Family 3-5 and δ-invariant of polarized del Pezzo surfaces.

Elena Denisova

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Motivation and Knowledge

A smooth Fano variety X admits a Kähler–Einstein metric

$$\Updownarrow$$

X is K-polystable.

$n = \dim(X)$

- $n = 1$: $\mathbb{P}^1$ is K-polystable
- $n = 2$: a del Pezzo surface is K-polystable if and only if it is not a blow up of $\mathbb{P}^2$ in one or two points
- $n = 3$: smooth Fano threefolds have been classified into 105 families

Calabi Problem:

Find all K-polystable smooth Fano threefolds in each family.

in *The Calabi Problem for Fano Threefolds*" (2021) by C. Araujo, A.-M. Castravet, I. Cheltsov, K. Fujita, A.-S. Kaloghiros, J. Martines-Garcia, C. Shramov, H. Suss, N. Viswanathan

δ-invariant

We define

$$\boxed{\delta(X) = \inf_{\mathbf{F}/X} \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})},}$$

Theorem

The following assertions holds:

- *X is K-stable $\Leftrightarrow \delta(X) > 1$*
- *X is K-semistable $\Leftrightarrow \delta(X) \geqslant 1$.*

We define

$$\boxed{\delta_P(X) = \inf_{\substack{\mathbf{F}/X \\ P \in C_X(\mathbf{F})}} \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})},}$$

Theorem

The following assertions holds:

- *X is K-stable $\Leftrightarrow \delta_P(X) > 1$ for all $P \in X$*
- *X is K-semistable $\Leftrightarrow \delta_P(X) \geqslant 1$ for all $P \in X$.*

Abban-Zhuang Theory via Kento Fujita formula

Let P be the point in X. We want to estimate $\delta_P(X)$:

1. Choose surface $S \subset X$ such that $P \in S$
2. Compute

$$\tau = \tau(S) = \sup\Big\{u \in \mathbb{Q}_{>0} \ \Big| \ -K_X - uS \text{ is big}\Big\}$$

3. For $u \in [0, \tau]$ let
   - $P(u)$ be the positive part of the Zariski decomposition of the divisor $-K_X - uS$
   - $N(u)$ be the negative part of the Zariski decomposition of the divisor $-K_X - uS$
4. Compute

$$S_X(S) = \frac{1}{(-K_X)^3} \int_0^{\tau} P(u)^3 du$$

Abban-Zhuang Theory via Kento Fujita formula

Theorem

$$\delta_P(X) \geqslant \min\left\{\frac{1}{S_X(S)}, \delta_P\big(S, W^S_{\bullet,\bullet}\big)\right\}$$

where

$$\delta_P\big(S, W^S_{\bullet,\bullet}\big) = \inf_{\substack{F/S \\ P\in C_S(F)}} \frac{A_S(F)}{S\big(W^S_{\bullet,\bullet}; F\big)},$$

the infimum is taken by all prime divisors F over the surface S such that $P \in C_S(F)$ and

$$S\big(W^S_{\bullet,\bullet}; F\big) = \frac{3}{(-K_X)^3}\int_0^\tau \big(P(u)^2\cdot S\big)\cdot \mathrm{ord}_F\Big(N(u)\big|_S\Big)du + \\ +\frac{3}{(-K_X)^3}\int_0^\tau\int_0^\infty \mathrm{vol}\big(P(u)\big|_S - vF\big)dvdu,$$

Local δ-invariant for surfaces

Let S be a smooth surface, let D be a big and nef divisor on S.
For every prime divisor F over S, set

$$\boxed{S_D(F) = \frac{1}{D^2}\int_0^{\infty} \mathrm{vol}\big(D - vF\big)\,dv.}$$

Let P be point in S, and let

$$\boxed{\delta_P\big(S, D\big) = \inf_{\substack{F/S \\ P \in C_S(F)}} \frac{A_S(F)}{S_D\big(F\big)},}$$

where the infimum is taken by all prime divisors over S whose center on S contains P.
If $D = -K_S$ then $\delta_P(S, -K_S)$ is denoted by $\delta_P(S)$.

How to estimate $\delta_P(S, D)$ from above?

- Fix a smooth curve $C \subset S$ that passes through P.
- Set

$$\tau = \sup\Big\{ v \in \mathbb{R}_{\geqslant 0} \ \Big| \ \text{the divisor } D - vC \text{ is pseudo-effective} \Big\}$$

- For $v \in [0, \tau]$, let $P(v)$ and $N(v)$ be the positive part and negative of the Zariski decomposition of the divisor $D - vC$.
- Then $A_S(C) = 1$ and

$$S_D(C) = \frac{1}{D^2} \int\limits_0^\infty \mathrm{vol}\big(D - vC\big) dv = \frac{1}{D^2} \int\limits_0^\tau P(v)^2 dv$$

Thus

$$\boxed{\delta_P(S, D) \leqslant \frac{1}{S_D(C)}}$$

How to estimate $\delta_P(S, D)$ from below?

- Set

$$S\big(W^{C}_{\bullet,\bullet}; P\big) = \frac{2}{D^2}\int\limits_0^{\tau} \mathrm{ord}_P\big(N(v)\big|_C\big)\big(P(v)\cdot C\big)dv +$$

$$+\frac{1}{D^2}\int\limits_0^{\tau}\big(P(v)\cdot C\big)^2 dv = \frac{2}{D^2}\int_0^{\tau} h(v)dv,$$

where

$$h(v) = \big(P(v)\cdot C\big)\times\big(N(v)\cdot C\big)_P + \frac{\big(P(v)\cdot C\big)^2}{2}.$$

Then it follows from Abban-Zhuang Theory that

$$\boxed{\delta_P\big(S, D\big) \geqslant \min\left\{\frac{1}{S_D(C)}, \frac{1}{S\big(W^{C}_{\bullet,\bullet}; P\big)}\right\}.}$$

How to estimate $\delta_P(S, D)$ from above using blowups?

- Let $f\colon \widetilde{S} \to S$ be the blow up of S at the point P, and let E be the f-exceptional curve.
- Set

$$\widetilde{\tau} = \sup\Big\{u \in \mathbb{R}_{\geqslant 0} \,\Big|\, \text{the divisor } f^*(D) - vE \text{ is pseudo-effective}\Big\}$$

- For $v \in [0, \widetilde{\tau}]$, let $\widetilde{P}(v)$ and $\widetilde{N}(v)$ be the positive and negative part of the Zariski decomposition of the divisor $f^*(D) - vE$.
- Then $A_S(E) = 2$ and

$$S_D(E) = \frac{1}{D^2}\int\limits_0^{\widetilde{\tau}} \widetilde{P}(v)^2 dv$$

Then

$$\boxed{\delta_P(S, D) \leqslant \frac{2}{S_D(E)}}$$

How to estimate $\delta_P(S, D)$ from below using blowups?

- for every point $O \in E$, we set

$$S\big(W^E_{\bullet,\bullet}; O\big) = \frac{2}{D^2}\int\limits_0^{\widetilde{\tau}} \mathrm{ord}_O\big(\widetilde{N}(v)|_E\big)\big(\widetilde{P}(v)|_E\big)dv +$$

$$+\frac{1}{D^2}\int\limits_0^{\widetilde{\tau}}\big(\widetilde{P}(v)\cdot E\big)^2 dv = \frac{1}{D^2}\int\limits_0^{\widetilde{\tau}} h(v)dv.$$

where

$$h(v) = \big(\widetilde{P}(v)\cdot E\big)\times\big(\widetilde{N}(v)\cdot E\big)_P + \frac{\big(\widetilde{P}(v)\cdot E\big)^2}{2}.$$

Then it follows from from Abban-Zhuang Theory that that

$$\boxed{\delta_P\big(S, D\big) \geqslant \min\left\{\frac{2}{S_D(E)}, \inf_{O\in E}\frac{1}{S\big(W^E_{\bullet,\bullet}; O\big)}\right\}.}$$

Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Let $S = \mathbb{P}^1 \times \mathbb{P}^1$. Suppose $P \in L_1$ where L_1 is one of the rulings.

$$P(v) = -K_S - vL_1 \text{ and } N(v) = 0 \text{ for } v \in [0,2]$$

$$P(v)^2 = 4(2-v) \text{ and } P(v) \cdot L_1 = 2 \text{ for } v \in [0,2]$$

Thus,

$$S_S(L_1) = \frac{1}{8}\int_0^2 4(2-v)dv = 1 \Rightarrow \delta_P(S) \leq 1$$

$$h(v) = (P(v) \cdot L_1) \times (N(v) \cdot L_1)_P + \frac{(P(v) \cdot L_1)^2}{2} = 2 \text{ for } v \in [0,2]$$

Thus,

$$S(W^{L_1}_{\bullet,\bullet}; P) = \frac{2}{8}\int_0^2 h(v)dv = \frac{2}{8}\int_0^2 2dv = 1$$

Thus, $\delta_P(S) \geq \min\{1,1\} = 1$.

Fano threefolds of Picard rank 3 and degree 20

Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, let C be a smooth curve in S of degree $(5, 1)$, and let $\epsilon \colon C \to \mathbb{P}^1$ be the morphism induced by the projection $S \to \mathbb{P}^1$ to the first factor.

$$\begin{array}{ccc} S = \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{proj_1} & \mathbb{P}^1 \\ \uparrow & & \uparrow = \\ C & \xrightarrow{\epsilon} & \mathbb{P}^1 \end{array}$$

- $\deg(\epsilon) = 5$
- Assume the points $([1:0],[0:1])$ and $([0:1],[1:0])$ are among ramifications points
- So the curve C is given by

$$u\left(x^5+a_1x^4y+a_2x^3y^2+a_3x^2y^3\right) = v\left(y^5+b_1xy^4+b_2x^2y^3+b_3x^3y^2\right)$$

Fano threefolds of Picard rank 3 and degree 20

The ramification index of the point $([1:0],[0:1])$ can be computed as follows:

$$\begin{cases} 2 \text{ if } a_3 \neq 0, \\ 3 \text{ if } a_3 = 0 \text{ and } a_2 \neq 0, \\ 4 \text{ if } a_3 = a_2 = 0 \text{ and } a_1 \neq 0, \\ 5 \text{ if } a_3 = a_2 = a_1 = 0. \end{cases}$$

Likewise, we can compute the ramification index of the point $([0:1],[1:0])$. We may assume that

- $([1:0],[0:1])$ has the largest ramification index among all ramifications points of ϵ
- the ramification index of the point $([0:1],[1:0])$ is the second largest index.

Fano threefolds of Picard rank 3 and degree 20

$$C:\ u\big(x^5+a_1x^4y+a_2x^3y^2+a_3x^2y^3\big)=v\big(y^5+b_1xy^4+b_2x^2y^3+b_3x^3y^2\big)$$

- if both indices are 5 then $a_1=a_2=a_3=b_1=b_2=b_3=0$

$$C: ux^5=vy^5 \text{ and } \mathrm{Aut}(S,C)\cong\mathbb{C}^*\rtimes\mathbb{Z}/2\mathbb{Z}$$

- otherwise $\mathrm{Aut}(S,C)<\infty$

Fano threefolds of Picard rank 3 and degree 20

Now, we consider embedding $S \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ given by

$$\big([u : v], [x : y]\big) \mapsto \big([u : v], [x^2 : xy : y^2]\big),$$

and identify S and C with their images in $\mathbb{P}^1 \times \mathbb{P}^2$.
Let $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ be the blow up of the curve C. We denote a strict transform of S by $\widetilde{S}$.

$$\begin{array}{ccc} & & \widetilde{S} \subset X \\ & & \quad\; C \Big\downarrow \pi \\ C \subset S = \mathbb{P}^1 \times \mathbb{P}^1 & \hookrightarrow & \mathbb{P}^1 \times \mathbb{P}^2 \end{array}$$

Then X is a smooth Fano threefold in the deformation family № 3.5 in the Mori–Mukai list and every smooth member of this family can be obtained in this way.

Known results (Book)

$$C: \ u\left(x^5+a_1x^4y+a_2x^3y^2+a_3x^2y^3\right) = v\left(y^5+b_1xy^4+b_2x^2y^3+b_3x^3y^2\right)$$

- X is K-stable if a_1, a_2, a_3, b_1, b_2, b_3 are general enough,
- X is K-polystable if $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$,
- X is not K-polystable if $(a_1, a_2, a_3) = (0, 0, 0) \neq (b_1, b_2, b_3)$,
- $\mathrm{Aut}(X)$ is finite $\Leftrightarrow$ $(a_1, a_2, a_3, b_1, b_2, b_3) \neq (0, 0, 0, 0, 0, 0)$,
- in this case X is K-polystable $\Leftrightarrow$ X is K-stable.

Let $\mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ be the projection to the first factor and $\phi_1 = \mathrm{pr}_1 \circ \pi$. Then ϕ_1 is a fibration into del Pezzo surfaces of degree four.

$$\begin{array}{ccccc} & & X & & \\ & & {\scriptstyle C}\Big\downarrow{\scriptstyle \pi} & \searrow^{\phi_1} & \\ C \subset S = \mathbb{P}^1 \times \mathbb{P}^1 & \hookrightarrow & \mathbb{P}^1 \times \mathbb{P}^2 & \xrightarrow{\mathrm{pr}_1} & \mathbb{P}^1 \end{array}$$

Conjecture

Conjecture (Book)

The Fano threefold X is K-stable $\Leftrightarrow (a_1, a_2, a_3) \neq (0, 0, 0)$.

Geometrically, this conjecture says that the following two conditions are equivalent:

1. the threefold X is K-stable,
2. the morphism $\epsilon\colon C \to \mathbb{P}^1$ does not have ramification points of ramification index five.

So it can be restated as follows:

Conjecture

The Fano threefold X is K-stable if and only if every singular fiber of ϕ_1 has only singular points of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$.

Goal

The goal is to prove the following (slightly weaker) result:

Theorem

If all ramification points of ϵ have ramification index two, then X is K-stable.

which can be restated as follows:

Theorem

If every singular fiber of ϕ_1 has only singular points of type $\mathbb{A}_1$, then X is K-stable.

Proof

Recall that X is K-stable $\Leftrightarrow \delta_O(X) > 1$ for all $O \in X$ where

$$\boxed{\delta_O(X) = \inf_{\substack{\mathbf{F}/X \\ O \in C_X(\mathbf{F})}} \frac{A_X(\mathbf{F})}{S_X(\mathbf{F})},}$$

for every prime divisor $\mathbf{F}$ over X such that $O \in C_X(\mathbf{F})$. Let's prove that if each singular fiber of the fibration ϕ_1 has one or two singular points of type $\mathbb{A}_1$ then $\delta_O(X) > 1$ for all $O \in X$!

- Fix a point $O \in X$

Reminder: Abban-Zhuang Theory via Kento Fujita formula

Theorem

Let X be a smooth Fano threefold, let Y be an irreducible normal surface in X. Suppose $P(u)$ and $N(u)$ are the positive part and negative parts of ZD of $-K_X - uY$. Then

$$\delta_P(X) \geqslant \min\left\{\frac{1}{S_X(Y)}, \delta_P\big(S, W^Y_{\bullet,\bullet}\big)\right\}$$

where

$$\delta_P\big(Y, W^Y_{\bullet,\bullet}\big) = \inf_{\substack{F/Y \\ P\in C_S(F)}} \frac{A_Y(F)}{S\big(W^Y_{\bullet,\bullet}; F\big)},$$

the infimum is taken by all prime divisors F/Y, $P \in C_S(F)$ and

$$S\big(W^Y_{\bullet,\bullet}; F\big) = \frac{3}{(-K_X)^3}\int_0^{\tau}\big(P(u)^2\cdot Y\big)\cdot \mathrm{ord}_F\Big(N(u)\big|_Y\Big)du + \\ + \frac{3}{(-K_X)^3}\int_0^{\tau}\int_0^{\infty}\mathrm{vol}\big(P(u)\big|_Y - vF\big)dvdu,$$

Proof: $O \in \widetilde{S}$

- $S_X(\widetilde{S}) = \frac{1}{(-K_X)^3} \int_0^\tau \mathrm{vol}(-K_X - u\widetilde{S})du = \frac{31}{40} < 1$
- Recall that $\widetilde{S} \cong \mathbb{P}^1 \times \mathbb{P}^1$ with rulings ℓ_1 and ℓ_2
- $\widetilde{S}|_{\widetilde{S}} = -\ell_1 - \ell_2$, $K_X|_{\widetilde{S}} = \ell_1 + \ell_2$
- Let $O \in \ell_2$
- Set $Y = \widetilde{S}$, compute τ, $P(u)$, $N(u)$, estimate $\delta_O(\widetilde{S}, P(u)|_{\widetilde{S}})$, and get $\delta_O(X) > 1$.

Proof: $O \notin \widetilde{S}$

- Let $\overline{T}$ be the fiber of ϕ_1 such that $O \in \overline{T}$
- $\overline{T}$ is a del Pezzo surface with at most Du Val singularities
- Set $\tau = \sup\{u \in \mathbb{R}_{>0} \big| -K_X - u\overline{T} \text{ is pseudo-effective}\}$
- For $u \in [0, \tau]$:
  - $P(u)$ be the positive part of the ZD of the divisor $-K_X - u\overline{T}$
  - $N(u)$ be its negative part of the ZD of the divisor $-K_X - u\overline{T}$
- $$P(u) = \begin{cases} -K_X - u\overline{T},\ u \in [0,1], \\ -K_X - u\overline{T} - (u-1)\widetilde{S}\ ,\ u \in [1,2] \end{cases} \quad N(u) = \begin{cases} 0\ ,\ u \in [0,1], \\ (u-1)\widetilde{S}\ ,\ u \in [1,2], \end{cases}$$
- $$S_X(\overline{T}) = \frac{1}{20} \int_0^2 P(u)^3 du = \frac{69}{80} < 1$$

Proof: $O \notin \widetilde{S}$

Since $O \notin \widetilde{S}$ then for any divisor F over $\overline{T}$ we get

$$S(W^{\overline{T}}_{\bullet,\bullet};F) = \frac{3}{(-K_X)^3}\Bigg(\int_0^\tau \big(P(u)^2\cdot\overline{T}\big)\cdot \mathrm{ord}_O\Big(N(u)\big|_{\overline{T}}\Big)du +$$

$$+\int_0^\tau\int_0^\infty \mathrm{vol}\big(P(u)\big|_{\overline{T}} - vF\big)dvdu\Bigg) = \frac{3}{20}\int_0^\tau\int_0^\infty \mathrm{vol}\big(P(u)\big|_{\overline{T}} - vF\big)dvdu =$$

$$= \frac{3}{20}\Bigg(\int_0^1\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}} - vF\big)dvdu + \int_1^2\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}} - (u-1)\overline{C}_2 - vF\big)dvdu\Bigg) =$$

$$= \frac{3}{20}\Bigg(\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}} - vF\big)dv + \int_0^\infty \mathrm{vol}\big(-K_{T} - (u-1)\overline{C}_2 - vF\big)dv\Bigg) \leq$$

$$= \frac{3}{20}\Bigg(\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}} - vF\big)dv + \int_0^\infty \mathrm{vol}\big(-K_{\overline{T}} - vF\big)dv\Bigg) =$$

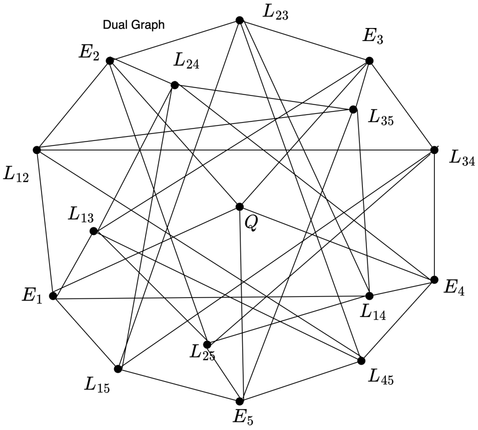
$$= \frac{3}{10}\Bigg(\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}} - vF\big)dv\Bigg) = \frac{6}{5}\Bigg(\frac{1}{4}\int_0^\infty \mathrm{vol}\big(-K_{\overline{T}} - vF\big)dv\Bigg) =$$

$$= \frac{6}{5}S_{\overline{T}}(F) \leq \frac{6}{5}\cdot\frac{A_{\overline{T}}(F)}{\delta_O(\overline{T})}$$

So if $\delta_O(\overline{T}) > 6/5$, then $\delta_O(X) > 1$.

Smooth dP_4

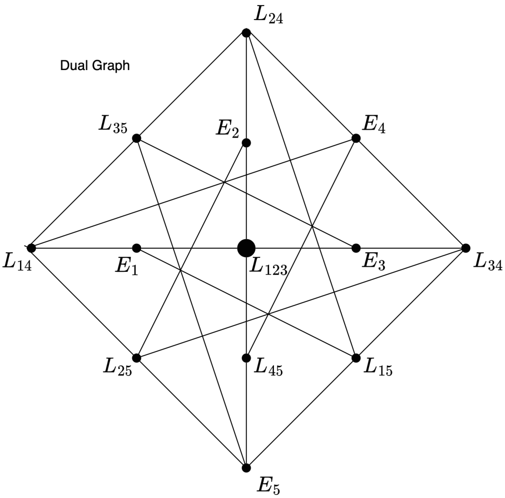
$$\delta_P(T) = \begin{cases} \frac{4}{3} \text{ if } P \in \text{two } (-1)\text{-curves}, \\ \frac{18}{13} \text{ if } P \in \text{one } (-1)\text{-curve}, \\ \frac{3}{2}, \text{ otherwise.} \end{cases}$$



$\mathbb{A}_1$

$\mathbf{E} := E_1 \cup E_2 \cup E_3 \cup L_{45}$,
$\mathbf{F} := \{(-1)\text{-curves}\} \backslash \mathbf{E}$ then

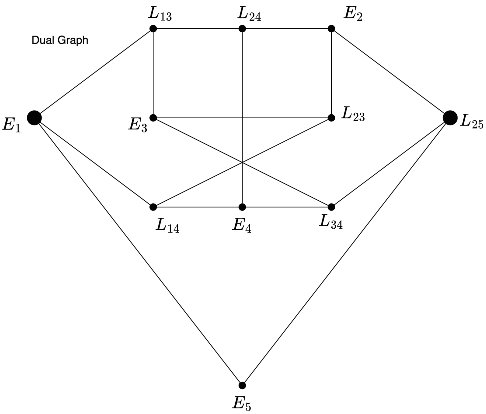
$$\delta_P(T) = \begin{cases} 1 \text{ if } P \in L_{123}, \\ \frac{6}{5} \text{ if } P \in \mathbf{E} \backslash L_{123}, \\ \frac{4}{3} \text{ if } P \in \text{two curves in } \mathbf{F}, \\ \frac{18}{13} \text{ if } P \in \text{one curve in } \mathbf{F}, \\ \frac{3}{2}, \text{ otherwise} \end{cases}$$



$\mathbb{A}_1 + \mathbb{A}_1$ (1)

Suppose
$\mathbf{E} := E_2 \cup L_{13} \cup L_{14} \cup L_{34}$,
$\mathbf{F} := (L_{23} \cap E_2) \cup (L_{24} \cap E_4)$,
$\mathbf{G} := L_{23} \cup E_2 \cup L_{24} \cup E_4$

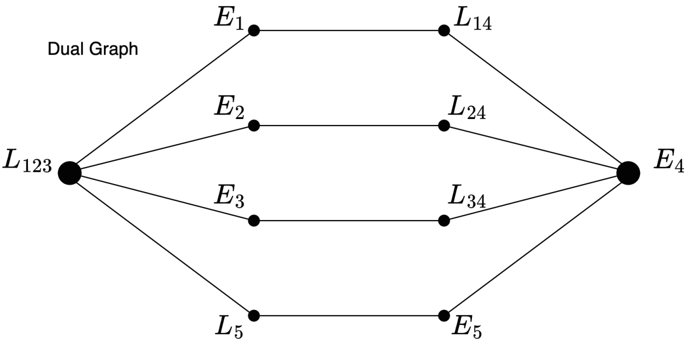
$$\delta_P(T) = \begin{cases} 1 \text{ if } P \in E_1 \cup L_{25} \cup E_5, \\ \frac{6}{5} \text{ if } P \in \mathbf{E} \backslash (E_1 \cup L_{25}), \\ \frac{4}{3} \text{ if } P \in \mathbf{F}, \\ \frac{18}{13} \text{ if } P \in \mathbf{G} \backslash (\mathbf{E} \cup \mathbf{F}), \\ \frac{3}{2}, \text{ otherwise} \end{cases}$$



$\mathbb{A}_1 + \mathbb{A}_1$ (2)

Suppose $\mathbf{E} := E_1 \cup E_2 \cup E_3 \cup E_5 \cup L_{14} \cup L_{24} \cup L_{34} \cup L_5$,

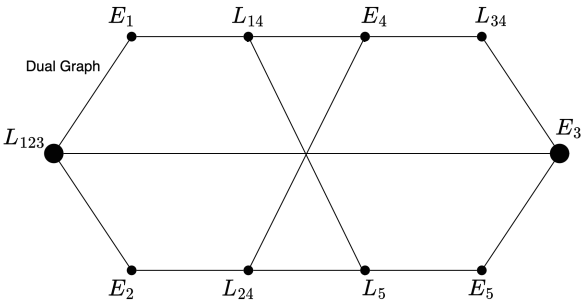
$$\delta_P(T) = \begin{cases} 1 \text{ if } P \in E_4 \cup L_{123}, \\ \frac{6}{5} \text{ if } P \in \mathbf{E} \backslash (E_4 \cup L_{123}), \\ \frac{3}{2}, \text{ otherwise} \end{cases}$$



$\mathbb{A}_2$

Suppose $\mathbf{E} := L_{123} \cup E_3$, $\mathbf{F} := E_1 \cup E_2 \cup E_5 \cup L_{34}$, $\mathbf{G} := (E_4 \cup L_5) \cap (L_{14} \cup L_{24})$, $\mathbf{H} := E_4 \cup L_{14} \cup L_{24} \cup L_5$

$$\delta_P(T) = \begin{cases} \frac{6}{7} & \text{if } P \in \mathbf{E}, \\ \frac{8}{7} & \text{if } P \in \mathbf{F} \backslash \mathbf{E}, \\ \frac{4}{3} & \text{if } P \in \mathbf{G}, \\ \frac{18}{13} & \text{if } P \in \mathbf{H} \backslash \mathbf{G}, \\ \frac{3}{2}, & \text{otherwise} \end{cases}$$

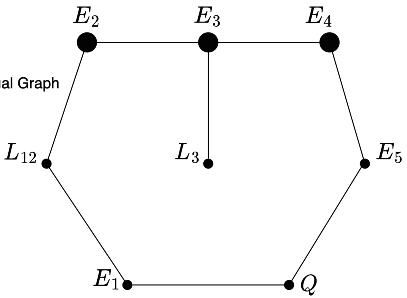


$\mathbb{A}_3$ (1)

Suppose $\mathbf{E} := E_2 \cup E_4$, $\mathbf{F} := E_5 \cup L_{12}$, $\mathbf{G} := E_1 \cup Q$

$$\delta_P(T) = \begin{cases} \frac{3}{4} \text{ if } P \in E_3, \\ \frac{24}{29} \text{ if } P \in \mathbf{E} \backslash E_3, \\ \frac{12}{11} \text{ if } P \in \mathbf{F} \backslash \mathbf{E}, \\ 1 \text{ if } P \in L_3 \backslash E_3, \\ \frac{4}{3} \text{ if } P = E_1 \cap Q, \\ \frac{18}{13} \text{ if } P \in \mathbf{G} \backslash (\mathbf{F} \cup (E_1 \cap Q)), \\ \frac{3}{2}, \text{ otherwise} \end{cases}$$

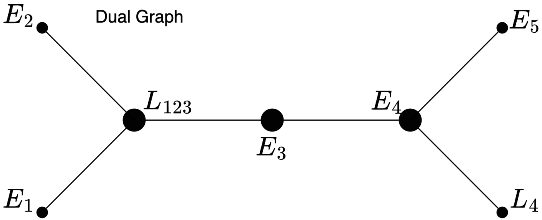
Dual Graph



$\mathbb{A}_3$ (2)

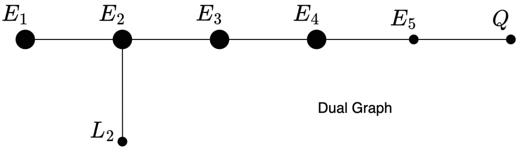
Suppose $\mathbf{E} := L_{123} \cup E_3 \cup E_4$, $\mathbf{F} := E_1 \cup E_2 \cup E_5 \cup L_4$

$$\delta_P(T) = \begin{cases} \frac{3}{4} & \text{if } P \in \mathbf{E}, \\ \frac{9}{8} & \text{if } P \in \mathbf{F} \setminus \mathbf{E}, \\ \frac{3}{2}, & \text{otherwise} \end{cases}$$



$\mathbb{A}_4$

$$\delta_P(T) = \begin{cases} \frac{6}{11} & \text{if } P \in E_2, \\ \frac{24}{37} & \text{if } P \in E_3 \backslash E_2, \\ \frac{4}{5} & \text{if } P \in E_4 \backslash E_3, \\ \frac{24}{29} & \text{if } P \in L_2 \backslash E_2, \\ \frac{9}{11} & \text{if } P \in E_1 \backslash E_2, \\ \frac{24}{23} & \text{if } P \in E_5 \backslash E_4, \\ \frac{18}{13} & \text{if } P \in Q \backslash E_5, \\ \frac{3}{2}, & \text{otherwise} \end{cases}$$



Dual Graph

Proof

We see that if $\overline{T}$ is smooth then $\delta_O(\overline{T}) > \frac{6}{5}$ so $\delta_O(X) > 1$.
So we may assume that $O \notin \widetilde{S}$ and $\overline{T}$ is singular.
Recall that

$$\delta_O\big(\overline{T}, \overline{D}\big) = \inf_{\substack{F/\overline{T} \\ O \in C_{\overline{T}}(F)}} \frac{A_{\overline{T}}(F)}{S_{\overline{D}}(F)} \text{ where } S_{\overline{D}}(F) = \frac{1}{\overline{D}^2} \int_0^{\tau} \mathrm{vol}\big(\overline{D} - vF\big)\, dv$$

where τ is the pseudo-effective threshold of F with respect to $\overline{D}$.

Proof: $\delta_O(\overline{T}) \leq 6/5$

We will prove that $\delta_O(\overline{T}, \overline{D}) \geq f(u)$ for every $u \in [1, 2]$:

$$f(u) = \begin{cases} \dfrac{15 - 3u^2}{16 + 3u - 9u^2 + 2u^3} & \text{for } u \in [1, a] \\ \dfrac{15 - 3u^2}{11 - u^3} & \text{for } u \in [a, 2] \end{cases}$$

$$S(W^{\overline{T}}_{\bullet,\bullet}; F) =$$

$$= \frac{3}{(-K_X)^3} \int_1^2 \int_0^\tau \mathrm{vol}\big(P(u)\big|_{\overline{T}} - vF\big) dv du + \frac{3}{(-K_X)^3} \int_0^1 \int_0^\tau \mathrm{vol}\big(P(u)\big|_{\overline{T}} - vF\big) dv du \leq$$

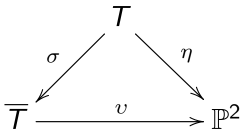
$$\leq \frac{3}{20} \left(\int_1^2 \frac{(5 - u^2)}{\delta_O(\overline{T}, \overline{D})} du \right) A_{\overline{T}}(F) + \frac{3}{20} \cdot \frac{4A_{\overline{T}}(F)}{\delta_O(\overline{T})} \leq$$

$$\leq \frac{3}{20} \left(\int_1^2 \frac{(5 - u^2)}{f(u)} du \right) A_{\overline{T}}(F) + \frac{3}{5} A_{\overline{T}}(F) \leq \frac{99}{100} A_{\overline{T}}(F)$$

Thus $\frac{A_{\overline{T}}(F)}{S\left(W^{\overline{T}}_{\bullet,\bullet}; F\right)} \geq \frac{100}{99}$ for $\forall$ prime F over $\overline{T}$, $O \in C_{\overline{T}}(F)$ so that $\delta_O(\overline{T}, \overline{D}) \geq \frac{100}{99}$ and X is K-stable.

Proof that $\delta_O(\overline{T}, \overline{D}) \geq f(u)$?

- $\overline{T}$ is a Du Val del Pezzo surface
- blow up π induces a birational morphism $\upsilon : \overline{T} \to \mathbb{P}^2$ which is weighted blow up:



- Suppose $u \in [1, 2]$:
  - $\overline{D} = -K_{\overline{T}} - (1-u)\overline{C}_2$ where $\overline{C}_2 := \widetilde{S}|_{\overline{T}}$
  - $\overline{C}_2$ is contained in the smooth locus of the surface $\overline{T}$
  - C_2 is the strict transform of the curve $\overline{C}_2$ on the surface T
  - $D = -K_T - (1-u)C_2 = \sigma^*(\overline{D})$ so D is big and nef and $D^2 = 5 - u^2$ for $u \in [1, 2]$

Reminder: δ-invariant

Recall that

$$\delta_O(\overline{T}, \overline{D}) = \inf_{\substack{F/\overline{T} \\ O \in C_{\overline{T}}(F)}} \frac{A_{\overline{T}}(F)}{S_D(F)}$$

where the infimum is run over all prime divisor F over $\overline{T}$ such that $O \in C_{\overline{T}}(F)$. For every point $P \in T$, we also define

$$\delta_P(T, D) = \inf_{\substack{E/T \\ P \in C_T(E)}} \frac{A_T(E)}{S_D(E)}$$

where the infimum is run over all prime divisor E over T such that $P \in C_T(E)$. Since $D = \sigma^*(\overline{D})$ and $K_T = \sigma^*(K_{\overline{T}})$, we have

$$\delta_O(\overline{T}, \overline{D}) = \inf_{P: O = \sigma(P)} \delta_P(T, D)$$

So, to estimate $\delta_O(\overline{T}, \overline{D})$ it is enough to estimate $\delta_P(T, D)$ for P all points P such that $\sigma(P) = O$.

Reminder: how to estimate $\delta_P(S, D)$?

- Fix a smooth curve $\mathcal{C} \subset T$ that passes through P.
- $\tau = \sup\{v \in \mathbb{R}_{\geqslant 0} \mid \text{the divisor } D - v\mathcal{C} \text{ is pseudo-effective}\}$
- For $v \in [0, \tau]$, let $P(v)$ and $N(v)$ be the positive part and negative of the Zariski decomposition of the divisor $D - vC$.
- Then $A_S(\mathcal{C}) = 1$ and $S_D(\mathcal{C}) = \frac{1}{D^2} \int\limits_0^\infty \mathrm{vol}(D - v\mathcal{C}) dv$

Thus

$$\boxed{\delta_P(T, D) \leqslant \frac{1}{S_D(\mathcal{C})}}$$

- Set $S(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{D^2} \int_0^\tau h(v) dv$ where

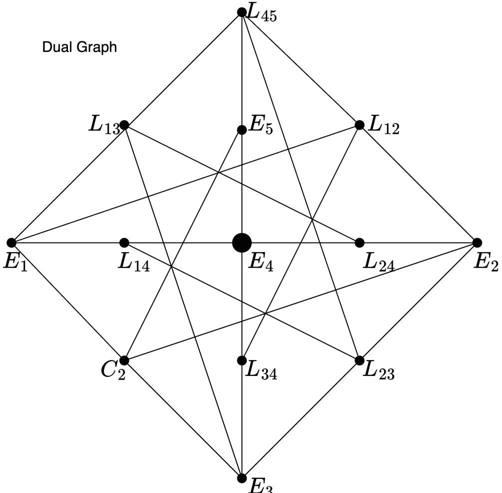
$$h_D(v) = \big(P(v) \cdot \mathcal{C}\big) \times \big(N(v) \cdot \mathcal{C}\big)_P + \frac{\big(P(v) \cdot \mathcal{C}\big)^2}{2}.$$

Then it follows from Abban-Zhuang Theory that

$$\boxed{\delta_P(T, D) \geqslant \min\left\{\frac{1}{S_D(\mathcal{C})}, \frac{1}{S(W^{\mathcal{C}}_{\bullet,\bullet}; P)}\right\}.}$$

One singular point of type $\mathbb{A}_1$

- $\overline{T}$ has one singular point of type $\mathbb{A}_1$
- blow up of $\mathbb{P}^2$ at points P_1, P_2, P_3 and P_4 in general position and a point P_5 in the exceptional divisor corresponding to P_4
- $\delta_P(T) \leq \frac{6}{5} \Leftrightarrow P \in E_4 \cup L_{14} \cup L_{24} \cup L_{24} \cup E_5$



One singular point of type $\mathbb{A}_1$

Suppose $P \in E_4$:

$$P(v) = \begin{cases} -K_T - (u-1)C_2 - vE_4 \text{ for } v \in [0, 2-u] \\ -K_T - (u-1)C_2 - vE_4 - (u+v-2)E_5 \text{ for } v \in [2-u, 1] \\ -K_T - (u-1)C_2 - vE_4 - (u+v-2)E_5 - (v-1)(L_{14} + L_{24} + L_{34}) \text{ for } v \in [1, 3-u] \end{cases}$$

$$N(v) = \begin{cases} 0 \text{ for } v \in [0, 2-u] \\ (u+v-2)E_5 \text{ for } v \in [2-u, 1] \\ (u+v-2)E_5 + (v-1)(L_{14} + L_{24} + L_{34}) \text{ for } v \in [1, 3-u] \end{cases}$$

$$P(v)^2 = \begin{cases} 5 - u^2 - 2v^2 \text{ for } v \in [0, 2-u] \\ 9 + 2uv - 4u - 4v - v^2 \text{ for } v \in [2-u, 1] \\ 2(2-v)(3-u-v) \text{ for } v \in [1, 3-u] \end{cases} \quad \text{and } P(v) \cdot E_4 = \begin{cases} 2v \text{ for } v \in [0, 2-u] \\ 2 - u + v \text{ for } v \in [2-u, 1] \\ 5 - u - 2v \text{ for } v \in [1, 3-u] \end{cases}$$

Thus,

$$S_D(E_4) = \frac{1}{5-u^2}\Big(\int_0^{2-u} 5 - u^2 - 2v^2 dv + \int_{2-u}^{1} 9 + 2uv - 4u - 4v - v^2 dv + \\ + \int_1^{3-u} 2(2-v)(3-u-v)dv\Big) = \frac{16 + 3u - 9u^2 + 2u^3}{15 - 3u^2}$$

and $\delta_P(T, D) \leq \frac{15-3u^2}{16+3u-9u^2+2u^3}$

One singular point of type $\mathbb{A}_1$

- if $P \in E_4 \backslash (E_5 \cup L_{14} \cup L_{24} \cup L_{34})$

$$h_D(v) = \begin{cases} 2v^2 \text{ for } v \in [0, 2-u] \\ \frac{(2-u+v)^2}{2} \text{ for } v \in [2-u, 1] \\ \frac{(5-u-2v)^2}{2} \text{ for } v \in [1, 3-u] \end{cases}$$

- if $P = E_4 \cap E_5$

$$h_D(v) = \begin{cases} 2v^2 \text{ for } v \in [0, 2-u] \\ \frac{(2-u+v)(u+3v-2)}{2} \text{ for } v \in [2-u, 1] \\ \frac{(u+1)(5-u-2v)}{2} \text{ for } v \in [1, 3-u] \end{cases}$$

- if $P \in E_4 \cap (L_{14} \cup L_{24} \cup L_{34})$

$$h_D(v) = \begin{cases} 2v^2 \text{ for } v \in [0, 2-u] \\ \frac{(2-u+v)^2}{2} \text{ for } v \in [2-u, 1] \\ \frac{(3-u)(5-u-2v)}{2} \text{ for } v \in [1, 3-u] \end{cases}$$

One singular point of type $\mathbb{A}_1$

So we have

- if $P \in E_4 \backslash (E_5 \cup L_{14} \cup L_{24} \cup L_{34})$ then

$$S_D(W^{E_4}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} 2v^2 dv + \int_{2-u}^{1} \frac{(2-u+v)^2}{2} dv + \int_1^{3-u} \frac{(5-u-2v)^2}{2} dv\Big) = = \frac{9+6u-9u^2+2u^3}{15-3u^2} \leq \frac{16+3u-9u^2+2u^3}{15-3u^2}$$

- if $P = E_4 \cap E_5$ then

$$S_D(W^{E_4}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} 2v^2 dv + \int_{2-u}^{1} \frac{(2-u+v)(u+3v-2)}{2} dv + + \int_1^{3-u} \frac{(u+1)(5-u-2v)}{2} dv\Big) = \frac{11-u^3}{15-3u^2}$$

- if $P \in E_4 \cap (L_{14} \cup L_{24} \cup L_{34})$ then

$$S_D(W^{E_4}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} 2v^2 dv + \int_{2-u}^{1} \frac{(2-u+v)^2}{2} dv + + \int_1^{3-u} \frac{(3-u)(5-u-2v)}{2} dv\Big) = = \frac{13+3u^3-12u^2+6u}{15-3u^2} \leq \frac{16+3u-9u^2+2u^3}{15-3u^2}$$

One singular point of type $\mathbb{A}_1$

We obtain that

$$\delta_P(T,D) = \frac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } P \in E_4 \backslash E_5 \text{ and } u \in [1,2]$$

and

$$\delta_P(T,D) \geq \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} & \text{for } P = E_4 \cap E_5 \text{ and } u \in [1,a] \\ \dfrac{15-3u^2}{11-u^3} & \text{for } P = E_4 \cap E_5 \text{ and } u \in [a,2] \end{cases}$$

where a is a root of $3u^3 - 9u^2 + 3u + 5$ on $[1,2]$. Note that $a \in [1.355, 1.356]$.