# Dataset comparison using persistent homology morphisms

Álvaro Torras Casas, Cardiff University

Online Machine Learning Seminar

31st May 2023

# Outline and scope of the talk:

- **Review:** Vietoris-Rips filtration and Persistent Homology
- **Example:** Morphisms between Vietoris-Rips filtrations.
- Motivation for induced partial matchings/Block Functions.
- **Review:** of Bauer-Lesnick matching
- Quick introduction to the induced block function.
- Explore **examples** of point-clouds embedded in $\mathbb{R}^2$.

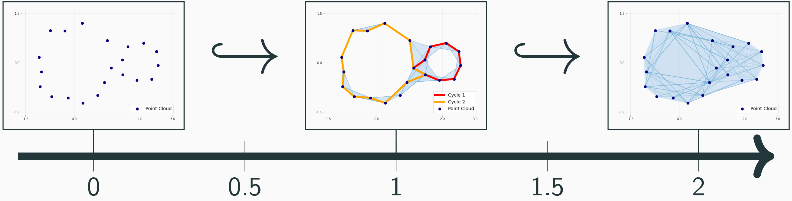
# Filtered Complexes: Vietoris-Rips filtration

- Consider a point sample $\mathbb{X} \subseteq \mathbb{R}^n$.
- Let $r \geq 0$, $\mathrm{VR}_r(\mathbb{X})$ is the maximal **simplicial complex** with edges

$$[x, y] \in \mathrm{VR}_r(\mathbb{X}) \iff \|x - y\|_n \leq 2r \ .$$

- Given a sequence $a_0 < a_1 < \cdots < a_n$ from $\mathbb{R}$, there are inclusions

$$\mathrm{VR}_{a_0}(\mathbb{X}) \hookrightarrow \mathrm{VR}_{a_1}(\mathbb{X}) \hookrightarrow \cdots \hookrightarrow \mathrm{VR}_{a_n}(\mathbb{X})$$

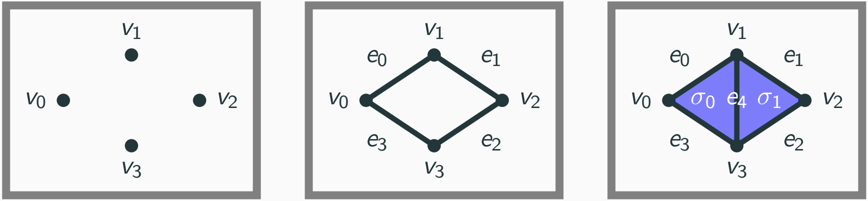



- **Category R** : objects $a \in \mathbb{R}$, arrows $a \to b$ iff $a \leq b$
- **Filtered Complex** : $\mathrm{VR}(\mathbb{X}) : \mathbf{R} \to \mathbf{SpCpx}$

# Computation of Persistence Barcode

- Pick up a maximum radius $R > 0$
- Given $\sigma \in \mathrm{VR}_R(\mathbb{X})$, define $\mathrm{filt}(\sigma) = \max\left\{\|x-y\|_n/2 \mid x, y \in \sigma\right\}$.
- Given $D \in \mathbb{Z}_{\geq 0}$, Consider $\mathrm{VR}_R^D(\mathbb{X})$, the $D$-**skeleton** given by simplices $\sigma \in \mathrm{VR}_R(\mathbb{X})$ such that $\dim(\sigma) \leq D$.
- Sort simplices from $\mathrm{VR}_R^D(\mathbb{X})$ by increasing filtration values and dimension, i.e. $\sigma_1 \leq \sigma_2 \Rightarrow \mathrm{filt}(\sigma_1) \leq \mathrm{filt}(\sigma_2)$ and $\dim(\sigma_1) \leq \dim(\sigma_2)$.
- Choose a field $k$; e.g. $\mathbb{Z}_{11}$
- Perform a Gaussian elimination on the boundary matrix of $\mathrm{VR}_R^D(\mathbb{X})$.
- We obtain the **persistence barcode** and **representatives**.

# Example of Computation of Persistent Homology

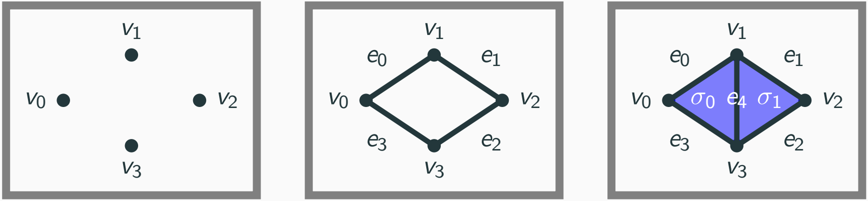



Boundary Matrix:

| | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $\sigma_0$ | $\sigma_1$ |
|---|---|---|---|---|---|---|---|
| $v_0$ | $-1$ | 0 | 0 | $-1$ | 0 | 0 | 0 |
| $v_1$ | 1 | $-1$ | 0 | 0 | $-1$ | 0 | 0 |
| $v_2$ | 0 | 1 | $-1$ | 0 | 0 | 0 | 0 |
| $v_3$ | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| $e_0$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $e_1$ | 0 | 0 | 0 | 0 | 0 | 0 | $-1$ |
| $e_2$ | 0 | 0 | 0 | 0 | 0 | 0 | $-1$ |
| $e_3$ | 0 | 0 | 0 | 0 | 0 | $-1$ | 0 |
| $e_4$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 |

$\rightarrow$ { **Persistence Pairs** **?** }

# Example of Computation of Persistent Homology




Reduced Boundary Matrix: obtain persistence pairs and representatives:

| | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $\sigma_0$ | $\sigma_1$ |
|---|---|---|---|---|---|---|---|
| $v_0$ | $-1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_1$ | 1 | $-1$ | 0 | 0 | 0 | 0 | 0 |
| $v_2$ | 0 | 1 | $-1$ | 0 | 0 | 0 | 0 |
| $v_3$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $e_0$ | 0 | 0 | 0 | 0 | 0 | 1 | $-1$ |
| $e_1$ | 0 | 0 | 0 | 0 | 0 | 0 | $-1$ |
| $e_2$ | 0 | 0 | 0 | 0 | 0 | 0 | $-1$ |
| $e_3$ | 0 | 0 | 0 | 0 | 0 | $-1$ | 1 |
| $e_4$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

$\rightarrow$

**Persistence Pairs**

$(v_1, e_0)$

$(v_2, e_1)$

$(v_3, e_2)$

$(e_4, \sigma_0)$

$(e_3, \sigma_1)$

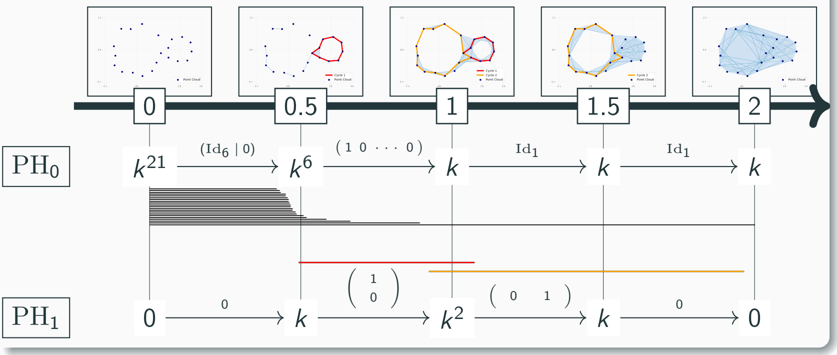
**Convention:**

(positive, negative)

# Example: Interval decomposition

- For each pair $(\tau, \sigma)$, we obtain an interval $I = [\mathrm{filt}(\tau), \mathrm{filt}(\sigma))$.
- $I$ is nontrivial iff $\mathrm{filt}(\tau) < \mathrm{filt}(\sigma)$
- $\mathrm{filt}(\tau)$ is the **birth value** and $\mathrm{filt}(\sigma)$ is the **death value** of $I$.

**Example**




- **Homology:** $\mathrm{H}_0$ "connected components", $\mathrm{H}_1$ "holes", etc.
- **Persistent Homology** : $\mathrm{PH}_n(\mathbb{X}) := \mathrm{H}_n(\mathrm{VR}(\mathbb{X}); k) : \mathbf{R} \to \mathbf{Vect}_k$

# Persistence Modules and Morphisms

- **Persistence Module:** a functor $V : \mathbf{R} \to \mathbf{Vect}_k$. Sometimes written as a pair $(V, \rho)$ where $\rho$ are the **structure maps** $\rho_{st} : V_s \to V_t$ for all $s < t$.
- **Morphism between Persistence Modules:** Given persistence modules $(V, \rho)$ and $(U, \tau)$, then $f : V \to U$ is a set of linear maps $f_t : V_t \to U_t$ for all $t \in \mathbf{R}$ s.t. $\tau_{st} f_s = f_t \rho_{st}$ for all $s < t$.
- **Alternative names:** "Persistence Morphism" or "Ladder Module".
- **Interval Module:** $k_{[a,b)} : \mathbf{R} \to \mathbf{Vect}_k$, with

$$k_{[a,b)}(r) = \begin{cases} k, \text{ for } r \in [a, b) \\ 0, \text{ otherwise.} \end{cases}$$

# A little more about Barcode Decompositions

- Let $(V, \rho)$ be a persistence module.
- If $V$ satisfies the **descending chain condition** for images and kernels then
$$V \simeq \bigoplus_{I \in S_V} \left( \oplus_{m_I} k_I \right),$$
as proved in[1].
- The **barcode** of $V$, $\mathbf{B}(V)$, is a **multiset** $(S_V, m)$ where $S_V$ is a set of intervals and $m : S_V \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is the multiplicity of bars.
- The *representation of a multiset* $(S, m)$ is the set
$$\mathrm{Rep}\,(S, m) = \{(I, i) \in S \times \mathbb{N} : i \leq m_I)\}.$$

[1] W. Crawley-Boevey *Decomposition of pointwise finite-dimensional persistence modules*, Journal of Algebra and Its Applications, 14 (5) (2015)

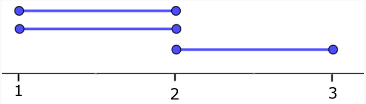
# Example of (Representation of) Barcodes

**Example**

Consider $U : \mathbf{R} \to \mathbf{Vect}_k$ such that

$$U \simeq k_{[1,2]} \oplus k_{[1,2]} \oplus k_{[2,3]} \ .$$

Then its barcode is $\mathbf{B}(U) = \{([1,2],2),([2,3],1)\}$ and the representation of its barcode is $\operatorname{Rep}\mathbf{B}(U) = \{[1,2]_1,[1,2]_2,[2,3]_1\}$, which can be displayed as:



# Persistence Morphisms

- Let a morphism between persistence modules $f : V \to U$.
  **Problem:** $f : V \to U$ has indecomposables of wild type[2]; i.e. there is no "barcode" for $f$.
  **Idea:** Use the barcode decompositions $\mathbf{B}(V)$ and $\mathbf{B}(U)$.
- A **barcode basis** for $V$ is a choice $V \simeq \bigoplus_{i\in\Gamma} k_{[a_i,b_i)}$
- Given a choice of bases for $V$ and $U$, we might understand $f$ by means of an associatd matrix $F$.

**Example**

- Let $\mathbb{X}$ and $\mathbb{Y}$ be two finite subsets from $\mathbb{R}^n$ such that $\mathbb{X} \subseteq \mathbb{Y}$.
- This induces an embedding $\mathrm{VR}(\mathbb{X}) \hookrightarrow \mathrm{VR}(\mathbb{Y})$.
- In turn, this induces a persistence morphism $f : V \to U$, where $V = \mathrm{PH}_n(\mathrm{VR}(\mathbb{X}))$ and $U = \mathrm{PH}_n(\mathrm{VR}(\mathbb{Y}))$ for some $n \in \mathbb{Z}_{\geq 0}$.

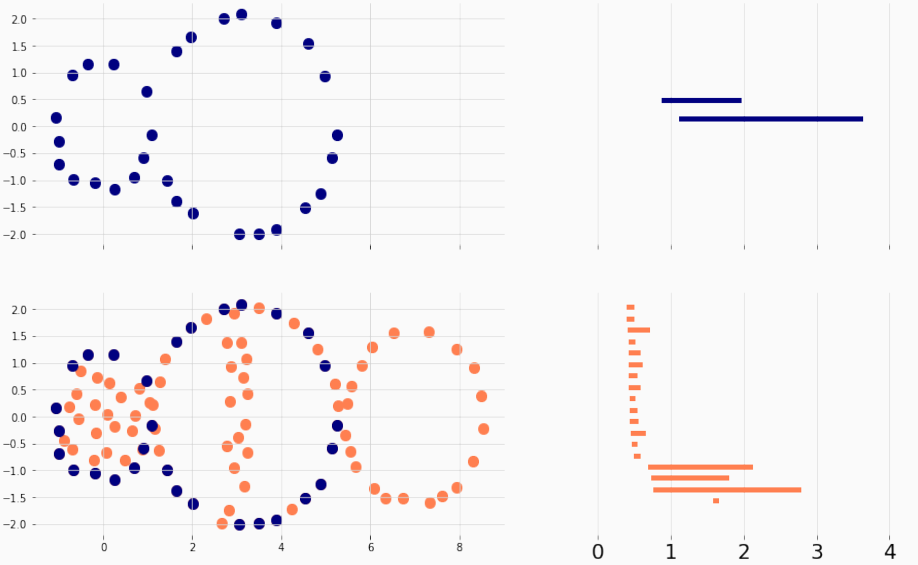
[2] E. Escolar, Y. Hiraoka. *Persistence modules on commutative ladders of finite type*, Discrete and Computational Geometry, 55 (2014), pp. 100-157

# Associated Matrix Computation (Skip?)

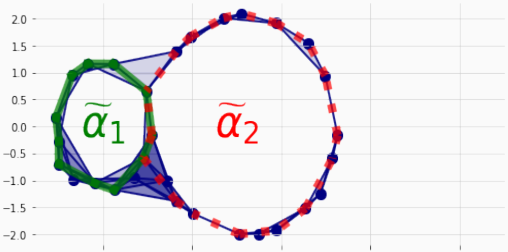
**Example**

- Consider the reduced matrices $R_V$ and $R_U$ that result from computing $V = \mathrm{PH}_*(\mathrm{VR}(\mathbb{X}))$ and $U = \mathrm{PH}_*(\mathrm{VR}(\mathbb{Y}))$ resp.
- Consider the **cycle representatives** of $V$, i.e. the submatrix $\widetilde{R_V}$ from $R_V$ that results from keeping the columns labelled by **negative simplices** from nontrivial intervals.
- Note that the rows from $\widetilde{R_V}$ correspond to simplices from $\mathrm{VR}_R^D(\mathbb{X})$.
- Using $\iota : C_*(\mathrm{VR}_R^D(\mathbb{X}); k) \hookrightarrow C_*(\mathrm{VR}_R^D(\mathbb{Y}); k)$, obtain the matrix product $E_V = M_\iota \widetilde{R_V}$, where $M_\iota$ is the matrix associated to $\iota$.
- Consider the matrix $(R_U \mid E_V)$ and reduce it; all columns from $E_V$ should vanish.
- Tracking the additions, one gets the associated matrix of $V \to U$.
- **Caveat:** One might need to do a little more work for "infinite bars".

# Example: Subset of a bigger Point Cloud

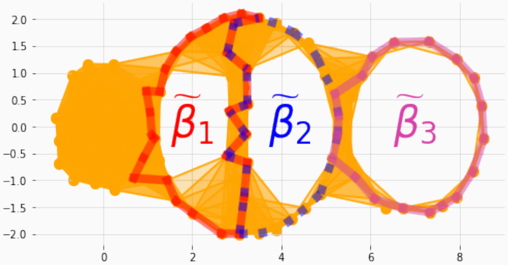


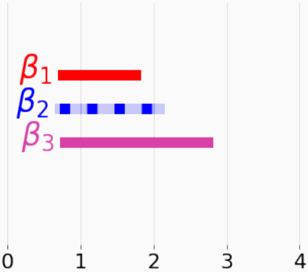
# Example: Subset of a bigger Point Cloud





$$F = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$



# Computing Images

- Let $f : V \to U$ be a persistence morphism with associated matrix $F$.
- Sort the intervals from $\mathbf{B}(V)$ following the **standard order**:

$$[a, b) \leq [c, d) \text{ iff } a < c, \text{ or if } a = c \text{ and } d \leq b\ .$$

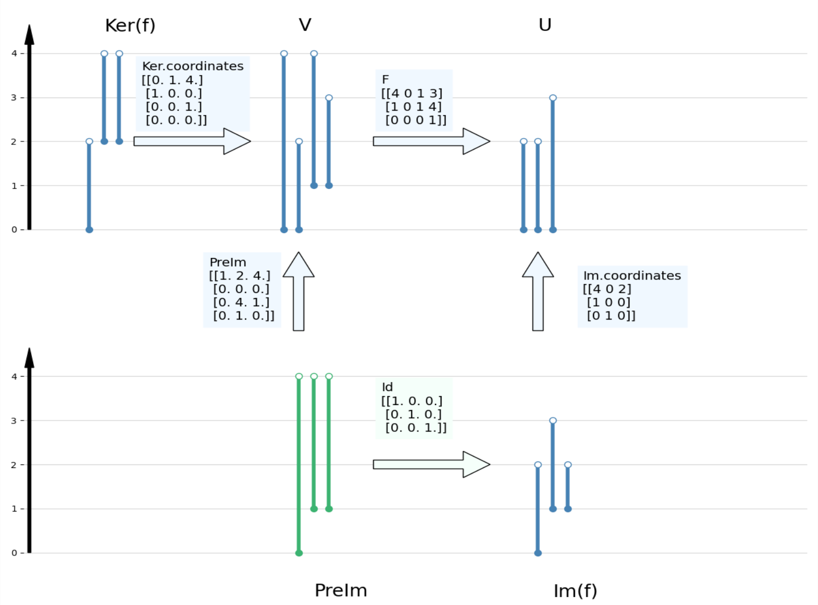
- Sort the intervals from $\mathbf{B}(U)$ following the **endpoint order**:

$$[a, b) \leq [c, d) \text{ iff } b < d, \text{ or if } b = d \text{ and } a \leq c\ .$$

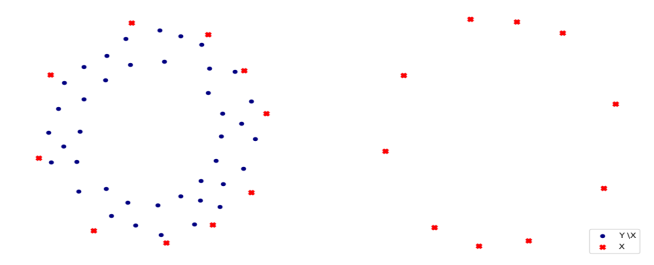
- Consider $F$ with rows and columns reordered.
- Let $R$ be the Gaussian column reduction of $F$.
- The columns from $R$ generate $\mathrm{Im}(f) \subseteq U$
- A pivot in a column associated to $[a, b)$ and a row associated to $[c, d)$ leads to a bar $[a, d)$ for $\mathbf{B}(\mathrm{Im}(f))$.
- Similarly one can compute kernels and quotients[3]

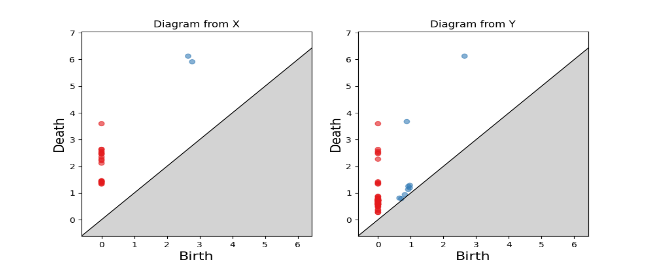
[3]Ch.4 Á Torras-Casas, *Persistence Spectral Sequences*, (2022) Cardiff University.

# Images and Kernels Illustration

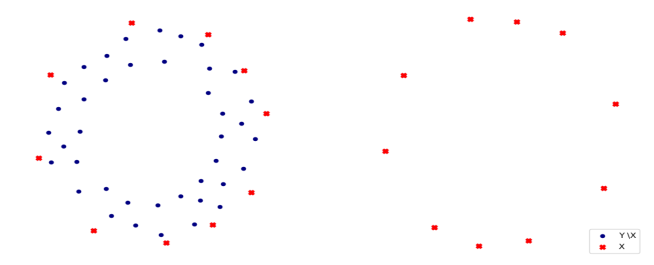


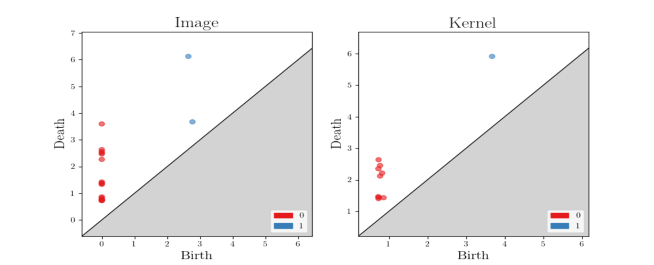
# Downside of Images and Kernels (example)



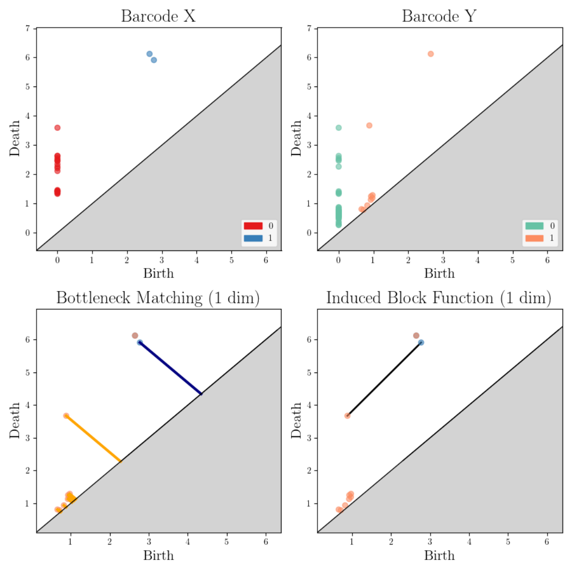



# Downside of Images and Kernels (example)



# Motivation for the Induced Block Function (example)



# Block Functions and Partial Matchings

- A **block function** between $\mathbf{B}_1 = (S_1, m)$ and $\mathbf{B}_2 = (S_2, n)$ is a function $\mathcal{M} : S_1 \times S_2 \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that:

$$\sum_{J \in S_2} \mathcal{M}(I, J) \leq m_I \ .$$

- **Assignment:** $\mathcal{M}_f \colon R_1 \to R_2$ between subsets $R_1 \subseteq \mathrm{Rep}\,\mathbf{B}_1$ and $R_2 \subseteq \mathrm{Rep}\,\mathbf{B}_2$. For ease, we write $\mathcal{M}_f : \mathrm{Rep}\,\mathbf{B}_1 \to \mathrm{Rep}\,\mathbf{B}_2$.
- A **partial matching** is a bijection $\sigma \colon R_1 \to R_2$.
- If a block function satisfies

$$\sum_{I \in S_1} \mathcal{M}(I, J) \leq n_J \ ,$$

it induces a partial matching $\mathrm{Rep}\,\mathbf{B}_1 \to \mathrm{Rep}\,\mathbf{B}_2$.

# Example: A block function NOT inducing a partial matching

**Example**

$\mathbf{B}_1 = (S_1, m) = \{([2,4],1),([1,5],2)\}$ and
$\mathbf{B}_2 = (S_2, n) = \{([2,3],1),([1,4],2)\}$
Consider $\mathcal{M} : S_1 \times S_2 \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ which is zero except for

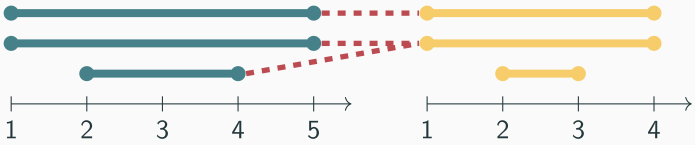
$$\mathcal{M}([2,4],[1,4]) = 1 \text{ and } \mathcal{M}([1,5],[1,4]) = 2\,.$$

$\mathcal{M}$ is a block function, since

$$\mathcal{M}([2,4],[1,4]) = 1 \leq m_{[2,4]} \text{ and } \mathcal{M}([1,5],[1,4]) = 2 \leq m_{[1,5]}$$

however $\mathcal{M}$ does not induce a partial matching since

$$\mathcal{M}([2,4],[1,4]) + \mathcal{M}([1,5],[1,4]) = 3 \not\leq n_{[1,4]} = 2\ .$$



# Example: A block function inducing a Partial Matching

**Example**

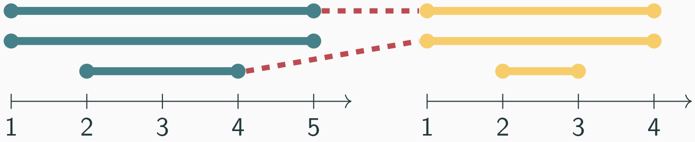
$\mathbf{B}_1 = (S_1, m) = \{([2,4],1),([1,5],2)\}$ and
$\mathbf{B}_2 = (S_2, n) = \{([2,3],1),([1,4],2)\}$
Consider $\mathcal{M} : S_1 \times S_2 \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ which is zero except for

$$\mathcal{M}([2,4],[1,4]) = \mathcal{M}([1,5],[1,4]) = 1\,.$$

$\mathcal{M}$ is a block function inducing a partial matching
$\sigma_{\mathcal{M}} : \mathrm{Rep}\,\mathbf{B}_1 \to \mathrm{Rep}\,\mathbf{B}_2$ given by:

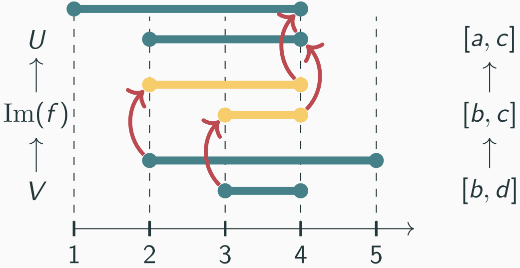
$$[2,4]_1 \mapsto [1,4]_1 \text{ and } [1,5]_1 \mapsto [1,4]_2$$

while $[2,3]_1 \in \mathrm{Rep}\,\mathbf{B}_2$ remains unmatched.



# The Bauer-Lesnick induced partial matching

- Let $f : V \to U$ be a persistence morphism.
- In 2015 Bauer and Lesnick introduced[4] an induced partial matching $\chi_f : \mathrm{Rep}\,\mathbf{B}(V) \to \mathrm{Rep}\,\mathbf{B}(U)$.
- $\chi_f$ is defined by using $\mathbf{B}(V)$, $\mathbf{B}(U)$ and $\mathbf{B}(\mathrm{Im}(f))$:




[4]U. Bauer, M. Lesnick. *Induced matchings and the algebraic stability of persistence barcodes*, Journal of Computational Geometry, 6 (2) (2015), pp. 162-191

# Downside to the Bauer-Lesnick Partial Matching

- $\chi_f$ might be "blind" to $f$.

**Example**

Consider the persistence morphism $f : V \to U$ given by:

$$f = \left(k_{[2,3]} \to 0\right) \oplus \left(\mathrm{Id} : k_{[2,2]} \to k_{[1,2]}\right)$$

i.e. $f : k_{[2,3]} \oplus k_{[2,2]} \to k_{[1,2]}$ with associated matrix:

$$F = \begin{pmatrix} 0 & 1 \end{pmatrix} .$$

One would expect: $[2,3]_1 \longmapsto \emptyset$ , $\qquad [2,2]_1 \longmapsto [1,2]_1$ .
However, $\mathrm{Im}(f) \simeq k_{[2,2]}$ and $\chi_f$ produces:

$$[2,3]_1 \xmapsto{\chi_f} [1,2]_1 \ , \qquad [2,2]_1 \xmapsto{\chi_f} \emptyset$$

- Additionally, when computing $\chi_f$ we might need to check equality between `double` type variables (!).

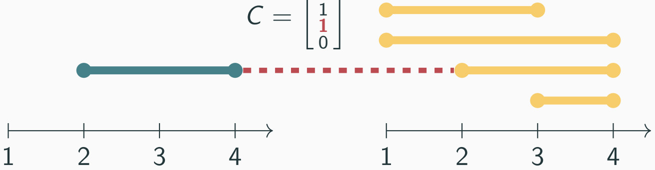
# Quick Introduction to the Induced Block Function $\mathcal{M}_f$

- Let $f\colon V \to U$ be a persistence morphism.
- There is an induced block funct.[5] $\mathcal{M}_f$ from $\mathbf{B}(V)$ to $\mathbf{B}(U)$ s. t.:
  - **Additivity:** Given a direct sum of morphisms:

$$f^1 \oplus f^2 : V^1 \oplus V^2 \longrightarrow U^1 \oplus U^2$$

    We have that, $\mathcal{M}_{f^1 \oplus f^2}(I, J) = \mathcal{M}_{f^1}(I, J) + \mathcal{M}_{f^2}(I, J)$.
  - **Pivots:** Let $f : k_I \to U$ with associated matrix $C$:

$$C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$




    then $\mathcal{M}_f(I, J) \neq 0$ where $J$ the "pivot" that results from the order: $[a, b] \leq [c, d]$ iff $b < d$ or, if $b = d$ then $a \leq c$.

[5] R. González-Díaz, M. Soriano-Trigueros, Á. Torras-Casas, *Partial Matchings induced by Morphisms between Persistence Morphisms*, Comput. Geom., Vol. 112, 2023.

# Revisiting the Example and additional property of $\mathcal{M}_f$

**Example**

Consider the persistence morphism $f : V \to U$ given by:

$$f = \big(k_{[2,3]} \to 0\big) \oplus \big(\mathrm{Id} : k_{[2,2]} \to k_{[1,2]}\big)$$

then,

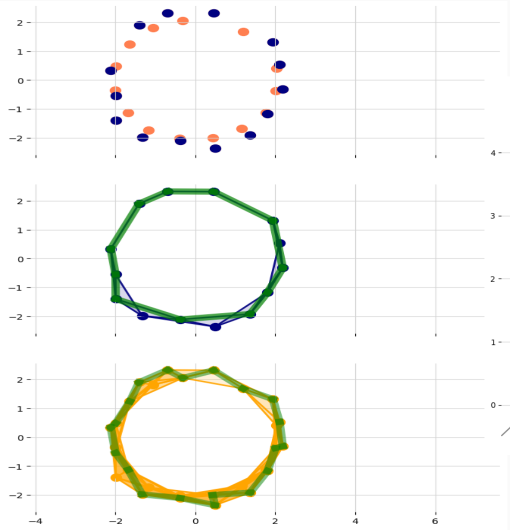
- by additivity $\mathcal{M}_f = \mathcal{M}_g$, where $g = \mathrm{Id} : k_{[2,2]} \to k_{[1,2]}$
- by the pivot property, $\mathcal{M}_f([2,2],[1,2]) = 1$.

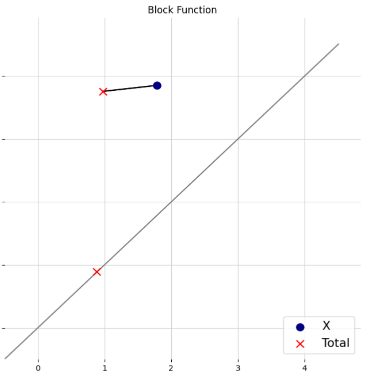
Altogether $\mathcal{M}_f$ is zero everywhere except $\mathcal{M}_f([2,2],[1,2]) = 1$ . Thus, $\mathcal{M}_f$ induces the expected matching:
$[2,3]_1 \longmapsto \emptyset$ , $\qquad [2,2]_1 \longmapsto [1,2]_1$ .

- **Interval Order condition:** given $I = [a,b]$ and $J = [c,d]$, if $\mathcal{M}_f(I,J) \neq 0$, then

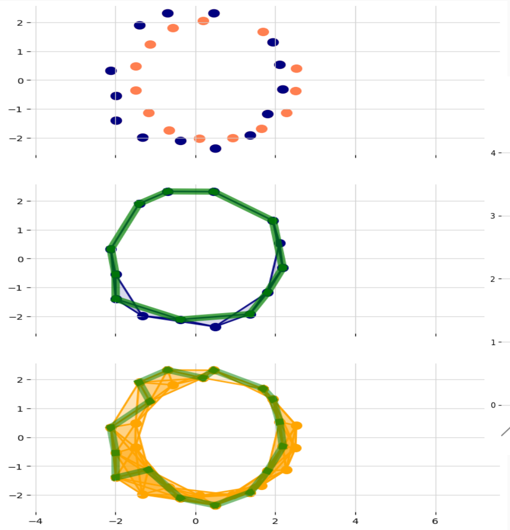
$$c \leq a \leq d \leq b \ .$$

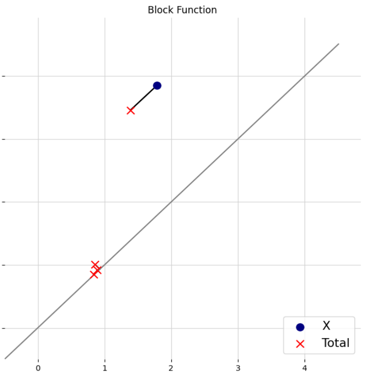
# Matching circles in the plane



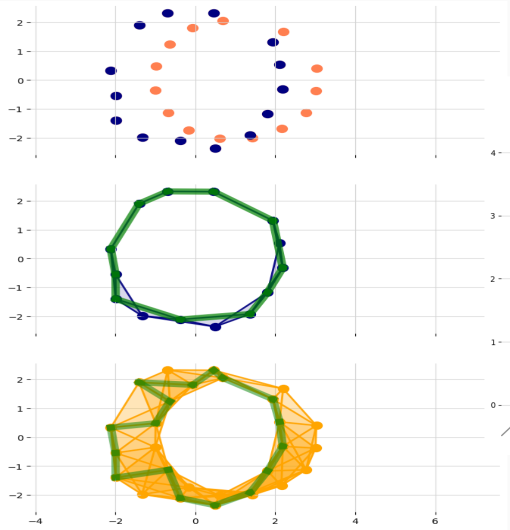



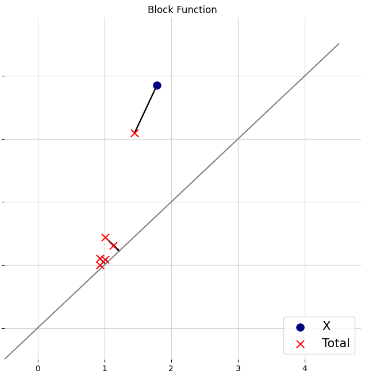
# Matching circles in the plane



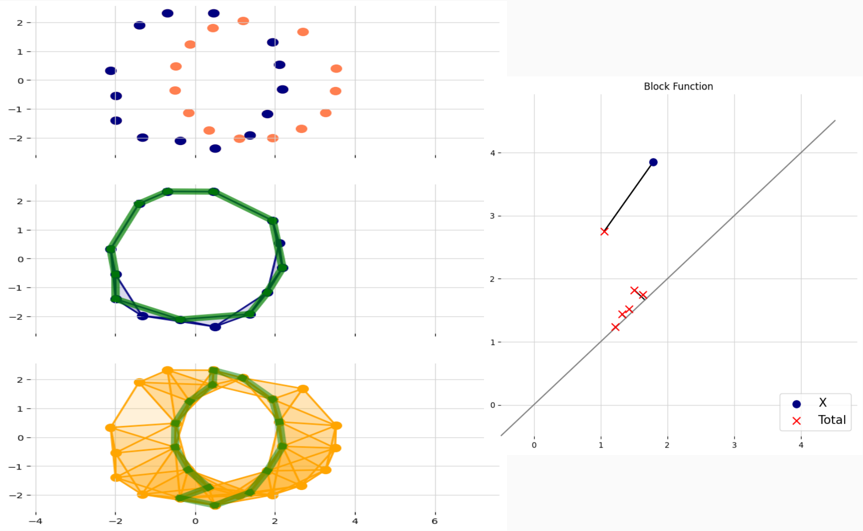


# Matching circles in the plane

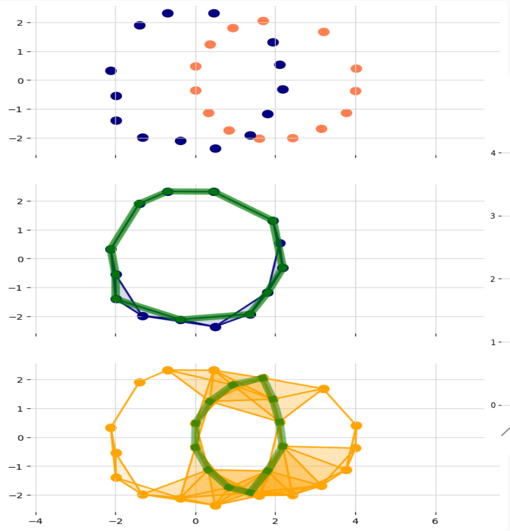


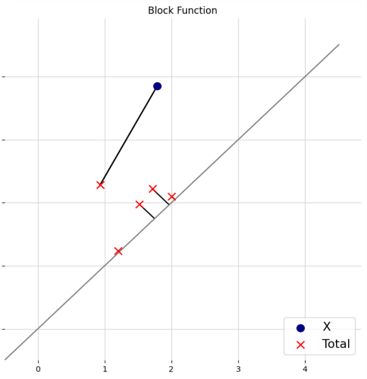


# Matching circles in the plane

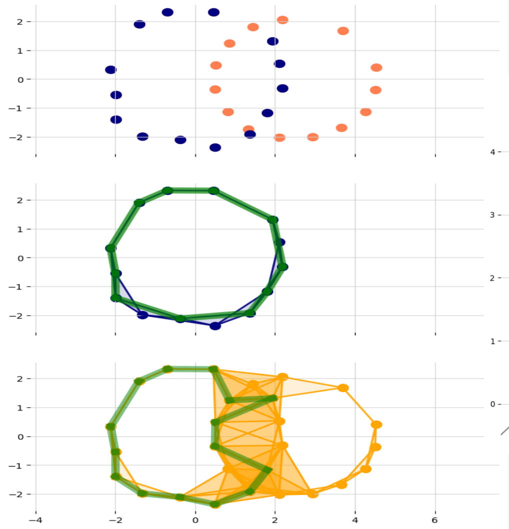


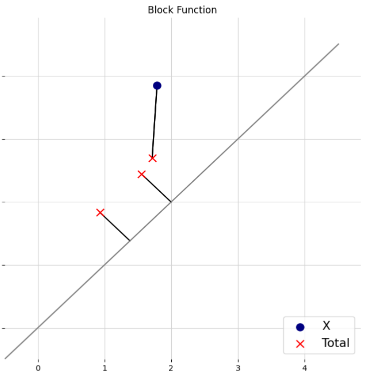
# Matching circles in the plane



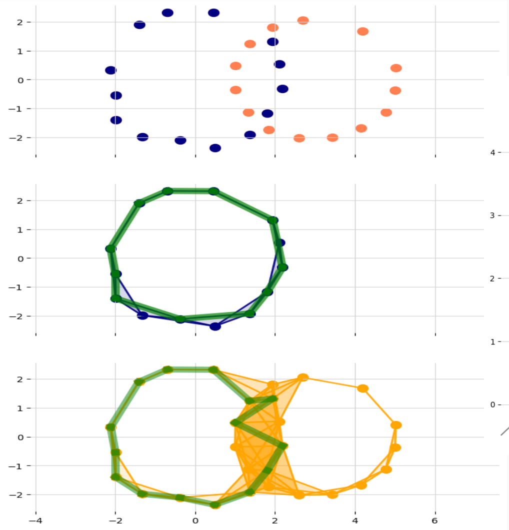



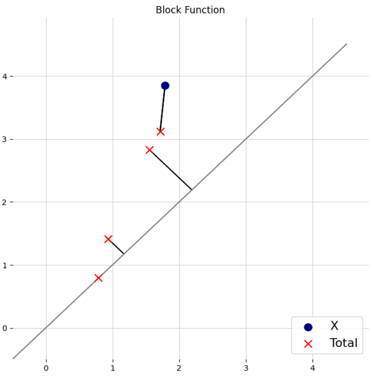
# Matching circles in the plane



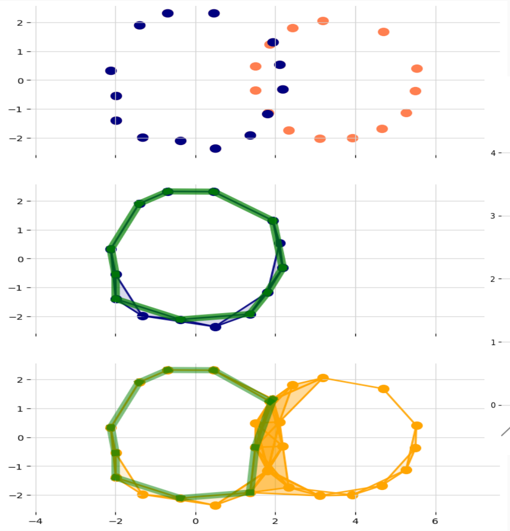



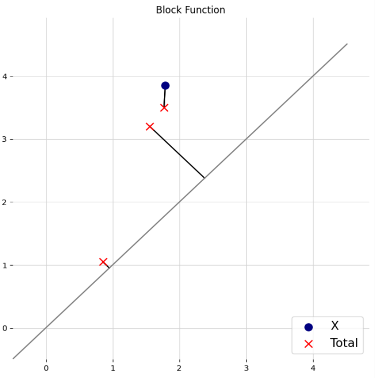
# Matching circles in the plane



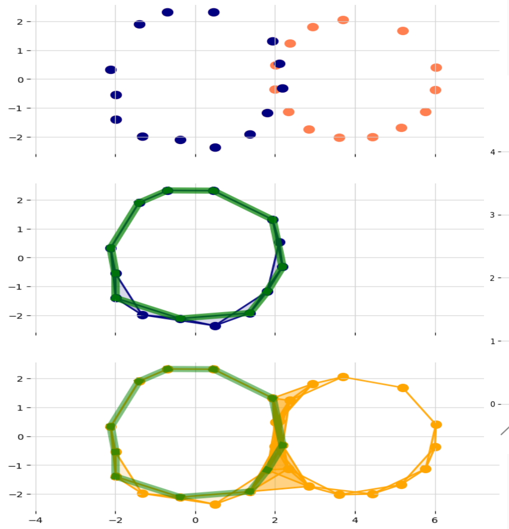


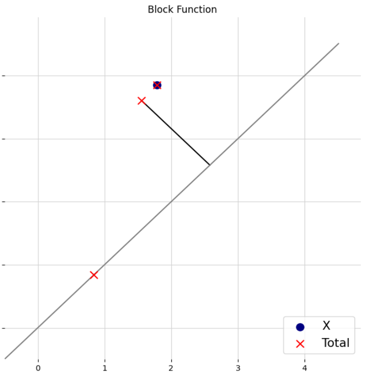
# Matching circles in the plane



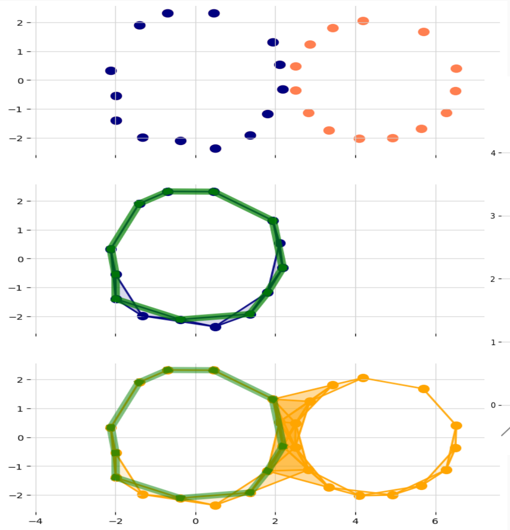



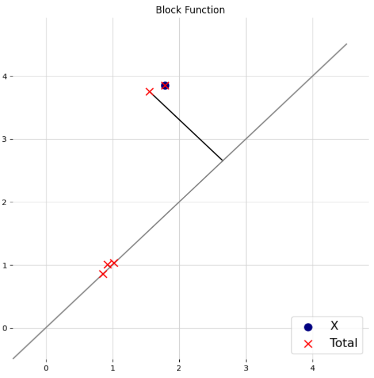
# Matching circles in the plane



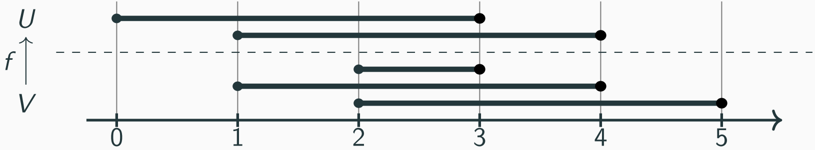
# Matching circles in the plane



# Example: Matrix computation

- Consider $V \simeq k_{[2,3]} \oplus k_{[1,4]} \oplus k_{[2,5]}$ and $U \simeq k_{[0,3]} \oplus k_{[1,4]}$.
- Order the intervals in $\mathbf{B}(V)$ and $\mathbf{B}(U)$ following the endpoint order.




- Suppose that $f$ is associated to the following matrix:

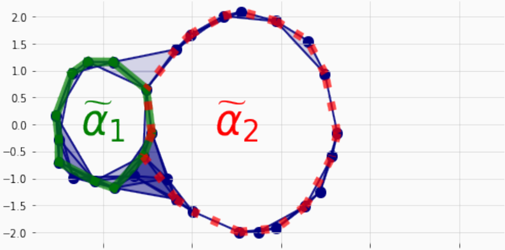
$$F = \left[\begin{array}{c|ccc} & [2,3] & [1,4] & [2,5] \\ \hline [0,3] & 1 & 1 & 0 \\ [1,4] & 0 & 1 & 1 \end{array}\right]$$

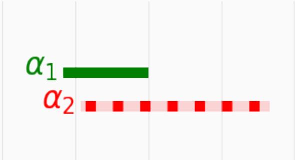
- Let $I = [a, b]$. Consider $F_I$, the reduced minor of $F$ restricted to columns associated to $[c, d]$ with $c \leq a$ and $d \leq b$:

$$F_{[2,3]} = \begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix}, \; F_{[1,4]} = \begin{bmatrix} 1 \\ \mathbf{1} \end{bmatrix}, \text{ and } F_{[2,5]} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \mathbf{0} \end{bmatrix}$$

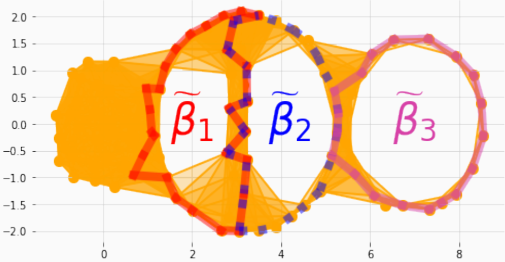
- $\mathcal{M}_f$ is given by $[2,3] \mapsto [0,3]$ and $[1,4] \mapsto [1,4]$ and $[2,5] \mapsto \emptyset$.

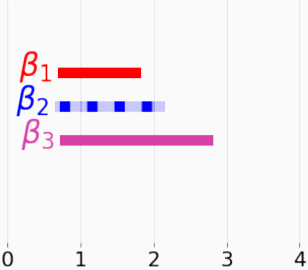
# Example: Subset of a bigger Point Cloud



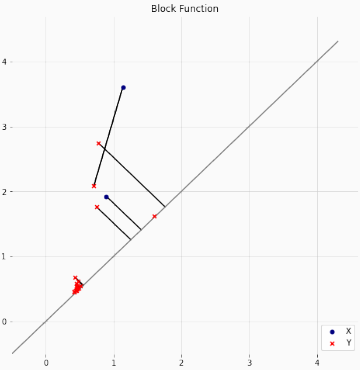



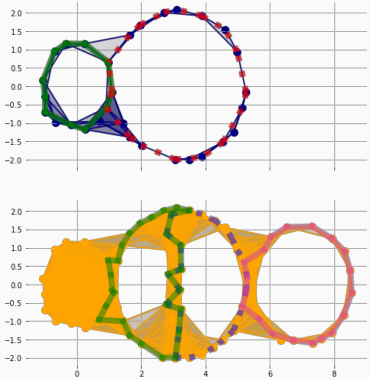

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$



# Example: Subset of a bigger Point Cloud



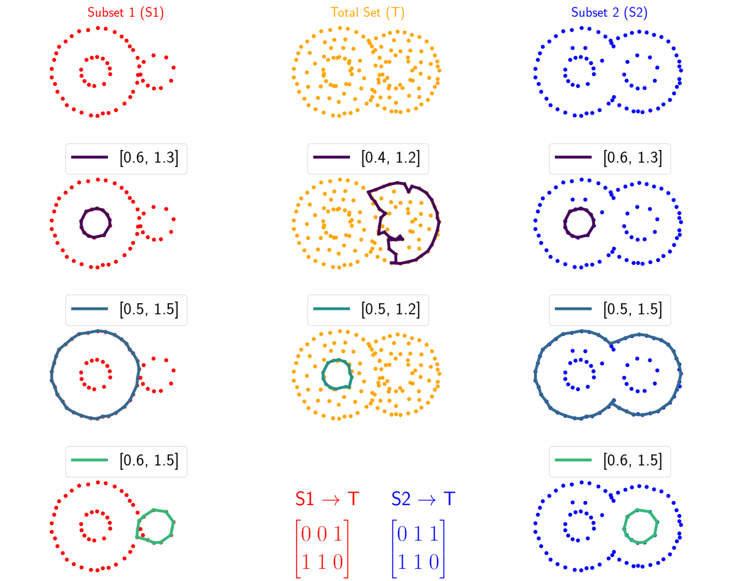
# Nested intervals and $\mathcal{M}_f$

- **Nested Intervals:** $[a, b]$ and $[c, d]$ are nested if $a < c < d < b$
- If for any set of intervals $S \subseteq S_V$ we have that

  $$\sum_{I \in S} \mathcal{M}_f(I, J) > n_J \,,$$

  then there exists a pair of nested intervals in $S$.
- **Corollary** If there are no two nested intervals in $S_V$ then $\mathcal{M}_f$ induces a partial matching.

# Example: Two subsets with the same intervals and image




S1 → T

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

S2 → T

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

## Example: Image computation for $S_1$

- $f_1 : S_1 \hookrightarrow T$ with
  $\mathbf{B}(\mathrm{PH}_1(\mathrm{VR}(S_1))) = \big\{[0.6, 1.3], [0.5, 1.5], [0.6, 1.5]\big\}$ and
  $\mathbf{B}(\mathrm{PH}_1(\mathrm{VR}(T))) = \{[0.4, 1.2], [0.5, 1.2]\}$.
- Order domain by standard order and codomain by endpoint order:

$F =$

| | [0.5, 1.5] | [0.6, 1.5] | [0.6, 1.3] |
|---|---|---|---|
| [0.4, 1.2] | 0 | 1 | 0 |
| [0.5, 1.2] | 1 | 0 | 1 |

- Obtain the reduction:

$R =$

| | [0.5, 1.5] | [0.6, 1.5] | [0.6, 1.3] |
|---|---|---|---|
| [0.4, 1.2] | 0 | 1 | 0 |
| [0.5, 1.2] | 1 | 0 | 0 |

- Image barcodes: $\mathbf{B}(\mathrm{Im}(f_1)) = \{[0.5, 1.2], [0.6, 1.2]\}$.

# Example: Image computation for $S_2$

- $f_2 : S_2 \hookrightarrow T$ with $\mathbf{B}(\mathrm{PH}_1(\mathrm{VR}(S_2))) = \{[0.6, 1.3], [0.5, 1.5], [0.6, 1.5]\}$ and $\mathbf{B}(\mathrm{PH}_1(\mathrm{VR}(T))) = \{[0.4, 1.2], [0.5, 1.2]\}$.
- Order domain by standard order and codomain by endpoint order:

$F =$

| | [0.5, 1.5] | [0.6, 1.5] | [0.6, 1.3] |
|---|---|---|---|
| [0.4, 1.2] | 1 | 1 | 0 |
| [0.5, 1.2] | 1 | 0 | 1 |

- Obtain the reduction:

$R =$

| | [0.5, 1.5] | [0.6, 1.5] | [0.6, 1.3] |
|---|---|---|---|
| [0.4, 1.2] | 1 | 1 | 0 |
| [0.5, 1.2] | 1 | 0 | 0 |

- Image barcodes: $\mathbf{B}(\mathrm{Im}(f_2)) = \{[0.5, 1.2], [0.6, 1.2]\}$.
- I.e. $\mathrm{Im}(f_1) \simeq \mathrm{Im}(f_2) \simeq k_{[0.5,1.2]} \oplus k_{[0.6,1.2]}$

# Example: Computation of $\mathcal{M}_{f_1}$

- Now, sort both $\mathbf{B}(S_1)$ and $\mathbf{B}(T)$ by endpoint order.
- We have a matrix

$$F = \left[\begin{array}{c|ccc} & [0.6, 1.3] & [0.5, 1.5] & [0.6, 1.5] \\ \hline [0.4, 1.2] & 0 & 0 & 1 \\ [0.5, 1.2] & 1 & 1 & 0 \end{array}\right]$$

- Obtain the matrices:

$$F_{[0.6,1.3]} = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}, \; F_{[0.5,1.5]} = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}, \; F_{[0.6,1.5]} = \begin{bmatrix} 0 & 0 & \mathbf{1} \\ 1 & 0 & 0 \end{bmatrix},$$

- Assignment: $[0.6, 1.3] \mapsto [0.5, 1.2]$, $[0.5, 1.5] \mapsto [0.5, 1.2]$ and $[0.6, 1.5] \mapsto [0.4, 1.2]$.

# Example: Computation of $\mathcal{M}_{f_2}$

- Now, sort both $\mathbf{B}(S_2)$ and $\mathbf{B}(T)$ by endpoint order.
- We have a matrix

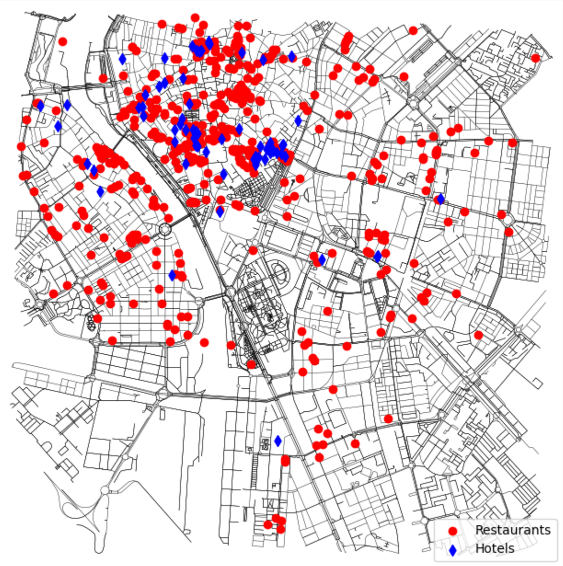
$$F = \left[\begin{array}{c|ccc} & [0.6, 1.3] & [0.5, 1.5] & [0.6, 1.5] \\ \hline [0.4, 1.2] & 0 & 1 & 1 \\ [0.5, 1.2] & 1 & 1 & 0 \end{array}\right]$$

- Obtain the matrices:

$$F_{[0.6,1.3]} = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}, \; F_{[0.5,1.5]} = \begin{bmatrix} 1 \\ \mathbf{1} \end{bmatrix}, \; F_{[0.6,1.5]} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

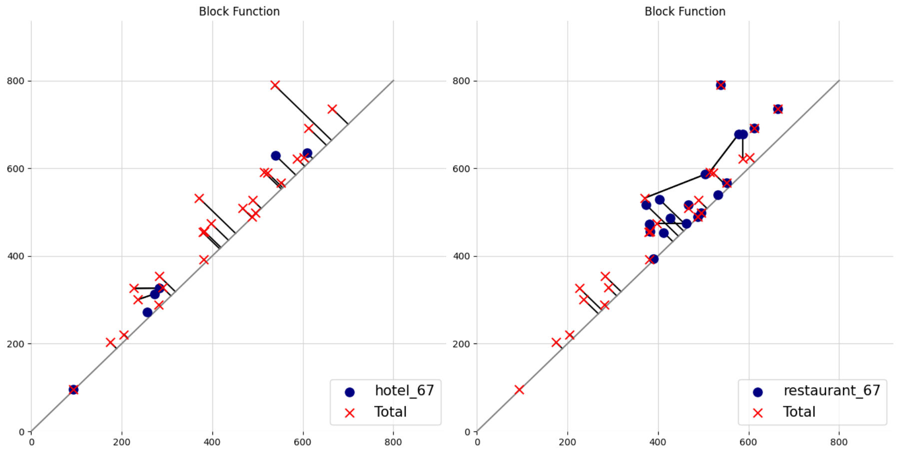
- Assignment: $[0.6, 1.3] \mapsto [0.5, 1.2]$ and $[0.5, 1.5] \mapsto [0.5, 1.2]$.
- We might distinguish $f_1$ and $f_2$ based on $\mathcal{M}_{f_1}$ and $\mathcal{M}_{f_2}$

# OSM Data Example: Hotels and Restaurants in Seville

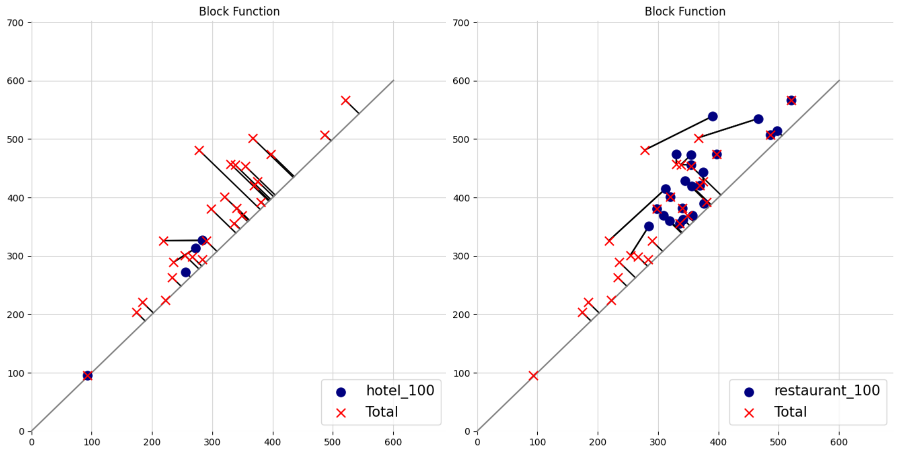



- There are 67 Hotels and 499 restaurants.

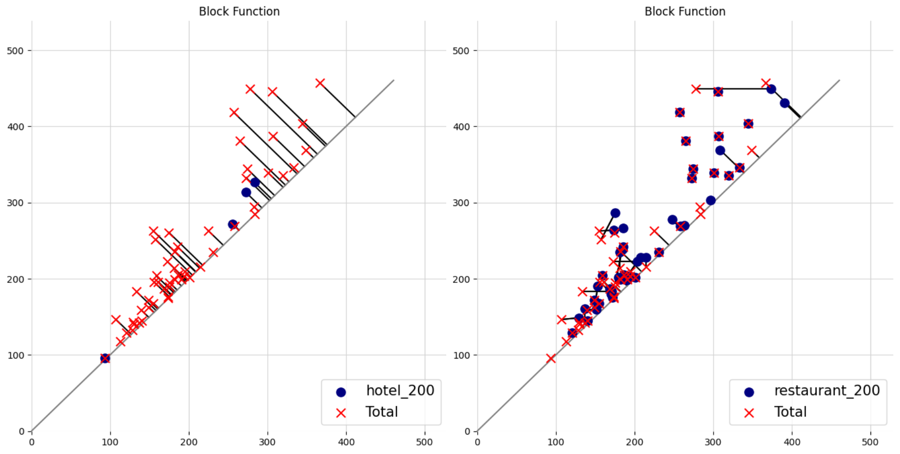
# Sample of 67 restaurants



# Sample of 100 restaurants



# Sample of 200 restaurants



# Future Work and Questions

- Can we obtain an alternative definition for an induced block funciton $\widetilde{\mathcal{M}_f}$ which always induces a partial matching? **yes, work in progress.**
- Optimal implementations for computing the associated matrix.
- Work with other filtrations; e.g. Block functions between alpha complexes.
- Find stability conditions for $\mathcal{M}_f$
- Find use-cases for this block function.

# Bibliography


- R. González-Díaz, M. Soriano-Trigueros, Á. Torras-Casas, *Partial Matchings induced by Morphisms between Persistence Morphisms*, Computational Geometry, Volume 112, June 2023.
- E. Escolar, Y. Hiraoka. *Persistence modules on commutative ladders of finite type*, Discrete and Computational Geometry, 55 (2014), pp. 100-157.
- W. Crawley-Boevey *Decomposition of pointwise finite-dimensional persistence modules*, Journal of Algebra and Its Applications, 14 (5) (2015)
- U. Bauer, M. Lesnick. *Induced matchings and the algebraic stability of persistence barcodes*, Journal of Computational Geometry, 6 (2) (2015), pp. 162-191
- Á Torras-Casas, *Persistence Spectral Sequences*, (2022) Cardiff University.

# Acknowledgements


- EPSRC grants: EP/W522405/1 (current) and EP/N509449/1.
- Ministerio Project: PID2019-107339GB-I00
- Junta Andalucía Project: P20_01145
- **CIMAgroup FQM-369** (Universidad de Sevilla)

# Thank You!

**Email:** TorrasCasasA@cardiff.ac.uk

**Website:** https://alvaro-torras-casas.org/

**Thesis:** https://orca.cardiff.ac.uk/id/eprint/149745/