From curve counting on Calabi-Yau 4-folds to quasimaps for quivers with potentials

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X: sm proj CY_4 $(K_X \simeq \mathcal{O}_X)$

$\overline{M}_{g,n}(X,\beta)$: moduli stack of genus g, n-pointed stable maps $f: C \to X$
w/ $f_*[C] = \beta \in H_2(X,\mathbb{Z})$

$\downarrow$

$v.d_{\mathbb{C}} = 1-g+n$

Gromov-Witten invs :

$$GW_{0,\beta}(\gamma) = \int_{[\overline{M}_{0,1}(X,\beta)]^{vir}} ev^*\gamma \in \mathbb{Q}.$$

$ev: \overline{M}_{0,1}(X,\beta) \to X$

$\gamma \in H^4(X)$

$$GW_{1,\beta} = \int_{[\overline{M}_{1,0}(X,\beta)]^{vir}} 1 \in \mathbb{Q}$$

Klemm–Pandharipande: Define $n_{0,\beta}(\gamma)$, $n_{1,\beta}$ by GW invs:
2007

$$GW_{0,\beta}(\gamma) = \sum_{k|\beta} \frac{1}{k^2} n_{0,\beta/k}$$

Gopakumar–Vafa type invs

$$\sum_{\beta} GW_{1,\beta}\, q^{\beta} = \sum_{\beta} n_{1,\beta} \cdot \sum_{d\geq 1} \frac{\sigma(d)}{d} q^{d\beta} + \frac{1}{24} \sum_{\beta} n_{0,\beta}(c_2(X)) \log(1-q^{\beta})$$
$$- \frac{1}{24} \sum_{\beta_1,\beta_2} m_{\beta_1,\beta_2} \log(1-q^{\beta_1+\beta_2})$$

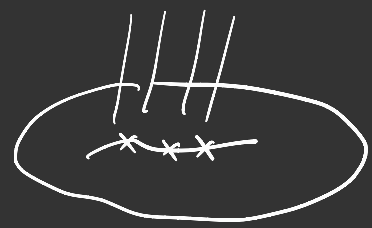
"meeting invs"
can be inductively
obtained by $g=0$ GW

Conj (KP): $n_{0,\beta}(\gamma)$, $n_{1,\beta} \in \mathbb{Z}$

proved by Ionel–Parker
using symp geo

many checks by examples

Sheaf theoretic approach:



$$P_n(X,\beta) = \left\{ (\mathcal{O}_X \xrightarrow{s} F) \in D^b(X) \;\middle|\; \begin{array}{ll} F: \text{pure 1-dim} & [F]=\beta \\ \text{coker: 0-dim} & \chi(F)=n \end{array} \right\}$$

Pandharipande-Thomas (PT) stable pairs

$$[P_n(X,\beta)]^{vir} \in H_{2n}(P_n(X,\beta), \mathbb{Z}) \qquad \text{DT4 virtual class}$$

$$P_{n,\beta}(\gamma) := \int_{[P_n(X,\beta)]^{vir}} \tau(\gamma)^n \in \mathbb{Z}, \qquad \begin{array}{l} \tau: H^4(X,\mathbb{Z}) \to H^2(P,\mathbb{Z}) \\ \quad \gamma \mapsto \pi_{P*}(\pi_X^*\gamma \cup ch_3(\mathbb{F})) \end{array}$$

<u>Conj</u> (CMT, 2019)

$$\sum_{n,\beta} \frac{P_{n,\beta}(\gamma)}{n!} q^n t^\beta = \prod_\beta \exp(q t^\beta)^{n_{0,\beta}(\gamma)} \cdot M(t^\beta)^{n_{1,\beta}}.$$

$$M(t) := \prod_{n\geq 1} \frac{1}{(1-t^n)^n}$$ is MacMahon function

Rk: 1. Integrality of $n_{g,\beta}(\gamma)$ follows from integrality of $P_{n,\beta}(\gamma)$

2. Conj is proved in "unobstructed case" curves deform in families of expected dim

explain what are

GV invs

in this picture

How about Non-compact CY_4, e.g. $\bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}(\delta_i)$, $\sum_{i=1}^{3} \delta_i = -2$

Problems: ① GW invs are rational functions (defined by torus localization)

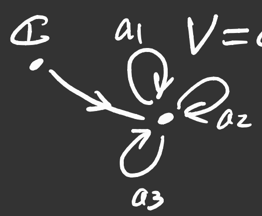
can have all higher genus invs

② Not clear how to define GV invs.

③ What stable pair invs could lead to.

Quiver w/ potential: (will work w/ this example)

w/ G. Zhao



$V = \mathbb{C}^n$

$$W = \mathrm{Hom}(V,V)^{\times 3} \times V \xrightarrow{\phi} \mathbb{C}$$

$$(a_1, a_2, a_3, v) \longmapsto \mathrm{tr}\, a_1[a_2, a_3]$$

$$G = GL(V) \curvearrowright W \quad \text{w/} \quad \theta : G \to \mathbb{C}^*$$
$$g \mapsto \det(g)$$

$$F = (\mathbb{C}^*)^3 \ \text{scale } a_1, a_2, a_3 \quad \text{w/} \quad \chi : F \to \mathbb{C}^*$$
$$(t_1, t_2, t_3) \mapsto t_1 \cdot t_2 \cdot t_3$$

$F_0 := \ker \chi$ CY torus

$$\mathrm{crit}\,\phi /\!\!/ G \cong \mathrm{Hilb}^n(\mathbb{C}^3) \hookrightarrow W /\!\!/ G$$

$$\begin{tikzcd}
QM \arrow[r, phantom, "\overset{\text{open}}{\subset}"] & \mathrm{Map}^{\delta}(\mathbb{P}^1, [\mathrm{crit}\,\phi / G]) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \{\mathcal{O}_{\mathbb{P}^1}(\delta_1) \oplus \mathcal{O}_{\mathbb{P}^1}(\delta_2) \oplus \mathcal{O}_{\mathbb{P}^1}(\delta_3)\} \arrow[r] \arrow[d, phantom, "\cap"] & \mathcal{O}_{\mathbb{P}^1}\left(\sum_{i=1}^{3} \delta_i\right) \arrow[d, phantom, "\cap"] \\
 & \mathrm{Map}(\mathbb{P}^1, [\mathrm{crit}\,\phi / G \times F]) \arrow[r] & \mathrm{Map}(\mathbb{P}^1, [pt/F]) = \mathrm{Bun}_F(\mathbb{P}^1) \arrow[r, "\chi"] & \mathrm{Bun}_{\mathbb{C}^*}(\mathbb{P}^1)
\end{tikzcd}$$

$$\begin{array}{c} P_{G\times F} \\ \uparrow \\ (P_G \times_{\mathbb{P}^1} P_F)\times_{G\times F} W \\ \downarrow \quad \uparrow u \\ \mathbb{P}^1 \end{array} \qquad E := P_G \times_G V$$

$$= \mathcal{E}nd\, E \otimes \sum_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}(\delta_i) \oplus E \quad \text{s.t. } \operatorname{Im} u \subseteq \operatorname{Crit}\phi \hookrightarrow W$$

$\Longleftrightarrow$ $s \in H^0(E)$ & comm hom

$\phi_i : E \to E \otimes \mathcal{O}_{\mathbb{P}^1}(\delta_i)$, $i = 1, 2, 3$.

$\Updownarrow$

$s : \mathcal{O}_X \to \mathcal{F}$ $(\pi_* \mathcal{F} = E)$. $X = \operatorname{Tot}(\bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}(\delta_i))$

$$\begin{array}{c} X \\ \downarrow \pi \\ \mathbb{P}^1 \end{array}$$

QM stability : $\exists$ finite set $B \subset \mathbb{P}^1$ s.t. $u(C \setminus B)$ contained in $P_{G\times F} \times_{G\times F} (\operatorname{Crit}\phi)^s$

[Ciocan-Fontanine, Kim, Maulik]

$\Longleftrightarrow$ s & ϕ_i generate E on $\mathbb{P}^1 \setminus B$

$\Longleftrightarrow$ ($\mathcal{F}$ is pure) cokers: 0-dim (PT stability)

Sum up: $QM_d^{\delta}(\mathbb{P}^1, \operatorname{Hilb}^n(\mathbb{C}^3)) \simeq P_{n+d}(X, n[\mathbb{P}^1])$

$\chi(E) = \int_{\mathbb{P}^1} c_1(E) + rk(E)$

$= d + n$

When $\delta_1+\delta_2+\delta_3=-2$, X is CY_4

above has a $\underset{\underset{F_0\times\mathbb{C}^*}{\parallel}}{T_0}$-equi virtual class $\left(F_0=\ker\chi\cong(\mathbb{C}^*)^2\right)$

WANT to play w/ $\mathrm{Hilb}^n(\mathbb{C}^3)$:

$$\begin{array}{c} QM \\ \text{open } \cup \ \infty\notin B \\ QM^{\delta}_{d,sm=\infty}(\mathbb{P}^1,\mathrm{Hilb}^n(\mathbb{C}^3)) \xrightarrow{ev_\infty} \mathrm{Hilb}^n(\mathbb{C}^3) \end{array}$$

$$\begin{array}{c} \mathrm{Hilb}^n(\mathbb{C}^3) \\ \uparrow \\ \{\lambda\}: \end{array}$$

F_0-fixed pts

plane partitions of size n

can put insertion

Vertex function : $V_{d,\lambda} := ev_{\infty *}\left(1 \cap [QM^{\circ}_{d,\infty\to\lambda}]^{vir}\right) \in A^{T_0}_{*}(\mathrm{Hilb}^n(\mathbb{C}^3))_{loc}$

(Okounkov school: Nakajima quiver var)

$$= ev_{\infty *} \frac{1}{\sqrt{e_{T_0}(T^{vir})}}$$

$$= \sum_{T_0\text{-fixed pts}} \frac{e_{T_0}(\chi(E)) \cdot e_{T_0}\left(\chi\left(\bigoplus_{i=1}^{3} \mathrm{End}\, E \otimes L_i\right)\right)}{e_{T_0}(\chi(\mathrm{End}\, E))} \to \text{quotient of Gamma functions}$$

2d picture

$$\begin{array}{|c|c|c|} \hline 4 & & \\ \hline 2 & 2 & \\ \hline 1 & 1 & 2 \\ \hline \end{array}$$

$z_{i_1,i_2,i_3} \in \mathbb{N}$ $\quad (i_1,i_2,i_3) \in \lambda$

s.t. $z_{i_1,i_2,i_3} \geq z_{i_1-1,i_2,i_3}, \dots, z_{i_1,i_2,i_3-1}$

$$\left(E = \bigoplus_{(i_1,i_2,i_3)\in\lambda} \mathcal{L}_1^{-i_1}\mathcal{L}_2^{-i_2}\mathcal{L}_3^{-i_3}\, \mathcal{O}(z_{i_1,i_2,i_3})\, \underset{= \mathbb{C}^*_{\mathbb{P}^1}}{q}{}^{z_{i_1,i_2,i_3}} \right)$$

In general can do : $V^{\tau}_{d,\lambda} = ev_{\infty *}\left(e^{T_0}(\tau(i^* V)) \cap [QM]^{vir}\right)$

$$\underset{\text{descendent insertion}}{=}$$

$$V_\lambda := \sum_d V_{d,\lambda}\, z^d = \sum_{(z_\square)_{\square\in\lambda}} \qquad = \int_C \Phi \cdot \prod_{\square\in\lambda} ds_\square$$

$e_{T_0}(-)$

$\downarrow$

(s.t relation above)

$C \searrow$ real n-cycle

Saddle pt equ: $\frac{\partial}{\partial s_\square}\Phi = 0, \quad \forall\, s_\square$

(Nekrasov–Shatashvili type limit)

(under $t \to 0$, $T_0 = \mathbb{C}_t^* \times F_0$)

$$\Rightarrow \qquad z = \frac{1}{s_i}\prod_{s=1}^{3}\prod_{j\neq i}\frac{s_i - s_j - \hbar_s}{s_i - s_j + \hbar_s},$$

$i = 1, 2, \cdots, n$, $\quad n = |\lambda|$

$\hbar_s$ $(s = 1, 2, 3)$

equi para of F_0

subject to $\sum_s \hbar_s = 0$

"Bethe equation" for $Y_1(\hat{gl}_1)$