \sigma, \delta_{0.3}, 256, \sigma, N_H, S)$
- linear regressor:
  $h^{1,1}$: $N_{4*4}(512, s, 256, s, 128, s, 32, s, 8, s, 1)$
  other: $N_{4*4}(1024, s, 1024, s, 512, s, 64, s, 16, s, 1)$
- convolutional regressor:
  $N_{4*4}(C_{5*5}180, r, BN_{0.99}, C_{5*5}100, r, BN_{0.99}, C_{5*5}40, r, BN_{0.99}, C_{5*5}20, r, BN_{0.99}, \delta_{0.4}, f, 1, r)$

Y.-H. He, A. Lukas, Machine Learning Calabi-Yau Four-folds, *Phys. Lett. B* **815** (2021) 136–139.

H. Erbin, R. Finotello, Machine learning for complete intersection Calabi-Yau manifolds: A methodological study, *Phys. Rev. D* **103** (2021).

Machine Learning CICY fivefolds : $h^{1,1}$

$h^{1,1}$	R^2	Accuracy
Classifier	n/a	88%
Linear Regressor	85%	93%
Convolutional Regressor	91%	96%

Remark

- *For four-folds, accuracy* 96%, 93%, 94% *respectively*
- *Number of values of* $h^{1,1}$: 12

Machine Learning CICY fivefolds : $h^{1,2}$

$h^{1,2}$	R^2	Accuracy
Classifier	n/a	98%
Linear Regressor	-ve	83%
Convolutional Regressor	-ve	97%

Remark

- *Number of values of* $h^{1,2}$*:* 33
- $h^{1,2} = 0$ *for* $97,7\%$

Machine Learning CICY fivefolds : $h^{1,3}$

$h^{1,3}$	R^2	Accuracy
Classifier	n/a	96%
Linear Regressor	61%	93%
Convolutional Regressor	79%	95%

Remark

- *Number of values of* $h^{1,3}$*:* 34
- $h^{1,2} = 0$ *for* $95,7\%$

Machine Learning CICY fivefolds : $h^{1,4}$

$h^{1,4}$	R^2	Accuracy	Acc w/ 10% Toler
Classifier	n/a	3%	n/a
Linear Regressor	98%	3%	86%
Convolutional Regressor	98%	2%	86%

Remark

- *Number of values of* $h^{1,4}$*:* 642
  Range: $52 \leq h^{1,4} \leq 1667$
- *Very high* R^2 *score*

Machine Learning CICY fivefolds : $h^{2,2}$

$h^{2,2}$	R^2	Accuracy
Classifier	n/a	22%
Linear Regressor	26%	25%
Convolutional Regressor	16%	27%

Remark

- *Wide range*
- *Most values close to* 0

Machine Learning CICY fivefolds : $h^{2,3}$

$h^{2,3}$	R^2	Accuracy	Acc w/ 10% Toler
Classifier	n/a	2.7%	n/a
Linear Regressor	97%	0.2%	77%
Convolutional Regressor	98%	0.1%	85%

Remark

- *Wide range:* $598 \leq h^{2,3} \leq 18327$
- *Very high R^2 score*

Machine Learning CICY fivefolds : χ

χ	R^2	Accuracy	Acc w/ 10% Toler
Classifier	n/a	3%	n/a
Linear Regressor	50%	0%	0%
Convolutional Regressor	98%	0.04%	83%

Remark

- *Wide range:* $-39984 \leq \chi \leq -1296$
- *Very high R^2 score*

Machine Learning CICY fivefolds : conclusions

- very good results for $h^{1,1}$
- probably guessing $h^{1,2} = 0$ and $h^{1,3} = 0$
- very poor performance for $h^{2,2}$
- regressors are good approximator for $h^{1,4}$, $h^{2,3}$, and χ

Work in progress. . .

- construct complete dataset
- extrapolate predictions from low to high Hodge numbers
- investigation across different complex dimensions
- unobserved clustering behaviour
- hints for unknown formulas
- include other topological properties
- approximation methods
- toric CY fivefolds
- non-Kähler CY
- . . .

Thank you!