McKay correspondence for isolated Gorenstein singularities

Ananyo Dan

University of Sheffield

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Goals

- Recall, the classical (geometric) McKay correspondence for Gorenstein quotient surface singularities.
- Recall, known generalization to higher dimensional quotient singularities.
- Discuss the general case for isolated Gorenstein singularities (in any dimension).
- Possibly, applications to matrix factorization.

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- Possibly, applications to matrix factorization.

Objects of interest

- $G \subset \mathrm{SL}_2(\mathbb{C})$ finite subgroup.
- $X := \mathbb{C}^2/G$ the associated quotient singularity.
- Simple example (A_1 singularity): Take the matrix

$$g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note g^2 is the identity matrix. Let $G := < g >$.

- g acts on $\mathbb{C}[X_1, X_2]$ by $X_i \mapsto -X_i$ for $i = 1, 2$.
- The G-invariant monomials are X_1^2, X_1X_2, X_2^2.
- Easy to check $\mathbb{C}^2/G$ is the hypersurface in $\mathbb{C}^3$ defined by $uv = w^2$.

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McKay correspondence

Theorem (McKay, Gonzalez-Sprinberg, Verdier)

Let X be a quotient surface singularity as before. Then, there is a bijection between the following three sets:

1. *isomorphism classes of indecomposable reflexive $\mathcal{O}_X$-modules (i.e., double dual of the module is isomorphic to itself e.g., vector bundles).*
2. *irreducible components of the exceptional divisor of the minimal resolution (i.e., every resolution factors through it).*
3. *isomorphism classes of irreducible representations of G (i.e. group homomorphism from G to $\mathrm{GL}(V)$ such that there is no non-trivial sub-representation).*

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Correspondence (1) $\longleftrightarrow$ (3)

- Consider the quotient map $\pi : \mathbb{C}^2 \to X$, $U \subset X$ the regular locus.
- Take a reflexive $\mathcal{O}_X$-module M.
- Take the pull-back to $\mathbb{C}^2$: $\pi^* M/(\mathrm{torsion})$.
- Standard argument: $\pi^* M/(\mathrm{torsion})$ is reflexive.
- reflexive module over a regular surface is locally-free.
- Moreover, as $\mathbb{C}^2$ is contractible, $\pi^* M/(\mathrm{torsion})$ is trivial.
- For any $g \in G$, the corresponding U-automorphism of $\pi^{-1}(U)$ induces an automorphism of $\pi^* M/(\mathrm{torsion})$ restricted to $\pi^{-1}(U)$. This gives us a representation of G.

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- Conversely, start with a (complex) representation $\rho : G \to \mathrm{GL}(V)$.
- By Riemann-Hilbert correspondence, we can uniquely associate to ρ a $\mathbb{C}$-local system L_G over the regular locus U of X.
- Take $M_U := L_G \otimes_{\mathbb{C}} \mathcal{O}_U$ the associated locally-free sheaf (with flat connection).
- Extend: $M := i_* M_U$ is a reflexive $\mathcal{O}_X$-module, where $i : U \to X$.

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Correspondence (1) $\longleftrightarrow$ (2)

- Take an indecomposable reflexive $\mathcal{O}_X$-module M.
- Let $\pi : \widetilde{X} \to X$ be the minimal resolution of X. Let E be the exceptional divisor.
- FACT: $\widetilde{M} := \pi^* M/(\mathrm{torsion})$ is a globally generated reflexive $\mathcal{O}_{\widetilde{X}}$-module. Hence, locally-free ($\widetilde{X}$ is a regular surface).
- Let $r = \mathrm{rank}(M)$.

Theorem (Artin-Verdier/Wunram)

For a general choice of r sections $s_1, ..., s_r$ of $\widetilde{M}$, the cokernel of the induced morphism $\mathcal{O}_{\widetilde{X}}^{\oplus r} \to \widetilde{M}$ is isomorphic to $\mathcal{O}_D$, where D is a smooth curve in $\widetilde{X}$ intersecting (transversally) an unique irreducible component of E.

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- Conversely, choose an irreducible component E_i of E and a smooth curve D (transversally) intersecting E at a single point on E_i.
- Let r be the minimum number of sections necessary to generate $\pi_* \mathcal{O}_D$ as an $\mathcal{O}_X$-module.
- Choose r sections $(t_1, .., t_r)$ generating $\pi_* \mathcal{O}_D$. Consider the resulting exact sequence:

$$0 \to N \to \mathcal{O}_{\widetilde{X}}^{\oplus r} \xrightarrow{(t_1,\ldots,t_r)} \mathcal{O}_D \to 0.$$

- Check that $M := \pi_*(N^\vee)$ is a reflexive $\mathcal{O}_X$-module and this construction gives the inverse correspondence.

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Example: A_1 singularity

- Recall, X is a hypersurface singularity defined by $uv = w^2$.
- Reflexive module: Ideal sheaf $(u - w, v - w)$.
- Exceptional divisor: $E \cong \mathbb{P}^1$.
- Non-trivial irreducible representations: Character $\rho : G \to \mathbb{C}^*$ defined by $g \mapsto (-1)$.

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What are the complications?

- What if $n \geq 3$ i.e., if $G \subset \mathrm{SL}_n(\mathbb{C})$ is a finite subgroup and $X := \mathbb{C}^n/G$ the associated quotient singularity?
- Problem I: No minimal resolution of singularity.
- Solution I: Replace minimal resolution by minimal model (in the sense of Mori). Partial results in this case.
- Problem II: Crepant resolution does not always exist in the case $n \geq 4$. Example: $\mathbb{C}^4/\pm$ does not admit a Crepant resolution.
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Crepant divisors and Juniors

- Crepant divisors: Let $f : Y \to X$ be a resolution of singularities, with exceptional divisor E. Write $K_Y := f^* K_X + \sum_i a_i E_i$, where E_i are the irreducible components of E. Recall, $a_i \geq 0$ for all i. We say E_i is a *Crepant divisor*, if $a_i = 0$.
- What should the Crepant divisors correspond to?
- Any element $g \in G$ has n eigenvalues $\lambda_1, ..., \lambda_n$, where
  - $\lambda_i = \zeta^{a_i}$ for some $a_i \in \mathbb{Z}$,
  - $\zeta := e^{2\pi i/r}$ i.e., r-th primitive root of unity,
  - and $r := \min\{a | g^a = 1\}$.
- Junior: Define $\mathrm{age}(g) := \frac{1}{r} \sum a_i$. If $\mathrm{age}(g) = 1$, then g is called a *junior*.

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Ito-Reid correspondence

Theorem (Ito-Reid)

There is a $1-1$ correspondence between:

$$\left\{ \begin{array}{c} \textit{junior elements of } G \\ \textit{upto conjugacy classes} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Crepant divisors in} \\ \textit{resolutions of } X \\ \textit{upto birational eq.} \end{array} \right\}$$

- Idea of the proof:
  - Step I: Reduce to the case when G is a cyclic subgroup.
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- Further generalization by Bridgeland-King-Reid to the case M/G, where M is a non-singular quasi-projective variety and $G \subset \mathrm{Aut}(M)$ a finite subgroup satisfying some condition.

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What's the problem?

- What if we do not have quotient singularity?
- Problem I: In the case $X = \mathbb{C}^2/G$, for every reflexive $\mathcal{O}_X$-module M, we have $R^1\pi_*\widetilde{M} = 0$, where $\pi : \widetilde{X} \to X$ is the minimal resolution and $\widetilde{M} := \pi^* M/(\text{torsion})$.
- This does not hold, even in the (rational) surface singularity case, for non-quotient singularities.
- Solution I: In the rational surface singularity case, instead of looking at all reflexive modules, restrict to those which satisfy $R^1\pi_*\widetilde{M}^\vee = 0$. Such modules are called *Wunram modules*.
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- Why does Wunram modules give the correct correspondence in the rational surface singularity case?
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- Problem III: Reflexive modules vs Maximal Cohen-Macaulay modules (i.e., depth of the module equals the dimension of the variety) in higher dimension. In the surface case, they coincide.
- Key step in the surface case: the correspondence uses degeneracy locus of globally generated vector bundles.
- Reflexive modules in dimension greater than 3 need not be locally-free
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Problem with intersection theory

- Key step in the RDP surface case: the first Chern class of $\widetilde{M}$ intersects an unique irreducible component of the exceptional divisor transversally at exactly one point.
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Main result

- Let (X, x) isolated, normal Gorenstein singularity (i.e., the canonical sheaf is invertible, e.g. local complete intersection subvarieties) of dimension n.
- Given an $\mathcal{O}_X$-module N, denote by $\mathrm{syz}^{n-2}(N)$, the $(n-2)$-th syzygy associated to N i.e., we have a minimal resolution of N of the form

$$0 \to \mathrm{syz}^{n-2}(N) \to \mathcal{O}_X^{\oplus a_{n-3}} \to \mathcal{O}_X^{\oplus a_{n-2}} \dots \mathcal{O}_X^{\oplus a_0} \to N \to 0.$$

- Depth comparison in short exact sequence implies that if N is Cohen-Macaulay of dimension 1, then $\mathrm{syz}^{n-2}(N)$ is a maximal Cohen-Macaulay module.

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Theorem (-, Fernández de Bobadilla, Velázquez)

There is a $1-1$ correspondence between

$$\left\{\begin{array}{c}\text{Crepant divisors in}\\ \text{resolutions of } X\\ \text{upto birational eq.}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Indecomposable generalized}\\ \text{Wunram } \mathcal{O}_X-\text{modules}\\ \text{modulo isomorphism}\end{array}\right\}$$

which associates to a Crepant divisor E_i contained in a resolution

$$\pi : \widetilde{X} \to X$$

of X, the $(n-2)$-th syzygy $\mathrm{syz}^{n-2}(\pi_* \mathcal{O}_D)$, where D is a smooth curve intersecting E_i at exactly one point.

What is Matrix factorization...

- Let $X \subset \mathbb{C}^n$ hypersurface singularity defined by f.
- A matrix factorization of f is a pair of $m \times m$-matrices A and B with coefficients in $\mathbb{C}[X_1, ..., X_n]$ such that $AB = BA = f.\mathrm{Id}_{m \times m}$.

Theorem (Eisenbud)

There is a one-to-one correspondence between:

1. *equivalence classes of reduced matrix factorizations of f*
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How to get the bijection?

- Start with a maximal Cohen-Macaulay $\mathcal{O}_X$-module M. So, $\mathrm{depth}(M) = \dim X = n - 1$.
- By Auslander-Buchsbaum theorem, the projective dimension of M is 1.
- So, we have a projective resolution of M of the form

$$0 \to \mathcal{O}_{\mathbb{C}^n}^{\oplus m} \xrightarrow{\phi} \mathcal{O}_{\mathbb{C}^n}^{\oplus m} \to M \to 0$$

- As $\mathrm{Supp}(M) = X$, we have $f.M = 0$. Hence, $f.\mathcal{O}_{\mathbb{C}^n}^{\oplus m} \subset \mathrm{Im}(\phi)$.
- In other words, for any $v \in \mathcal{O}_{\mathbb{C}^n}^{\oplus m}$ there is an unique $w \in \mathcal{O}_{\mathbb{C}^n}^{\oplus m}$ such that $f.v = \phi(w)$. Set $\psi(v) = w$.
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$$0 \to \mathcal{O}_{\mathbb{C}^n}^{\oplus m} \xrightarrow{\phi} \mathcal{O}_{\mathbb{C}^n}^{\oplus m} \to M \to 0$$

- As $\mathrm{Supp}(M) = X$, we have $f.M = 0$. Hence, $f.\mathcal{O}_{\mathbb{C}^n}^{\oplus m} \subset \mathrm{Im}(\phi)$.
- In other words, for any $v \in \mathcal{O}_{\mathbb{C}^n}^{\oplus m}$ there is an unique $w \in \mathcal{O}_{\mathbb{C}^n}^{\oplus m}$ such that $f.v = \phi(w)$. Set $\psi(v) = w$.
- ψ gives a $\mathcal{O}_{\mathbb{C}^n}$-linear morphism from $\mathcal{O}_{\mathbb{C}^n}^{\oplus m}$ to itself.
- Note that, $\phi \circ \psi = \psi \circ \phi = f.\mathrm{Id}$.

Wild representation types

- Singularities can be classified into three categories:
  1. (finite type) there exists finitely many indecomposable Cohen-Macaulay modules over the singularity. E.g. Du Val surface singularities.
  2. (tame type) for each fixed r, the Cohen-Macaulay modules of rank r over the singularity form a finite set of one parameter families. E.g. simple elliptic surface singularity i.e., the exceptional divisor of the minimal resolution is an irreducible elliptic curve.
  3. (wild type) for almost all (in terms of density) positive integer n, there exists a n-parameter family of non-isomorphic indecomposable Cohen-Macaulay modules over the singularity. E.g. Minimally elliptic surface singularities that are neither simple elliptic nor a cusp.

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Parametrization of matrix factorization

- How do we parameterize the matrix factorization corresponding to hypersurface singularities of wild representation type?

Theorem (-,Velázquez)

Let X be a quasi-homogeneous surface singularity of weights (a, b, c) (i.e., hypersurface singularity defined by f satisfying $f(\lambda^a X_1, \lambda^b X_2, \lambda^c X_3) = \lambda^d f(X_1, X_2, X_3)$ for some d). Then, the matrix factorization associated to any generalized Wunram modules of rank 1 is given by a 2×2-matrix $(m_{i,j})$ of the form:

1. $m_{1,1} = X_1^b y_0 - X_2 x_0^b$,
2. $m_{1,2} = X_3 x_0^c - X_1^c z_0$
3. $m_{2,1}$ and $m_{2,2}$ are certain linear combination of $X_1^{ic} X_3^j$ and $X_1^{ib} X_2^j$ depending on (x_0, y_0, z_0),

where $(x_0, y_0, z_0) \in X$.

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Thank you for your attention !