Invariance of plurigenera and KSBA moduli in positive and mixed characteristic

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February 9, 2023

Notation

- R is an excellent DVR with residue field $k = k^{\text{perf}}$ of characteristic $p > 0$, and fraction field K.
- A *pair* (X, B) consists of a reduced, pure dimensional, $G1$, excellent scheme X over a field or DVR, and a $\mathbb{Q}$-divisor $B = \sum a_i B_i$, where B_i are distinct prime divisors none of which is contained in $\text{Sing}(X)$, and $K_X + B$ is $\mathbb{Q}$-Cartier (most of the time our pairs will be normal and integral). **Note**: we are not requiring X to be $S2$.
- A scheme is *demi-normal* if it is $S2$ and at worst nodal in codimension one.
- A *family of pairs* consists of a pair (X, B) and a flat morphism $X \to T$, where T is a regular one-dimensional scheme, such that (X_t, B_t) is a pair, for all $t \in T$.
- If X is an R-scheme, X_k, X_K will denote the closed and generic fiber, respectively. Same for subschemes, coherent sheaves,....
- Confuse notation between line bundles and Cartier divisors.

Siu's Theorem and its applications

Definition

Let (X, B) be a proper pair over a field $\mathbb{K}$, and let m be a positive integer such that mB is integral. The m-plurigenus of (X, B) is $h^0(X, m(K_X + B)) := \dim_{\mathbb{K}} H^0(X, m(K_X + B))$.

Theorem (Siu '00, Berndtsson-Paun '12, Hacon-McKernan '14)

Let $\pi\colon (X, B) \to T$ be a projective family of normal integral complex pairs. Assume that

- *π is log smooth and (X_t, B_t) is klt for all $t \in T$; or*
- *π is log smooth and (X_t, B_t) is lc and of general type for all $t \in T$; or*
- *(X_t, B_t) has canonical singularities for all $t \in T$.*

Then $h^0(X_t, m(K_{X_t} + B_t))$ is independent of $t \in T$ for all $m \geq 0$ such that mB is integral.

Remarks:

- Equivalently, for all such $m \geq 0$ the restriction map
$$H^0(X, m(K_X + B)) \to H^0(X_t, m(K_{X_t} + B_t))$$
is surjective
- Heavily analytic proof (Ohsawa-Takegoshi's L^2-extension theorem).
- Application to moduli spaces for varieties of general type.

KSBA moduli

- "Higher-dimensional version of moduli of weighted stable curves."
- Idea: moduli for integral lc pairs of general type (X, B), s.t. $\dim(X) = n$ and $\mathrm{vol}(K_X + B) = v$.These have very poorly behaved moduli spaces (non-separated).
- Solution: to such (X, B) one can associate its *log canonical model*

$$\phi\colon (X, B) \dashrightarrow (X^c := \mathrm{Proj} R(K_X + B), B^c := \phi_* B),$$

  where $R(K_X + B) := \bigoplus_{m \geq 0} H^0(X, \lfloor m(K_X + B) \rfloor)$ is the *canonical ring* of (X, B). This is still an lc pair and $K_{X^c} + B^c$ is now ample.
- Objects: (X, B) log canonical model of dimension n and volume v.
- Families: families of log canonical models $(X, B) \to T$ of volume v and dimension n.
- The corresponding moduli functor $\mathcal{S}_{n,v}$ is separated but not proper$\Longrightarrow$ stable pairs.

KSBA moduli

Definition

A pair over a field of characteristic zero (X, B) is *slc* if

- X is demi-normal; and
- letting $\pi\colon \overline{X} \to X$ be the normalization, $\overline{D} \subset \overline{X}$ the double locus, and $\overline{B} := \pi^{-1}(B)$, then $(\overline{X}, \overline{B} + \overline{D})$ is lc.

If (X, B) is slc, projective, and $K_X + B$ is ample, we call it a *stable pair*. A *stable family* is a pair (X, B) with a flat proper morphism $\pi\colon X \to T$ such that

(a') (X_t, B_t) is slc for all $t \in T$; or equivalently

(a") $(X, B + X_t)$ is slc for all $t \in T$; and

(b) $K_X + B$ is π-ample.

KSBA moduli

Theorem (Kollár, Hacon-Xu, Hacon-McKernan-Xu,...)

Over the complex numbers, the functor $\overline{\mathcal{S}}_{n,v}$ of stable pairs is representable, separated, proper, bounded, and it admits a projective coarse moduli space.

Remark: Siu's theorem $\implies$ functoriality of log canonical models.

Let $(X, B) \to T$ be a log smooth family of lc pairs of general type. Consider the relative canonical model over T

$$\phi\colon (X, B) \dashrightarrow (X^c := \mathrm{Proj}_T R(K_X + B/T), B^c := \phi_* B)/T.$$

Then we have

$$(X^c, B^c) \times_T \{t\} = ((X_t)^c, (B_t)^c)$$

for all $t \in T$. In particular, all the fibers of the relative canonical model are (s)lc, hence $S2$.

Positive and mixed characteristic results

Well known: $\exists$ smooth projective families of surfaces $X \to \operatorname{Spec} R$ such that $h^0(X_K, K_{X_K}) < h^0(X_k, K_{X_k})$ (Lang '83, Katsura-Ueno '85, Suh '08).

Question (A.I.P.)

Let $(X, B) \to \operatorname{Spec} R$ be a "nice" projective family of integral normal pairs. Does $h^0(X_K, m(K_{X_K} + B_K)) = h^0(X_k, m(K_{X_k} + B_k))$ hold for all $m \geq 0$ sufficiently divisible?

No: in any characteristic $p > 0$ there are examples of

- $(X, B) \to \operatorname{Spec} R$ projective families of minimal surface pairs of Kodaira dimension one such that A.I.P. fails.
- $(Y, D) \to \operatorname{Spec} R$ log smooth projective families of plt 3-fold pairs of general type such that A.I.P. fails.

In both cases the log canonical divisor is semiample.

Positive and mixed characteristic results

Lemma

Let $X \to \operatorname{Spec} R$ be a contraction with integral normal fibers, let L be a semiample line bundle on X and let $f : X \to Y/R$ be the semiample contraction. TFAE:

1. $h^0(X_k, L_k^m) = h^0(X_K, L_K^m)$ *for all* $m \geq 0$ *divisible enough;*
2. $f_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{Y_k}$*;*
3. *(if L is big) Y_k is normal.*

Proof sketch.

1 holds $\Rightarrow$ f_t SA cont of L_t $\forall t \in \operatorname{Spec} R$. In particular $f_{t,*}\mathcal{O}_{X_t} = \mathcal{O}_{Y_t}$ & Y_t is normal.

$(2 \Rightarrow 1)$ $L \sim_{\mathbb{Q}} f^*A$ A ample $\mathbb{Q}$-div & $h^0(X_t, L_t^m) = h^0(Y_t, f_{t,*}\mathcal{O}_{X_t} \otimes \mathcal{O}_{Y_t}(mA_t))$ (proj formula)

$= h^0(Y_t, mA_t)$ (by 2)

$= \chi(Y_t, mA_t)$ Serre Van. ($m \gg 0$ div enough)

↑ invariant in flat families.

$(3 \Rightarrow 2)$ $f_k : X_k \xrightarrow{\bar{f}_k} \bar{Y}_k \xrightarrow{u_k} Y_k$ Stein fact

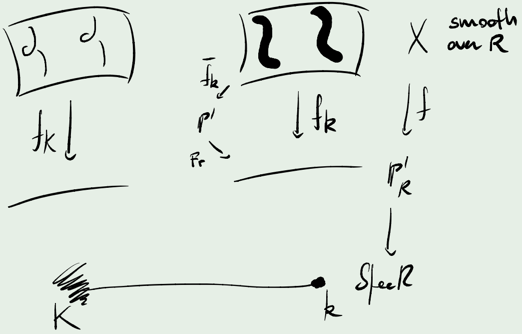
↑ contr. ↑ birational universal homeomorphism

$\bar{Y}_k$ always normal, then Zariski's Main Thm $\Rightarrow$ u_k is an isomorphism

□

Example (B-,'20)

E/R ell. curve, M nontrivial p-torsion line bundle such that $M_k = \mathcal{O}_{E_k}$.
$\mathbb{P}^1_R$ with $N = \mathcal{O}_{\mathbb{P}^1_R}(1)$, and homogeneous coordinates $[S : T]$.
$Z := E \times_R \mathbb{P}^1_R$, $L := M \boxtimes N$, $\sigma = 1_M \boxtimes S^{p-1}T \in H^0(Z, L^p)$.
$X := (Z[\sigma^{1/p}])^\nu \to Z$ normalized p-cover.
The induced morphism $f \colon X \to \mathbb{P}^1_R$ looks as follows:

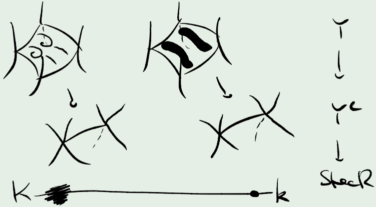


$\kappa(X_t) = -\infty \ \forall t \in \operatorname{Spec} R$
Set $B := f^*(\sum a_i P_i)$, $a_i > 0$ small enough
$\Rightarrow$ get $(X, B) \to \operatorname{Spec} R$ family of terminal surface pairs: $K_X + B$ is semiample of Kodaira dim 1 & f is its sa-contraction.

Pathologies in positive and mixed characteristic

Example (Kollár '22)

$X \to \mathrm{Spec} R$ the family from the previous example, $\Lambda \geq 0$ on X such that (X_t, Λ_t) is CY and terminal for all $t \in \mathrm{Spec} R$.
$L := K_X + B$ with notation as before.
$Y := \mathbb{P}(\mathcal{O}_X + A) \xrightarrow{\tau} X$, with mobile and fixed sections X_∞ and X_0.
Let $\Lambda_Y := \tau^*\Lambda$, $L_Y := \tau^* L$, and let $X'_\infty \in |2X_\infty|_{\mathbb{Q}}, L'_Y \in |L_Y|_{\mathbb{Q}}$ be general divisors.
Set $D := X_0 + X'_\infty + \Lambda_Y + L'_Y$, so that $(Y, D) \to \mathrm{Spec} R$ is a log smooth family of plt 3-fold pairs, and $K_Y + D \sim_{\mathbb{Q}} X_\infty + L_Y$ is semiample, with Iitaka fibration as follows:



Computation using adjunction
$\Rightarrow (Y, D) \to \mathrm{Spec} R$ doesn't satisfy A.I.P.
$\Rightarrow (Y^c)_k$ is not normal.

Pathologies in positive and mixed characteristic

Remark: The pair $(Y^c, D^c + (Y^c)_k)$ is (s)lc, however $(Y^c)_k$ is not $S2$ (in particular, $((Y^c)_k, (B^c)_k)$ is not slc). The equivalence $(a') \Leftrightarrow (a'')$ no longer holds!

Fact: it can be shown that stable families in the sense of (a'') still form a separated functor.

Consequence: in positive and mixed characteristic the moduli functor of stable pairs $\overline{\mathcal{S}}_{n\geq 3,v}$ is no longer proper.

Pathologies in positive and mixed characteristic

On the positive side we have (assuming resolution of singularities):

Theorem (B-,'21)

Let $(X, B) \to \mathrm{Spec} R$ be a projective family of normal integral klt 3-fold pairs. Assume $p > 5$ and

- $K_{X_k} + B_k$ *is nef; or*
- X *is $\mathbb{Q}$-factorial, and every non-canonical center V of $(X, B + X_k)$ such that $V \subset \mathbf{B}_-(K_X + B/R)$ is horizontal over R.* (∗)

Then A.I.P. holds.

Proof sketch.

$\rho : (X, B) \dashrightarrow (\bar{X}, \bar{B})$ K_X+B-MMP/R

(∗) ⇒ ρ_t is a $K_{X_t}+B_t$-MMP $\forall t \in \mathrm{Spec} R$

Cascini-Tanaka or Witaszek } ⇒ $K_{\bar{X}}+\bar{B}$ semiample /R

MMP preserves klt singularities ⇒ wlog K_X+B is semiample

$f : X \to T$ sa-contr of K_X+B

$0 \to \mathcal{O}_X(-X_k) \to \mathcal{O}_X \to \mathcal{O}_{X_k} \to 0$ Apply f_*

⇒ ETS $R^1 f_* \mathcal{O}_X = 0$ ⇒ ETS $R^1 f_{k,*} \mathcal{O}_{X_k} = 0$ by upper-semicontinuity. Follows by rationality of klt 3fold sings in char p > 5 (Arvidsson, Bernasconi, Lacini) □

What next?

Maximalistic approach: enlarge the category of stable pairs to allow for more general limits.

Definition

We say X is **quasi-demi-normal** if it is reduced, $S1$, at most nodal in codimension one, and the demi-normalization morphism $\tilde{X} \to X$ is an universal homeomorphism.

Question 1

Define **quasi-stable pairs** by replacing demi-normal with quasi-demi-normal. Is the functor $\overline{\mathcal{QS}}_{n,v}$ of quasi-stable pairs proper?

Takes care of Kollár's example, however quasi-stable-pairs of fixed volume are not bounded.

What next?

Example (Unboundedness)

Consider

$$\varphi_e \colon S := E \times \mathbb{P}^1 \xrightarrow{\mathrm{pr}_2} \mathbb{P}^1 \xrightarrow{F^e} \mathbb{P}^1,$$

and let $L_e := \varphi_e^* \mathcal{O}(1)$, so that φ_e is induced by a base-point-free linear system $V_e \subset H^0(S, L_e)$.
Let $Z := \mathbb{P}(\mathcal{O}_S + A) \xrightarrow{\tau} S$ be a $\mathbb{P}^1$ bundle with mobile and fixed sections S_∞ and S_0 as before.
Let $m \geq 1$ be sufficiently divisible and let

$$\Phi_e \colon Z \to Z_e$$

be the morphism induced by $\tau^* V_e \otimes H^0(Z, mS_\infty)$.
Then $\{Z_e\}_{e \in \mathbb{N}}$ does not form a bounded family.

What next?

Minimalistic approach: restrict the stable pairs we consider, so that non-$S2$ schemes are not allowed.

Definition

Let $\overline{\mathcal{S}}^{klt}_{3,v} \subset \overline{\mathcal{QS}}_{3,v}$ be the subfunctor of three-dimensional quasi-stable pairs (X, B) which are a limit of stable klt pairs.

Question 2

Let $(X, B) \in \overline{\mathcal{QS}}_{3,v}$ and assume $p > 5$. Is X an $S2$ scheme?

Thank you for your attention