Online Algebraic Geometry.

14/10/2021

"Hyperpolygon spaces: beyond the movable cone"

OR

"All 81 projective crepant resolutions of

$\mathbb{C}^4/G$ as hyperpolygon spaces."

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(jt. with G. Bellamy, S. Rayan, T. Schedler, H. Weiß.)

§1 Crepant resolutions of symplectic quotients.

V complex vector space of dimension $2n$

$G \subset Sp(V)$ finite group.

$V/G = \mathrm{Spec}(\mathbb{C}[V]^G)$.

$$\begin{tikzcd} X \arrow[dr, "f"'] & & V \arrow[dl] \\ & V/G & \end{tikzcd}$$

resolution

V/G has <u>symplectic singularities</u> [Beauville]

- smooth locus admits symplectic form ω s.t. $f^*(\omega)$ extends to regular 2-form on X.

f <u>symplectic resolution</u> iff $f^*(\omega)$ nondeg. on X

$\Updownarrow$

f <u>crepant resolution</u> iff $K_X = f^*(K_{V/G}) = 0$.

e.g. Minimal resolution $f: X \to \mathbb{C}^2/G$ for $G \subset SL(2,\mathbb{C})$

Symplectic resolutions of symplectic quotient singularities are very rare...

1. Fix $n \geq 1$ and finite $\Gamma < SL(2,\mathbb{C})$.

Wreath product $\Gamma_n := \Gamma^n \rtimes S_n < Sp(2n,\mathbb{C})$

[Kuznetsov, Nakajima] give symplectic res

$$\mathrm{Hilb}^{[n]}(X) \xrightarrow{f} \mathrm{Sym}^n(\mathbb{C}^2/\Gamma) \cong \mathbb{C}^{2n}/\Gamma_n$$

where $X \longrightarrow \mathbb{C}^2/\Gamma$ is minimal resol'n.

2. $S_n \curvearrowright V = \mathfrak{h} \oplus \mathfrak{h}^* \cong \mathbb{C}^{2n}$.

There are symplectic resolutions:

$$\begin{tikzcd}
\mathrm{Hilb}^{[n]}(\mathbb{C}^2) \arrow[r, "\tau"] & \mathrm{Sym}^n(\mathbb{C}^2) \cong \mathbb{C}^{2n}/S_n \arrow[r] & \mathbb{C}^2 \\
\tau^{-1}(V/S_n) \arrow[u, hook] \arrow[r, "f"] & V/S_n \arrow[u, hook] \arrow[r] & 0 \arrow[u, hook]
\end{tikzcd}$$

3. Binary tetrahedral group

$$G_4 = Q_8 \rtimes \mathbb{Z}/3 \subset Sp(4, \mathbb{C})$$

where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ quaternionic gp.

Symplectic resolution $X \xrightarrow{f} \mathbb{C}^4/G_4$

- exists [Bellamy]
- constructed [Lehn–Sorger]

4. $\boxed{\text{The key example for today:}}$

$$G = Q_8 \times_{\mathbb{Z}/2} D_8 \subset Sp(4, \mathbb{C})$$

of order 32.

Symplectic resolutions $f : X \longrightarrow \mathbb{C}^4/G$

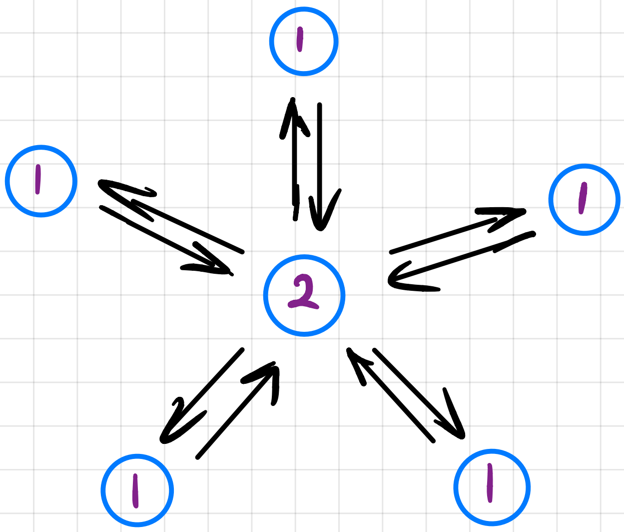
- shown to exist [Bellamy–Schedler]
- counted [Bellamy]
- all 81 projective crepant resolutions constructed by VGIT using Cox rings. [Donten-Bury–Wiśniewski].

Theorem 1 [BCRSW] c.f. [Mekareeya]

For $G = Q_8 \times_{\mathbb{Z}_2} D_8 \subset Sp(4,\mathbb{C})$, $\exists$ isom.

$$\mathbb{C}^4/G \xrightarrow{\sim} X_5(0)$$

to the affine Nakajima quiver variety for



$$\mathfrak{M}_0(\underline{v},\underline{w})$$

$$\underline{v} = (2,1,1,\dots 1)$$
$$\underline{w} = (0,\dots,0).$$

Theorem 2 [BCRSW]

- All 81 projective crepant resolutions of $\mathbb{C}^4/G$ are 'hyperpolygon spaces' (= quiver varieties)

$$\mathfrak{M}_\theta(\underline{v},\underline{w}) = X_5(\theta) \longrightarrow X_5(0) \cong \mathbb{C}^4/G,$$

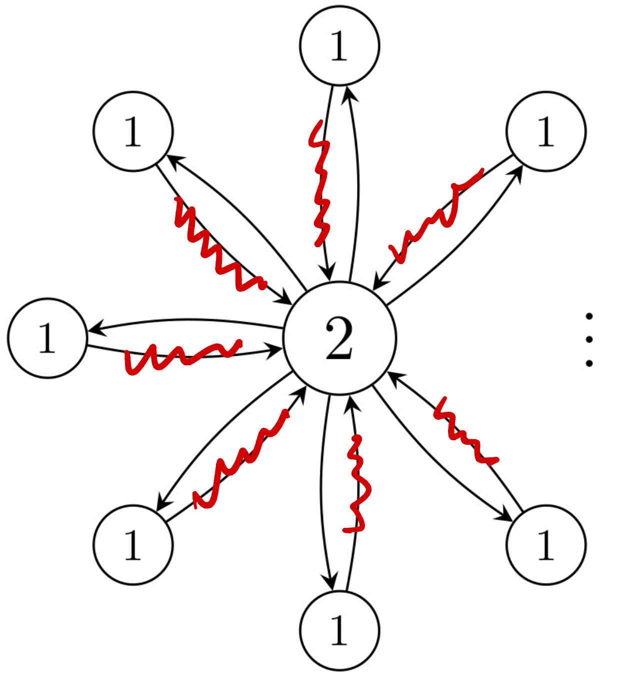
where wall-crossings are Mukai flops.

- Movable cone described explicitly.

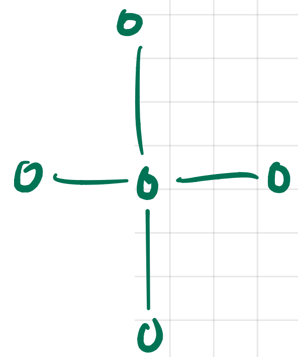
§2 Hypervpolygon spaces $(n \geq 3)$

Quiver Q with $n+1$ vertices, $\underline{v} = (2,1,\dots,1)$:

$\mathfrak{M}_\theta(\underline{v},\underline{w})$



$n=4$.



$$\mathrm{Rep}(Q,\underline{v}) := \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}) \oplus \bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2)$$

$$GL_{\underline{v}} := GL(2) \times (\mathbb{C}^\times)^n$$

$\rightsquigarrow$ moment map $\mu : \mathrm{Rep}(Q,\underline{v}) \longrightarrow \mathfrak{gl}_{\underline{v}}^*$

$$\underline{(GL_{\underline{v}} / \mathbb{C}^\times)^\vee} \cong \mathbb{Z}^n . \ni \theta$$

$$\boxed{X_n(\theta) := \mu^{-1}(0)^{\theta\text{-ss}} / GL_{\underline{v}} \quad \text{for } \theta \in \mathbb{Q}^n.}$$

Examples

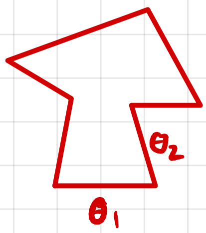
- $n=4$, McKay quiver of type D_4 & $v=\delta$
  $\therefore X_4(0) \cong \mathbb{C}^2/Q_8$ [Kronheimer].
- $n=5$, $X_5(0) \cong \mathbb{C}^4/G$ is Theorem 1.
- $n=6$, we show $X_6(0)$ admits 1684 projective crepant resolutions.

For 'generic' GIT stability parameters in

$$F := \{\theta = (\theta_1, \dots, \theta_n) \in \mathbb{Q}^n \mid \theta_i \geq 0 \ \forall\, 1 \leq i \leq n\},$$

$X_n(\theta)$ is "hyperpolygon space".

[Konno, Harada–Proudfoot] Hyperkähler analogues of polygon spaces, where ind. component of core in $X_n(\theta)$ is



$$M_n(\theta) := \left\{ \begin{matrix} n\text{-sided polygons in } \mathbb{R}^3 \\ \text{with edge-lengths } (\theta_1, \dots, \theta_n) \end{matrix} \right\} \Big/ SO(3)$$

Hyperplane arrangement $\mathcal{A}$:

$$H_i := \{\theta \in \mathbb{Q}^n \mid \theta_i = 0\} \qquad 1 \leq i \leq n,$$

$$H_I := \{\theta \in \mathbb{Q}^n \mid \sum_{i \in I} \theta_i = \sum_{j \notin I} \theta_j\} \qquad \{1\} \subseteq I \subseteq \{1, \ldots, n\}$$

Proposition T.F.A.E. for $\theta \in \mathbb{Q}^n$:

(i) θ does not lie in a hyperplane in $\mathcal{A}$;

(ii) every θ-semistable point of $\mu^{-1}(0)$ is θ-stable;

(iii) the VGIT morphism

$$X_n(\theta) \longrightarrow X_n(0)$$

is a projective, symplectic resolution.

Rmks

- $\dim X_n(\theta) = 2n-6$
- the positive orthant

$$F := \{\theta \in \mathbb{Q}^n \mid \theta_i \geq 0 \text{ for } 1 \leq i \leq n\}$$

is the union of closures of GIT chambers.

- VGIT $X_n(\theta) \dashleftarrow\dashrightarrow X_n(\theta')$ for $\theta, \theta' \in F$ (generic)

studied by [Godinho-Mandini]

Fix GIT chamber

$$C_+ := \left\{ \theta \in \mathbb{Q}^n \;\middle|\; \begin{array}{l} \theta_1 > \theta_2 + \dots + \theta_n \\ \theta_i > 0 \quad 2 \leq i \leq n \end{array} \right\}$$

$$X := X_n(\theta_+) \xrightarrow{f} Y := X_n(0)$$

for $\theta_+ \in C_+$.

Quiver construction means X carries tautological loc. free sheaf

$$T_0 \oplus \bigoplus_{i=1}^{n} T_i$$

$\mathbb{C}^\times \subset GL_{\underline{v}}$ diagonal.

of rank $(2,1,1,\dots 1)$ where $\det(T_0) \cong \mathcal{O}_X$.

The linearisation map for X is

$$\boxed{L_C : \mathbb{Q}^n \longrightarrow N^1(X/Y)}$$

$N^1(X) \cong \mathrm{Pic}(X)_{\mathbb{Q}} \cong H^2(X, \mathbb{Q})$

$$\theta \longmapsto \bigotimes_{i=1}^{n} T_i^{\otimes \theta_i}$$

§3 Theorem 2 revisited (c.f. [Bellamy–C])

The linearisation map identifies the fans:

$$\begin{tikzcd}
\mathbb{Q}^n \arrow[rr, "L = L_C", "\sim"'] & & N^1(X/Y) \\
F \arrow[u, hook] \arrow[rr, "\sim"] & & \mathrm{Mov}(X/Y) \arrow[u, hook] \\
C \arrow[u, hook] \arrow[rr, "\sim"] & & \mathrm{Amp}(X_n(\theta)/Y) \arrow[u, hook]
\end{tikzcd}$$

s.t. $\forall$

$\forall \theta \in C$. ✓ ?

<u>Cor</u> Every proj. crepant res of $X_n(0)$ is a hyperpolygon spaces $X_n(\theta)$ for some generic $\theta \in F$.

<u>Cor</u> (Beyond movable cone for $n \geq 5$).

Namikawa Weyl group $\mathbb{Z}_2^{\oplus n}$ gen'd by reflections in coord. hyperplanes $\{H_i\}$, and

$X_n(\theta) \cong X_n(\theta')$ $\Leftrightarrow$ for $\theta \in C$, $\theta' \in C'$, $\exists\, \omega \in \mathbb{Z}_2^{\oplus n}$ s.t. $\omega(C) = C'$

$$\begin{tikzcd}
X_n(\theta) \arrow[dr] & & X_n(\theta') \arrow[dl] \\
& Y &
\end{tikzcd}$$

$\theta' \rightsquigarrow \theta \in F$ with $\theta_i = |\theta_i'|$

Cor Count proj. crepant resolutions of $X_n(0)$

via $\dfrac{\#\{\text{chambers of hyperplane arrangement}\}}{2^n}$

(for $n \geq 5$)

$\rightsquigarrow$ $X_n(0)$ admits $\begin{cases} 81 & n=5; \\ 1684 & n=6. \end{cases}$

Ideas in the proof of Theorem 2

Adapt proof from [Bellamy-C] for resolutions of $\mathbb{C}^{2n}/\Gamma_n$. A key point is the étale-local description of wall crossing:

- interior walls of F lie in some H_I, and étale-locally π

$$\begin{tikzcd} T^*\mathbb{P}(V) \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & T^*\mathbb{P}(V^*) \arrow[dl] \\ & \mathcal{M}_0 & \end{tikzcd}$$

[BC, §3].

with unstable locus codim ≥ 2 $\therefore$ linearisation maps of adjacent chambers in F agree.

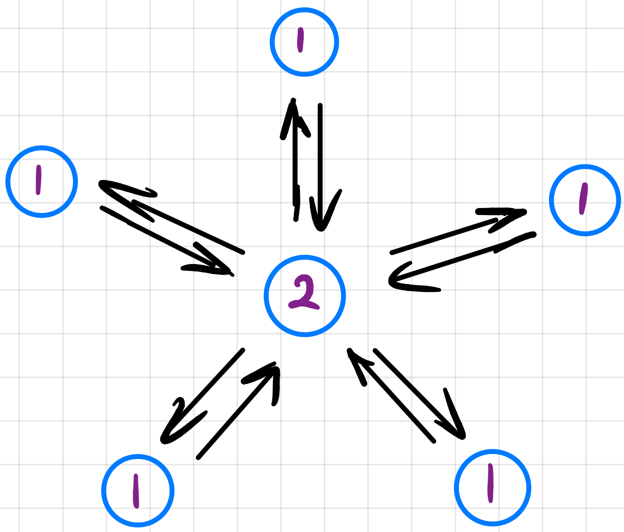
- boundary walls of F lie in H_i $\rightsquigarrow$ divisorial.

§4 Theorem 1 revisited

For $G = Q_8 \times_{\mathbb{Z}_2} D_8 \subset Sp(4, \mathbb{C})$, $\exists$ isom

$$\mathbb{C}^4/G \cong X_5(0)$$

to the Nakajima quiver variety for:



Idea of the proof

$$X_5(0) = \operatorname{Spec}\left(\mathbb{C}[\mu^{-1}(0)]^{GL_{\underline{v}}}\right)$$

$$V/G = \operatorname{Spec}\left(\mathbb{C}[V]^G\right). \qquad V = \mathbb{C}^4$$

We construct a map

$$\mathbb{C}[\mu^{-1}(0)]^{GL_{\underline{v}}} \twoheadrightarrow \mathbb{C}[V]^G.$$

The idea is to first construct explicitly a map

$$\begin{tikzcd}
\mathbb{C}[\mu^{-1}(0)]^{SL_2} \arrow[r, two heads] & \mathbb{C}[V]^{[G,G]} \\
\mathbb{C}[\mu^{-1}(0)]^{GL_{\underline{v}}} \arrow[u, hook] \arrow[r, two heads] & \mathbb{C}[V]^{G} \arrow[u, hook]
\end{tikzcd}$$

- $\mathrm{Cox}(V/G) := \bigoplus_{\chi \in G^\vee} \mathbb{C}[V]^G_\chi$

$$\cong \mathbb{C}[V]^{[G,G]}$$

contains $\mathbb{C}[V]^G \cong H^0(V/G, \mathcal{O}_{V/G})$.

- $\mathbb{C}[\mu^{-1}(0)]^{SL_2}$ is not (quite) the Cox ring of $X_5(\theta)$ for generic θ, but it's $(\mathbb{C}^\times)^5 \cong GL_{\underline{v}}/SL_2$-graded. with invariant part $\mathbb{C}[\mu^{-1}(0)]^{GL_{\underline{v}}}$.

§5 Where next?

Hyperpolygon spaces are a nice testing ground for questions about Nakajima quiver varieties....

1. Study birational geometry & Cox rings of all Nakajima quiver varieties [BCS].

2. Asymptotic geometry of the hyperkähler metric on $X_n(\theta)$ inherited from HKLR quotient (ALE $n=4$, QALE $n=5$, ...) [RW].

3. Thm 1 begs the question (which I bet you've already asked...): is there a quiver moduli space description of $\mathbb{C}^4/G_4$ for binary tetrahedral gp G_4 (studied by Bellamy + Lehn-Sorger)?