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Fock-Goncharov dual cluster varieties and Gross-Siebert mirrors.

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joint work with Hülya Argüz

1) FG duality for cluster varieties / Combinatorial duality

$(\mathcal{A}, \mathcal{X})$

2) Mirror symmetry / Enumerative geometry of (X,D)

$\uparrow$

Gross-Siebert

$(X,D) \rightsquigarrow X^\vee$

log CY pair — mirror family

3) Main result:

Mirror of a log CY compactification of $\mathcal{X}$ (or $\mathcal{A}$) $=$ Degeneration of $\mathcal{A}$ (or $\mathcal{X}$)

1) Cluster varieties. $\begin{cases} \text{Cluster algebras (Fomin-Zelevinsky)} \\ \text{Fock-Goncharov} \end{cases}$

↑

obtained by gluing tori along explicit birational transformations.

$N \simeq \mathbb{Z}^n$

$M = \mathrm{Hom}(N, \mathbb{Z})$

$$\mathcal{A} = \underbrace{\bigcup}\ \underbrace{\mathrm{Spec}\,\mathbb{C}[M]}_{\simeq (\mathbb{C}^*)^n} \qquad \mathcal{X} = \underbrace{\bigcup}\ \mathrm{Spec}\,\mathbb{C}[N]$$

<u>Seed</u> s:

- $\{,\} : N \times N \to \mathbb{Z}$ skew-symmetric $\quad (\{e_i, e_j\})$
- basis (e_i) of N

Birational transformations are volume preserving $\Big| \Rightarrow \mathcal{A}, \mathcal{X}$ non-compact Calabi-Yau varieties.

Alternative point of view on $\mathcal{A}$ & $\mathcal{X}$ from log Calabi-Yau varieties.

Log Calabi-Yau variety: (X, D)

smooth proj variety

normal crossing divisor

$K_X + D = 0$

$U = X \setminus D$

Calabi-Yau variety

Ex: (X_Σ, D_Σ) $\quad U = X_\Sigma \setminus D_\Sigma = (\mathbb{C}^*)^n$

Toric variety fan Σ

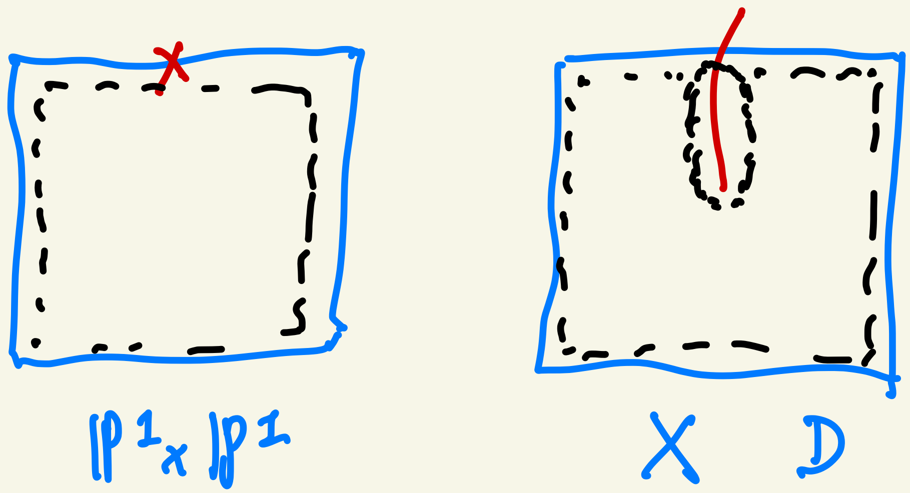
Toric boundary divisor

Ex: (X_Σ, D_Σ) Hypersurfaces $H_i \subset D_\Sigma$

$$\left(X := \mathrm{Blowup}_{\bigcup_i H_i}(X_\Sigma) \;,\; D := \text{strict transform of the toric boundary} \right) \text{ log CY pair.}$$

$$U = X \setminus D$$

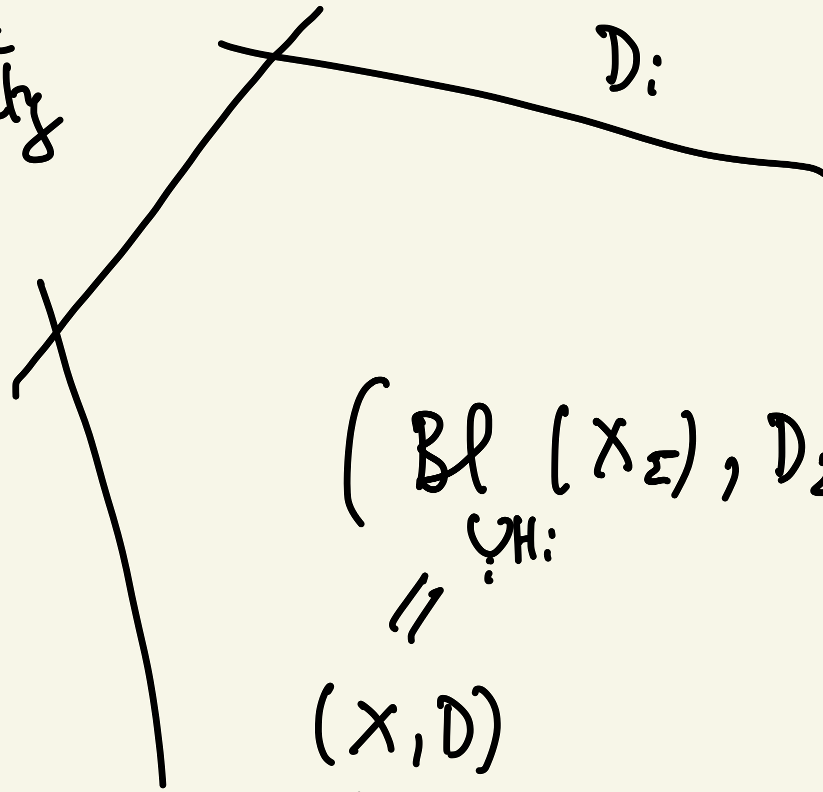
Ex: (X_Σ, D_Σ)



Ex: $\mathcal{A}$ $\quad\Sigma$ FAN in N

contains rays $\mathbb{R}_{\geq 0} e_i \to$ divisor D_i

↳ Toric variety



$$H_i \subset D_i$$
$$\shortparallel$$
$$\{1 + z^{v_i} = 0\}$$

where $v_i = \{e_i, -\} \in M$

$$\left(Bl_{\cup H_i}(X_\Sigma), D_\Sigma\right)$$
$$\shortparallel$$
$$(X, D)$$

<u>Thm</u> (Gross-Hacking-Keel)

$$U_{\mathcal{A}} := X \setminus D \simeq \mathcal{A}$$

up to codimension 2.

$$U_{\mathcal{X}} = X' \setminus D' \simeq \mathcal{X}$$

$\mathcal{X}$ FAN in M

(X', D')

Rays $\mathbb{R}_{\geq 0} v_i$

Blow-up $1 + z^{e_i} = 0$.

Fock-Goncharov duality: properties exchanged

$\mathcal{A} \longleftrightarrow \mathcal{X}$

[Canonical basis for algebras of regular functions on $\mathcal{A}$ & $\mathcal{X}$]

2) Mirror symmetry

X Calabi-Yau variety $\qquad$ $X^\vee$ Calabi-Yau variety

Complex geometry of X $\longleftrightarrow$ Symplectic geometry of $X^\vee$

Gross-Siebert mirror construction:

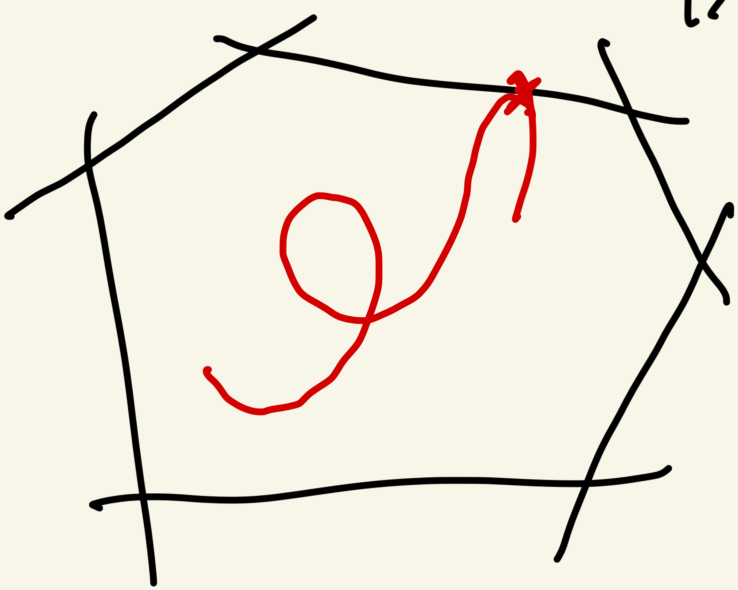
$$\left.\begin{array}{l} (X, D) \text{ log Calabi-Yau variety} \\ \text{s.t. } D \text{ is "maximal", meaning} \\ \text{containing a } \underline{0\text{-dim}} \text{ stratum.} \\ \left\{ \begin{array}{l} \text{Always true: if } (X,D) \text{ is toric} \\ \qquad\qquad\quad \cdot \text{ if } (X,D) \text{ is obtained} \\ \qquad\qquad\qquad \text{as blow-up} \end{array} \right. \end{array}\right\} \longrightarrow \begin{array}{c} \text{Mirror family} \\ X^\vee \\ \downarrow \\ \underbrace{\mathrm{Spf}\, \mathbb{C}[[NE(X)]]} \end{array}$$

where $NE(X)$ is the monoid spanned by effective curve classes in the group of curve classes / numerical equivalence.

$$\begin{tikzcd} (X, D) \arrow[r, squiggly] & X^\vee \arrow[d] \\ & \mathrm{Spf}\ \underbrace{\mathbb{C}[[NE(X)]]} \end{tikzcd}$$

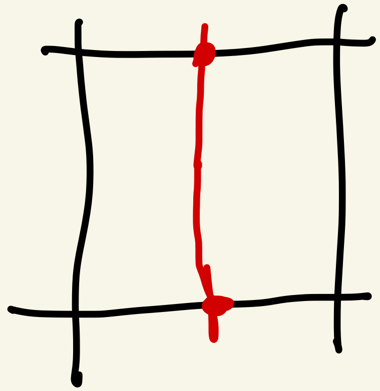
Enumerative geometry of (X, D)

(X, D)

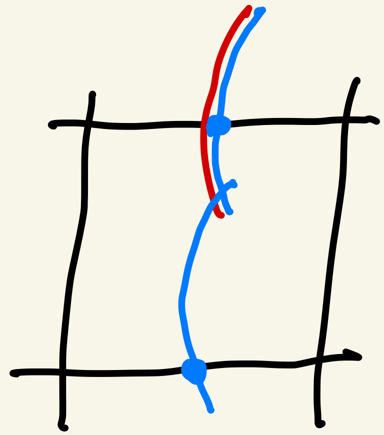


Counts rational curves in X meeting D in a single point.

$$\sum_{\beta \in NE(X)} N_\beta t^\beta \Big]$$



$\mathbb{P}^1 \times \mathbb{P}^1$



If $U = X \setminus D$ is affine $\rightsquigarrow$

$$\begin{tikzcd} X^\vee \arrow[d] & X^\vee \arrow[d] \\ \operatorname{Spf} \mathbb{C}[[NE(X)]] & \operatorname{Spec} \mathbb{C}[NE(X)] \end{tikzcd}$$

If not, might only get this formal family.

3) Comparison ?

$\mathcal{X}$ cluster variety

$\parallel$

$X \setminus D$

(X, D) log Calabi-Yau pair

$\parallel$

$Bl_H(X_\Sigma)$

Fock-Goncharov duality

$\mathcal{A}$ cluster variety

$\updownarrow$?

Gross-Siebert mirror construction

Mirror family

$$X^\vee \downarrow \mathrm{Spf}\, \mathbb{C}[[NE(X)]]$$

"Easy case"

If $\mathcal{X}$ is affine $\rightsquigarrow$ Mirror family

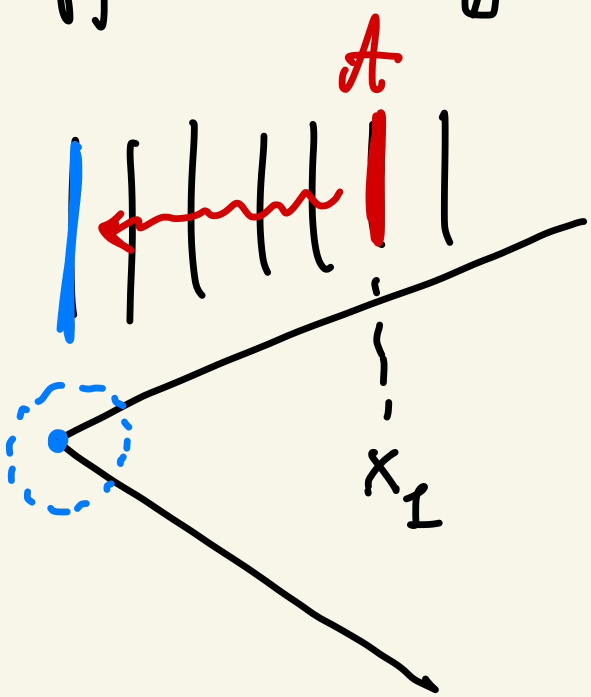
$$\begin{array}{c} X^\vee \\ \pi \downarrow \\ \operatorname{Spec} \mathbb{C}[NE(X)] \\ \cup \\ 1 \in \text{Big Torus} \end{array}$$

Thm (Keel-Yu)

$$\underbrace{\pi^{-1}(1)}_{\text{Fiber over } 1 \text{ of mirror to } \mathcal{X}} \simeq A$$

What to do in general? Do not assume $\mathcal{X}$ affine.

$$\begin{array}{c} X^\vee \\ \downarrow \\ \operatorname{Spf} \mathbb{C}[[NE(X)]] \end{array} \qquad \text{versus} \qquad \mathcal{A} \ ?$$



Spec $\mathbb{C}[NE(X)]$ Toric variety

$$\mathcal{A} \rightsquigarrow \mathcal{A}_{prin}$$

$\mathcal{A}$- cluster variety with principal coefficients.

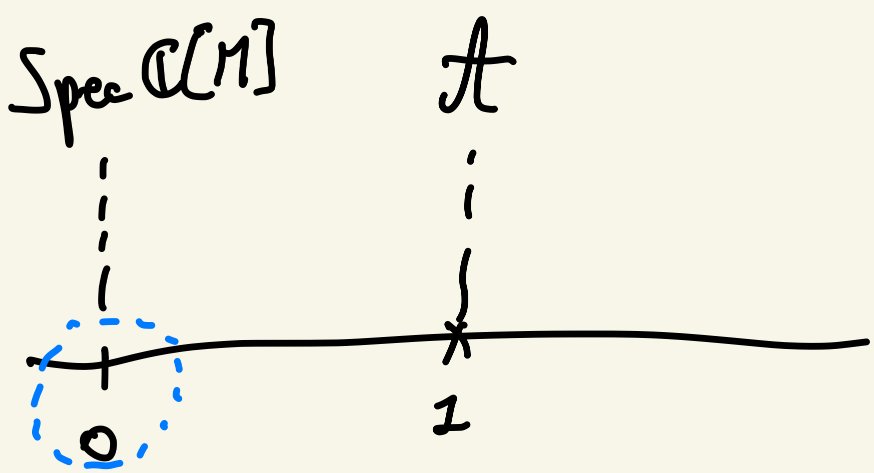
$$\begin{array}{c} \mathcal{A}_{prin} \\ \downarrow \\ \operatorname{Spec} \mathbb{C}[N^{\oplus}] \simeq \mathbb{A}^{\cdots} \\ t_i \end{array}$$

N

e_i

$$N^{\oplus} = \bigoplus_i \mathbb{N} e_i$$

$\mathcal{A}$ blowing-up $\underbrace{1 + t_i z^{v_i} = 0}$



$$\begin{array}{c} \hat{\mathcal{A}}_{prin} \\ \downarrow \\ \operatorname{Spf} \mathbb{C}[[N^{\oplus}]] \end{array}$$

$$\begin{array}{ccc} X^\vee & \overset{?}{\longleftrightarrow} & \hat{\mathcal{A}}_{prin} \\ \downarrow & & \downarrow \\ \underbrace{\operatorname{Spf} \mathbb{C}[[NE(X)]]} & & \operatorname{Spf} \mathbb{C}[[N^\oplus]] \end{array}$$

$$\begin{array}{c} X \\ \downarrow \\ X_\Sigma \end{array}$$

exceptional curves.

$$N_1(X) = N_1(X_\Sigma) \oplus \bigoplus_i \mathbb{Z} E_i$$

$$\underbrace{NE(X)} \subset \underbrace{NE(X_\Sigma)} \oplus \bigoplus_i \mathbb{Z} E_i$$

$$\cup \qquad (-a_i E_i)$$

$$\underbrace{\underbrace{NE(X_\Sigma)} \oplus N^\oplus}_{(a_i)_i} \qquad (a_i)_i \mapsto (-a_i E_i)$$

$\operatorname{Spec} \mathbb{C}[NE(X)]$ and $\operatorname{Spec} \mathbb{C}[NE(X_\Sigma) \oplus N^\oplus]$

along $\operatorname{Spec} \mathbb{C}[NE(X_\Sigma) \oplus \bigoplus \mathbb{Z}E_i)$

Thm [Argüz–B]. The mirror family canonically extends over

. Extended mirror family | $\simeq \hat{\mathcal{X}}_{\mathrm{prin}}$.

Proof?

[Gross-Siebert mirror construction] → [Canonical scattering diagram]

↕ ← Argüz-Gross "The higher dim tropical vertex".

[Cluster varieties Gross-Hacking-Keel-Kontsevich] ← [Cluster scattering diagram]