Quantum geometry of log Calabi–Yau surfaces

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Overview

The main character: log-Calabi–Yau surfaces with nef maximal boundary (X, D) (*nef Looijenga pairs*).

- X smooth complex projective surface
- $|-K_X| \ni D = D_1 + \cdots + D_l$, $l > 1$ s.n.c. divisor, D_i irreducible, smooth and nef

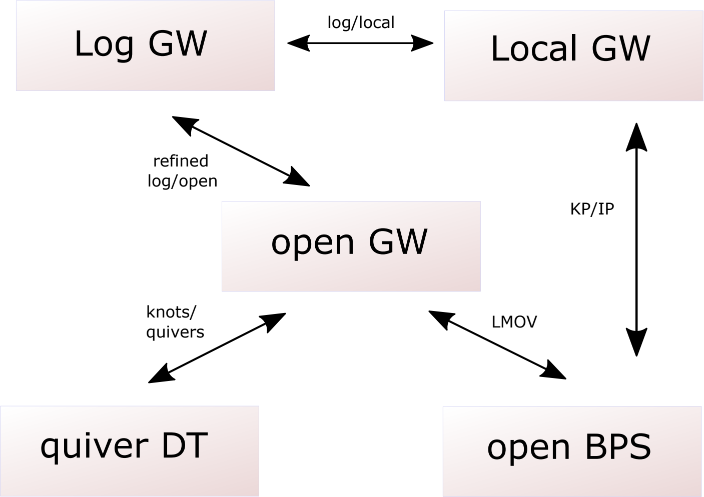
Overview

The two main messages:

1. many (different, but equivalent) enumerative theories of curves built from (X, D)
2. they are all closed-form solvable

Joint with P. Bousseau (ETH/Orsay) and M. van Garrel (Birmingham); +ongoing work with Y. Schüler (Sheffield).

Overview


Log GW
log/local
Local GW
refined
log/open
KP/IP
open GW
knots/
quivers
LMOV
quiver DT
open BPS

Counting curves

- Fundamental subset of Q's in geometry: enumerative problems.
- Examples:
  - How many lines on a smooth cubic surface?
  - How many rational degree-d plane curves through $3d - 1$ points?
  - How many degree-d curves on a quintic threefold?
- relevance: both intrinsic and for other domains of maths (MathsPhys/Topology/Number Theory/...)

Counting curves

- Typical setup:
  - X: algebraic variety$|_{\mathbb{C}}$ (e.g. $X = \mathbb{P}^2_{\mathbb{C}}$)
  - $\mathcal{M}(X)$ curves in X (e.g. plane conics)
  - $\int_{\mathcal{M}(X)}(\dots)$ numbers (“quantum invariants")
  - $(\dots)$= “incidence cond’n" (e.g. conics thru 5 pts $\to$ 1)
- No sense ($\mathcal{M}(X)$ non-compact)
- Different compactifications $\overline{\mathcal{M}}(X) \rightsquigarrow$ different invariants

Gromov–Witten theory

- Today (mostly):

$$\begin{aligned}\overline{\mathcal{M}}(X) &= \overline{\mathcal{M}}_{g,n}(X,d) \\ &= \overline{\{(C,p_1,\dots p_n) \xrightarrow{\phi} X | h^1(C,\mathcal{O}_C)=g, \phi_*[C]=d\}/\sim}\end{aligned}$$

- Compactification: smooth $C \rightsquigarrow$ nodal + stability
- Proper, expdim = $(\mathrm{dim} X - 3)(1-g) - K_X \cdot d + n$
- $[\overline{\mathcal{M}}_{g,n}(X,d)]^{\mathrm{vir}} \in A_{\mathrm{expdim}}(\overline{\mathcal{M}}_{g,n}(X,d))$

$$\begin{aligned} n_{X,g,d}[B_1,\dots,B_n] &= \text{“\# degree-}d\text{, gen-}g\text{ curves in } X \text{ through } B_i\text{”} \\ &:= \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*[B_i] \end{aligned}$$

Example: counting rational plane curves

Q. $N_d =$ *rational degree-d plane curves through* $3d-1$ *points?*

d	$3d-1$	$N_d := n_{\mathbb{P}^2,0,d}[\overbrace{\mathrm{pt},\ldots,\mathrm{pt}}^{3d-1}]$
1	2	1
2	5	1
3	8	12
4	11	620
5	14	87304
6	17	26312976
$\vdots$	$\vdots$	$\vdots$

- $N_d \sim (3d-1)! x_0^d$

[Di Francesco–Itzykson, Tian–Wei, Zinger]

- $\exists$ (non-linear) recursion, g.f. solution of Painlevé VI

[Kontsevich, Dubrovin]

Nef pairs

(X, D) smooth nef pair:

- X smooth complex projective, $\dim X = m$
- $|-K_X| \ni D = D_1 + \cdots + D_l$ s.n.c. divisor with D_i irreducible, smooth and nef $\forall i$

Examples:

1. $X = \mathbb{P}^1_{\mathbb{C}}$, $D = \{0\} + \{\infty\}$
2. $X = \mathbb{P}^2_{\mathbb{C}}$, $D =$
   - I. smooth cubic ($l = 1$)
   - II. conic+line ($l = 2$)
   - III. $\sum_{i=1}^{3} (\mathrm{axes})_i$ ($l = 3$)

Nef pairs

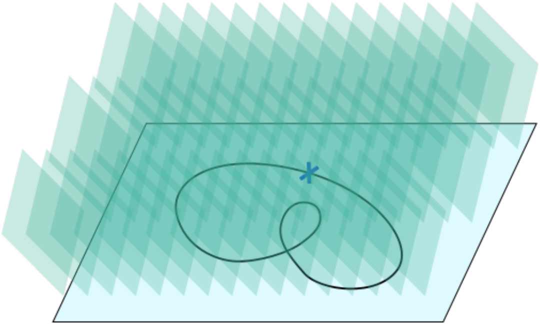
Jargon: a smooth nef pair is

- *toric* if X is, and $X \setminus D \simeq (\mathbb{C}^\star)^m$
- *Looijenga* (log-CY surface with maximal boundary) if $m = 2$, D is singular ($\Rightarrow l > 1$)
- a nef Looijenga pair is **tame** if either $l > 2$, or $D_i^2 > 0\ \forall i$

Finite catalogue of deformation families of nef Looijenga pairs; most are tame.
From now: stick to Looijenga pairs (X, D).

Local ($g = 0$) Gromov–Witten theory of (X, D)

- $E_{X,D} := \mathrm{Tot}\big(\oplus_{i=1}^{l} \mathcal{O}_X(-D_i)\big)$ (CY$(2+l)$-fold)
- $N^{\mathrm{loc}}_{(X,D),d}$ = "# degree-d, $g = 0$ curves in $E_{X,D}$ through $l-1$-points in X"



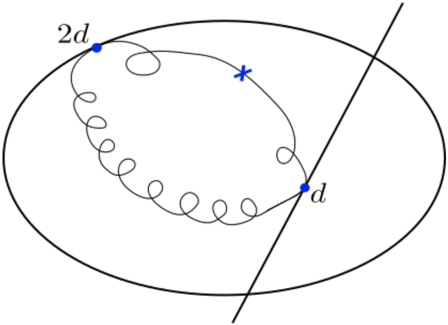
Local ($g = 0$) Gromov–Witten theory of (X, D)

- $N^{\mathrm{loc}}_{(X,D),d}$ = "# degree-d, $g = 0$ curves in $E_{X,D}$ through $l-1$-points in X"
  - $\mathrm{Obs}^{(X,D)}_d \to \overline{\mathcal{M}_{0,n}}(X,d)$, $\quad \mathrm{Obs}^{(X,D)}_{[C,\phi]} = H^1(C, \oplus_{i=1}^{l} \phi^* \mathcal{O}_X(-D_i))$
  - $[\overline{\mathcal{M}_{0,n}}(E_{X,D}, d)]^{\mathrm{vir}} = [\overline{\mathcal{M}_{0,n}}(X,d)] \cap \mathrm{c_{top}}(\mathrm{Obs}^{(X,D)})$
  - expdim = $l + n - 1$

$$N^{\mathrm{loc}}_{(X,D),d} := \int_{[\overline{\mathcal{M}_{0,l-1}}(E_{X,D},d)]^{\mathrm{vir}}} \prod_{i=1}^{l-1} \mathrm{ev}_i^*[\mathrm{pt}_X]$$

Log ($g = 0$) Gromov–Witten theory of (X, D)

$N^{\log}_{(X,D),d} =$ "# degree-d, $g = 0$ curves in X through $l - 1$ points with maximal tangency at $\{D_i\}_i$"



Log ($g = 0$) Gromov–Witten theory of (X, D)

- $N^{\log}_{(X,D),d}$ = "# degree-d, $g = 0$ curves in X through $l - 1$ points with maximal tangency at $\{D_i\}_i$"
- view X as log-scheme with D-log structure
- moduli of log-stable maps $\overline{\mathcal{M}}^{\log}_{0,n}((X, D), d)$
  - proper, expdim = $l - 1 + n$
  - $[\overline{\mathcal{M}}^{\log}_{0,n}((X, D); d)]^{\mathrm{vir}} \in A_{\mathrm{expdim}}(\overline{\mathcal{M}}^{\log}_{0,n}((X, D); d), \mathbb{Q})$

[Chen, Abramovich–Chen, Gross–Siebert]

$$N^{\log}_{(X,D),d} := \int_{[\overline{\mathcal{M}}^{\log}_{0,l-1}((X,D);d)]^{\mathrm{vir}}} \prod_{i=1}^{l-1} \mathrm{ev}_i^* [\mathrm{pt}_X]$$

Example II: $(\mathbb{P}^2, L + C)$

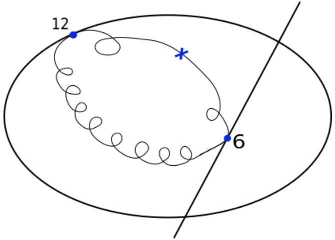


Figure: A degree 6 rational curve in $\mathbb{P}^2$ passing through 1 point and maximally tangent to $\mathrm{line} + \mathrm{conic}$.

d	$N_d^{\log}$	$N_d^{\log}/N_d^{\mathrm{loc}}$	N_d
1	2	-2	1
2	6	8	1
3	20	-18	12
4	70	32	620
5	252	-50	87304
6	924	72	26312976
$\vdots$	$\vdots$	$\vdots$	

$$N_d^{\log} \sim 4^d \text{ (in fact } = \binom{2d}{d})$$
$$N_d^{\log}/N_d^{\mathrm{loc}} = (-1)^d 2d^2$$

Example III: $(\mathbb{P}^2, L_1 + L_2 + L_3)$

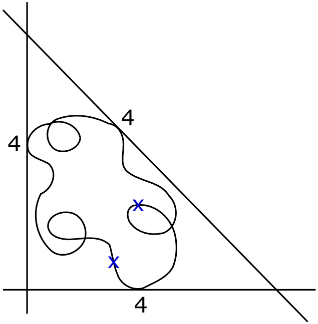


Figure: A degree 4 rational curve in $\mathbb{P}^2$ passing through 2 points and maximally tangent to three lines.

d	$N_d^{\log}$	$N_d^{\log}/N_d^{\mathrm{loc}}$	N_d
1	1	1	1
2	4	-8	1
3	9	27	12
4	16	-64	620
5	25	125	87304
6	36	-216	26312976
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$N_d^{\log} \sim d^2 \text{ (in fact } = d^2)$$
$$N_d^{\log}/N_d^{\mathrm{loc}} = (-1)^{d+1} d^3$$

The log-local correspondence

Conjecture (The log-local correspondence, vGGR '17)

For a smooth nef pair (X, D),

$$N^{\log}_{(X,D),d} = \prod_{i=1}^{l} (-1)^{d \cdot D_i + 1} (d \cdot D_i) N^{\mathrm{loc}}_{(X,D),d}$$

Evidence:

- $l = 1$

  [van Garrel–Graber–Ruddat]

- toric pairs

  [Bousseau–B–van Garrel]

The log-local correspondence for log-CY surfaces

Theorem (Bousseau–B–van Garrel ’20)

The descendent log & local GW $g = 0$ GW theory of nef Looijenga pairs is closed-form solvable.
In particular, the log-local correspondence holds.

The log-local correspondence for log-CY surfaces

- The local side:
  - main idea: GW theory of $E_{X,D}$ reconstructed from that of X
    [Coates–Givental]
  - degeneration+toric mirror symmetry+big QH reconstruction
    [Givental, Coates–Corti–Iritani–Tseng]
  - tameness $\leftrightarrow$ trivial mirror map
- The log side:
  - comparison theorem for $l = 2$
    [Abramovich–Chen–Gross–Siebert]
  - explicit solution for tame via (finite) scattering diagrams
    [Gross–Hacking–Keel, Gross–Pandharipande–Siebert, Mandel, Keel–Yu]

Higher genus logarithmic invariants

$$N^{\log}_{(X,D),d} = \int_{[\overline{\mathcal{M}}^{\log}_{0,l-1}(X,D,d)]} \prod_{i=1}^{l-1} \mathrm{ev}_i^* [\mathrm{pt}]$$

$$\rightsquigarrow$$

$$\begin{aligned} N^{\log}_{(X,D),g,d} &= \int_{[\overline{\mathcal{M}}^{\log}_{g,l-1}(X,D,d)]} \prod_{i=1}^{l-1} \mathrm{ev}_i^* [\mathrm{pt}] (-1)^g \lambda_g \\ &= [\log^{2g} q] \widetilde{\mathrm{N}}^{\log}_{(X,D),d}(q) \in \mathbb{Q}(q^{1/2}) \end{aligned}$$

Scattering calculation of $\mathrm{N}^{\log}_{(X,D),d}$

$\rightsquigarrow$

(q-deformed) scattering calculation of $\widetilde{\mathrm{N}}^{\log}_{(X,D),d}(q)$

[Bousseau]

The $g > 0$ log-(local)-open correspondence

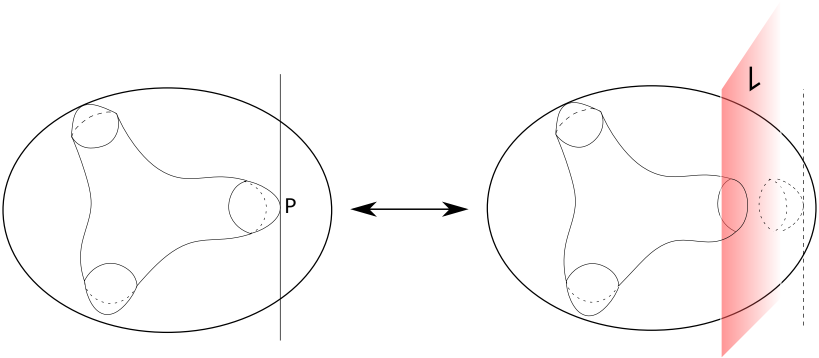
Higher genus local invariants?! (expdim< 0 for $g > 1$)

Proposal: GW invariants of local CYn-folds

$$=$$

$(n-3)$-holed open GW invariants of sLags in local CY3-folds

Symplectic heuristics



Max tangency $d \cdot D_j$ with D_j $\longleftrightarrow$ Winding $d \cdot D_j$ around L near D_j, multiply by $(-1)^{d \cdot D_j + 1} d \cdot D_j$

Physics heuristics: QFT engineering

1. GW potential of local CY3 $\to$ Nekrasov instanton partition function on $\mathbb{R}^4 \times S^1$

   [Katz–Klemm–Vafa, Lawrence–Nekrasov, Goettsche–Nakajima–Yoshioka]

2. GW potential of local CY4 $\to$ superpotential terms in LEET on $\mathbb{R}^2 \times S^1 \ldots$

   [Greene–Morrison–Plesser, Gukov–Vafa–Witten, Mayr]

3. $\cdots \leftarrow$ disk GW potential of local CY3

   [Ooguri–Vafa, Mayr, Aganagic–Beem]

The $g > 0$ log-(local)-open correspondence

The drill: starting from Looijenga (X, D),

1. replace max tgcy on D_l with twist by $\mathcal{O}_X(-D_l)$;
2. replace max tgcy on D_i, $i < l$ by open condition on sLags $L_i \subset Y := \mathcal{O}_{X \setminus \cup_i D_i}(-D_l)$.

The expectation:

1. $\exists$ sensible definition of open GW invariants of (Y, L)
2. $g = 0$: $N^{\rm open}_{(Y,L);d} = N^{\rm loc}_{(X,D),d}$
3. $g > 0$: $\widetilde{\rm N}^{\rm open}_{(Y,L),d}(q) \leftrightarrow \widetilde{\rm N}^{\rm log}_{(X,D),d}(q)$, refining log-local.

The $g > 0$ log-(local)-open correspondence

More precisely:

Conjecture ($g = 0$ log-local-open correspondence)

$$N^{\mathrm{loc}}_{(X,D),d} = N^{\mathrm{open}}_{(Y,L),d} = \left(\prod_{i \leq l} \frac{(-1)^{d \cdot D_i + 1}}{d \cdot D_i} \right) N^{\mathrm{log}}_{(X,D),d}.$$

Conjecture (All-genus log-open correspondence)

$$\widetilde{\mathrm{N}}^{\mathrm{open}}_{(Y,L),d}(q) = \left(\prod_{i < l} \frac{(-1)^{d \cdot D_i + 1}}{d \cdot D_i} \right) \frac{(-1)^{d \cdot D_l + 1} [1]_q^{l-2}}{[d \cdot D_l]_q} \widetilde{\mathrm{N}}^{\mathrm{log}}_{(X,D),d}(q)$$

where $[n]_q = q^{n/2} - q^{-n/2}$.

The $g > 0$ log-(local)-open correspondence

$(X, D = D_1 \cup \cdots \cup D_l)$ **tame** Looijenga pair

$$\rightsquigarrow$$

$(Y, L = L_1 \cup \ldots \cup L_{l-1})$ semi-projective Aganagic–Vafa pair

Aganagic–Vafa (toric) A-branes

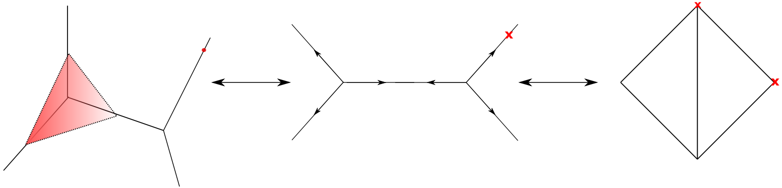
Harvey–Lawson fibration:

$$\begin{aligned} \mu : \mathbb{C}^3 &\longrightarrow \mathbb{R}^3 \\ (z_1, z_2, z_3) &\longrightarrow \left(|z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2, \mathrm{Im}(z_1 z_2 z_3)\right) \end{aligned}$$

1. generic fibre $\simeq \mathbb{T}^2 \times \mathbb{R}$
2. special fibres ($z_i = z_j = 0$, $i \neq j$) $\simeq \mathbb{R}^2 \times S^1$ (Aganagic–Vafa).

For Y semi-projective toric CY3:

Critical value set $\longleftrightarrow$ planar lattice graph $\longleftrightarrow$ height-1 slice of the fan



Open Gromov–Witten theory

1. Several approaches to defining open invariants of Aganagic–Vafa A-branes in class $(\beta, \vec{\nu}) \in H_2(Y;\mathbb{Z}) \oplus H_1(L;\mathbb{Z}) \simeq H_2(Y, L;\mathbb{Z})$

[Katz–Liu, Li–Song, Li–Liu–Liu–Zhou]

2. Upshot: $\overline{\mathcal{M}}_{g,|\vec{\nu}|}(Y, L; \beta, \vec{\nu})$, expdim $= 0\ \forall\ g$.

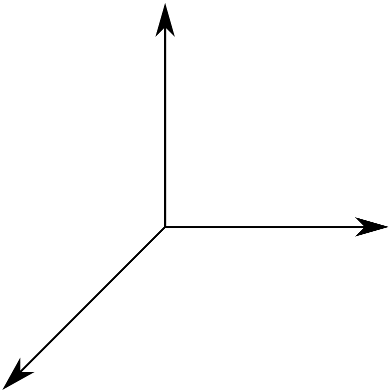
$$\begin{aligned} N^{\rm open}_{(Y,L);g,\beta,\vec{\nu}} &= \int_{[\overline{\mathcal{M}}_{g,|\vec{\nu}|}(Y,L;\beta,\vec{\nu})]^{\rm vir}} 1 \\ &= [\log^{2g-2+|\nu|} q]\widetilde{\mathrm{N}}^{\rm open}_{(Y,L);g,\beta,\vec{\nu}}(q) \end{aligned}$$

The $g > 0$ log-(local)-open correspondence

$$(X, D = D_1 \cup \cdots \cup D_l) \text{ tame Looijenga pair}$$
$$\rightsquigarrow$$
$$(Y, L = L_1 \cup \ldots \cup L_{l-1}) \text{ semi-projective Aganagic–Vafa pair}$$

- $Y := \mathrm{Tot}\big(\mathcal{O}_{X \setminus \cup_{i=1}^{l-1} D_i}(-D_l)\big)$
- L_i incident to torus 1-orbit intersecting D_i
- $d \in H_2(X) \leftrightarrow (\beta(d), \vec{\nu}(d)) \in H_2(Y, L)$ canonically

Example: $(X, D) = (\mathbb{P}^2, L + C)$



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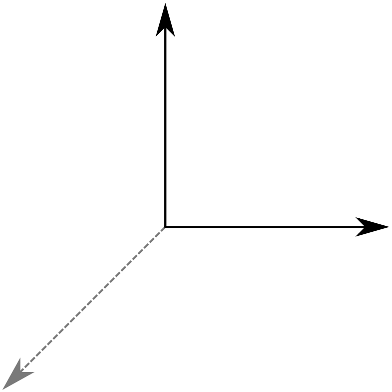

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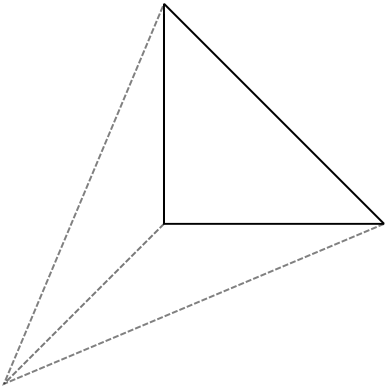

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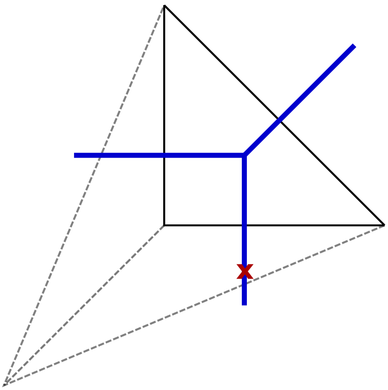
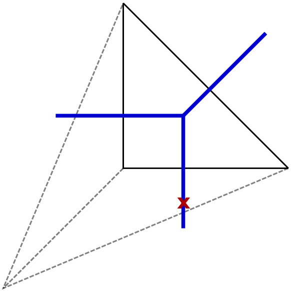


The $g > 0$ log-(local)-open correspondence

Theorem (BBvG '20)

The higher genus log-GW theory of tame pairs (X, D) and open GW theory of the associated (Y, L) are closed-form solvable. In particular, the higher genus log-open correspondence holds for tame (X, D).

Example: ($X = \mathbb{P}^2$, $D = L + C$), the open CY3 side



One-legged topological vertex:

$$\widetilde{\mathrm{N}}^{\mathrm{open}}_{(Y,L),d}(q) = \frac{1}{d}\sum_{R\vdash d}\chi_R((d))q^{\kappa(R)/2}(-1)^{|R|}s_R(q^\rho)$$

$$= \frac{(-1)^d}{d[d]_q}\sum_{s=0}^{d-1}(-1)^s q^{\frac{3}{2}\binom{d}{2}}\begin{bmatrix} d-1 \\ s \end{bmatrix}_q (-q^d)^s q^{-ds/2} = \frac{(-1)^d}{d[2d]_q}\begin{bmatrix} 2d \\ d \end{bmatrix}_q$$

Implications for log GW theory

To a tame Looijenga pair, we can assign:

1. large N dual Chern–Simons theory interpretation
   [Gopakumar–Vafa, Ooguri–Vafa]
2. all-genus calculation scheme (localisation, topological vertex)
   [Graber–Zaslow, Diaconescu–Florea–Grassi, Aganagic–Klemm–Mariño–Vafa, Li–Liu–Liu–Zhou]
3. random matrix/crystal/free fermion models
   [Mariño, Okounkov–Reshetikhin–Vafa, Saulina–Vafa]
4. classical integrable hierarchy (2-Toda reduction)
   [AB; AB–Carlet–Rossi–Romano]
5. higher genus mirror reconstruction theorem (remodelled B-model)
   [Bouchard–Klemm–Mariño–Pasquetti; Eynard–Orantin; Fang–Liu–Zong]
6. integral structure via open BPS counts
   [Ooguri–Vafa, Labastida–Mariño–Vafa]
7. symmetric quiver DT reformulation
   [Kucharski–Reineke–Stošić–Sułkowski, Panfil–Sułkowski]
8. gauge theory interpretation
   [Kozçaz–Pasquetti–Wyllard; Dimofte–Gukov–Hollands]

Implications for log GW theory

Example: $X = \mathbb{P}^2$, $D_1 = H$, $D_2 = 2H$.

coloured extremal HOMFLY of the unknot
SW/GKM resolvent

$\widetilde{\mathrm{N}}^{\log}_{(X,D),d}(q)$: discrete KdV/Volterra τ-function

$\omega_{g,1}[\mathcal{S}]$ in TR, $\mathcal{S} = \{1 + \mathrm{e}^{x-y} + \mathrm{e}^{y} = 0\}$

DT(2-loop quiver)

Implications for log GW theory

To a tame Looijenga pair, we can assign:

1. large N dual Chern–Simons theory interpretation
2. all-genus calculation scheme (localisation, topological vertex)
3. random matrix/crystal/free fermion models
4. classical integrable hierarchy (2-Toda reduction)
5. higher genus mirror reconstruction theorem (remodelled B-model);
6. integral structure via open BPS counts
7. symmetric quiver DT reformulation
8. gauge theory interpretation (4d surface operator/vortex partition function)

Today: 2) + 6) + 7)

Application I: log/local GW $\leftrightarrow$ quiver DT

Gopakumar–Vafa invariants of CY($l+2$)-folds:

$$\begin{aligned}\Omega_d(X,D) &= \sum_{k|d} \frac{\mu(k)}{k^{3-(l-1)}} N^{\mathrm{loc}}_{(X,D),d/k} \\ &= \frac{1}{\prod_{i=1}^{l} d\cdot D_i} \sum_{k|d} \frac{\mu(k)(-1)^{d\cdot D_i/k+1}}{k^2} N^{\mathrm{log}}_{(X,D),d/k}\end{aligned}$$

[Mayr, Klemm–Pandharipande, Ionel–Parker]

Conjecture (Klemm–Pandharipande)

$$\Omega_d(X,D) \in \mathbb{Z}$$

(Ionel–Parker: symplectic proof for compact CYn-folds)

Application I: log/local GW $\leftrightarrow$ quiver DT

$l-1$-holed open Gopakumar–Vafa invariants of CY(3)-folds:

$$\Omega_d^{\mathrm{open}}(Y,L) = \sum_{k|d} \frac{\mu(k)}{k^{3-(l-1)}} N^{\mathrm{open}}_{(Y,L),d/k}$$

[Ooguri–Vafa, Labastida–Mariño–Vafa]

Theorem (BBvG '20, after Panfil–Sułkowski '17)

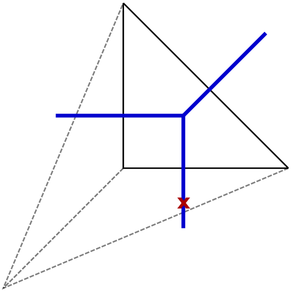
Let (Y,L) be the AV pair corresponding to a tame Looijenga pair (X,D). $\exists$ symmetric quiver Q such that

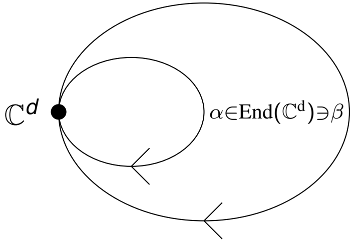
$$|\Omega_d(X,D)| = |\Omega_d^{\mathrm{open}}(Y,L)| = \mathrm{DT}_d(Q)$$

Corollary (BBvG '20, after Efimov '11)

$$\Omega_d(X,D) \in \mathbb{Z}$$

Example: $(X, D) = (\mathbb{P}^2, L + C)$





Application II: log/local GW & DT/PT

- Cao–Leung, Borisov–Joyce, Oh–Thomas: invariants $\mathrm{DT}_d^{(4)}$ from moduli of stable sheaves on CY4-folds
- Cao–Maulik–Toda: conjecturally $\mathrm{DT}_d^{(4)} = \Omega_d$
- for $\mathcal{O}(-D_1) \oplus \mathcal{O}(-D_2) \to X$: checks by Cao–Kool–Monavari, based on BBvG

Application III: higher genus log GW & higher genus LMOV

$$\widetilde{\Omega}_{(X,D),d}(q) := \sum_{k|d} \mu(k)(-1)^{\sum_i d\cdot D_i/k+1} \frac{[1]_q^2}{[k]_q^2} \prod_{i=1}^{l} \frac{[k]_q}{[d\cdot D_i]_q} \widetilde{\mathrm{N}}^{\mathrm{log}}_{(X,D),d/k}(q^k)$$

$$= [1]_q^2 \left(\prod_{i=1}^{l-1} \frac{d\cdot D_i}{[d\cdot D_i]_q} \right) \sum_{k|d} \mu(k) \frac{1}{k} \widetilde{\mathrm{N}}^{\mathrm{open}}_{(Y,L),\beta(d)/k,\vec{\mu}(d)/k}(q^k)$$

Theorem (The higher genus open BPS property, BBvG)

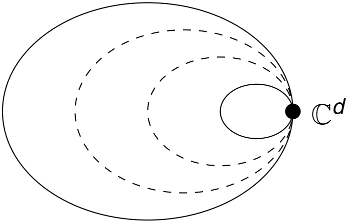
$$\widetilde{\Omega}_{(X,D),d}(q) \in \mathbb{Z}[q, q^{-1}]$$

Strategy: direct arithmetic proof from the log/open calculation.

Orbifolds

Whole story generalises to orbifolds $\Rightarrow$ infinite list.

Example: $(X = \mathbb{P}(1,1,n), D_1 = L, D_2 = -K_X - L)$
$\rightsquigarrow Q = (n+1)$-loop quiver



Conclusion

For nef/tame Looijenga pairs:

1. log GW are local GW are open GW are quiver DT are KP/IP/LMOV invariants

&

2. invariants are closed-form computed.

Food for thought

Some open questions (in very random order):

- $\dim X > 2$?
- D_i non nef? Non-tame cases?
- deduce higher genus log-(local)-open principle
  - geometrically?
  - algebraically?
- origin/meaning of quiver?