The 3-dimensional Lyness map and an explicit mirror for the Fano 3-fold V_{12}

Nottingham algebraic geometry seminar

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The Lyness map

The Lyness map

The *d*-dimensional Lyness map is the birational map $\sigma_d \in \mathrm{Bir}(\mathbb{C}^d)$ given by

$$\sigma_d(x_1, x_2, \ldots, x_{d-1}, x_d) = \left(x_2, x_3 \ldots, x_d, \frac{1 + x_2 + \cdots + x_d}{x_1}\right)$$

If we iterate by $\sigma_d^{\pm 1}$ we can define a sequence of rational functions $(x_i \in \mathbb{C}\,(x_1, \ldots, x_d) : i \in \mathbb{Z})$ where

$$x_i x_{i+d} = 1 + x_{i+1} + \cdots + x_{i+d-1} \qquad \forall i \in \mathbb{Z}$$

is the *d*-dimensional Lyness recurrence relation.

Behaviour in low dimensions

When $d = 2$ the recurrence relation is 5-periodic

$$x_1,\ x_2,\ x_3 = \frac{1+x_2}{x_1},\ x_4 = \frac{1+x_1+x_2}{x_1x_2},\ x_5 = \frac{1+x_1}{x_2},\ x_6 = x_1, \ldots.$$

When $d = 3$ the recurrence relation is 8-periodic

$$x_1,\ x_2,\ x_3,\ x_4 = \frac{1+x_2+x_3}{x_1},\ x_5 = \frac{1+x_1+x_2+x_3+x_1x_3}{x_1x_2},$$

$$x_6 = \frac{(1+x_1+x_2)(1+x_2+x_3)}{x_1x_2x_3},\ x_7 = \frac{1+x_1+x_2+x_3+x_1x_3}{x_2x_3},$$

$$x_8 = \frac{1+x_1+x_2}{x_3},\ x_9 = x_1 \ldots.$$

Also note that there is a Laurent phenomenon, i.e.

$$x_i \in \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_d^{\pm 1}\right] \subset \mathbb{C}(x_1, \ldots, x_d) \qquad \forall i \in \mathbb{Z}.$$

Integrability

When $d \leq 3$ this is an integrable system—in other words, this recurrence has the maximum number $d-1$ of first integrals (functionally independent invariant functions).

When $d \geq 4$ the recurrence relation is neither periodic, nor possesses a Laurent phenomenon. It is no longer integrable, but it does still preserve a system of $\lfloor \frac{d+1}{2} \rfloor$ Laurent polynomials (Tran–van der Kamp–Quispel).

Dimension 2: del Pezzo surface dP_5

The del Pezzo surface of degree 5

Recall the five functions from the 2-dimensional recurrence

$$x_1,\ x_2,\ x_3 = \frac{1+x_2}{x_1},\ x_4 = \frac{1+x_1+x_2}{x_1 x_2},\ x_5 = \frac{1+x_1}{x_2}.$$

As is well-known, these are coordinates on an affine del Pezzo surface U of degree 5

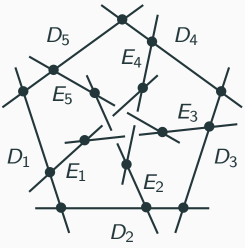
$$U = \operatorname{Spec}\left(\frac{\mathbb{C}[x_1,\dots,x_5]}{(x_{i-1}x_{i+1} = x_i + 1 : i \in \mathbb{Z}/5\mathbb{Z})}\right) \subset \mathbb{A}^5.$$

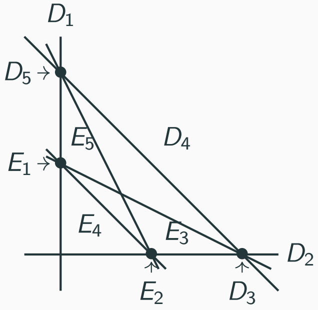
The projective closure $Y = \overline{U} \subset \mathbb{P}^5$ is a (projective) dP_5 where the complement $Y \setminus U$ is a pentagon of lines $D = \sum_{i=1}^{5} D_i$.

The configuration of lines inside Y

Note that $U \subset \mathbb{A}^5$ contains five straight lines $E_i = U \cap \{x_i = 0\}$, obtained by intersecting U with a coordinate hyperplane.

Taken together with D, these ten lines (i.e. (-1)-curves) intersect in a very beautiful configuration, obtained by blowing up $\mathbb{P}^2$ in the four points shown on the right.





Three themes

We want to pull out three themes from this example which will generalise to the dimension 3 case:

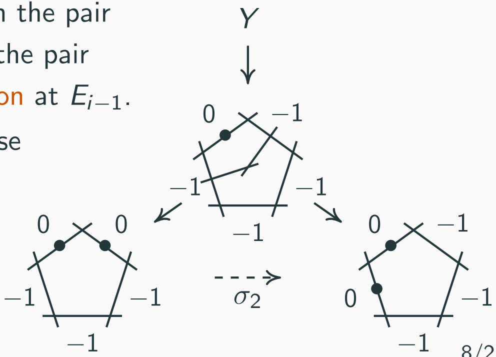
1. U is a cluster variety,
2. U *'comes from'* the Grassmannian $\mathrm{Gr}(2,5)$,
3. U can be used to construct a mirror for dP_5.

1. U is a cluster variety

The variety U is a cluster variety, i.e. it is the interior of a log Calabi–Yau pair (Y, D) which admits a toric model $\pi\colon (Y, D) \to (\overline{Y}, \overline{D})$. In other words we can blow down two disjoint (-1)-curves $\{E_i, E_{i+1}\}$ inside U to get a map to a toric pair.

Changing from blowing down the pair $\{E_{i-1}, E_i\}$ to blowing down the pair $\{E_i, E_{i+1}\}$ is called a mutation at E_{i-1}.

The induced map on the dense open torus is the Lyness map $\sigma_2(x_{i-1}, x_i) = (x_i, \frac{1+x_i}{x_{i-1}})$.



2. Relationship with $\mathrm{Gr}(2,5)$

We can write the equations of U as the 4×4 Pfaffians of a 5×5 skew matrix

$$\mathrm{Pfaff}_4 \begin{pmatrix} 1 & x_1 & x_4 & 1 \\ & 1 & x_2 & x_5 \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix} \xrightarrow[\text{nicely}]{\text{homogenise}} \mathrm{Pfaff}_4 \begin{pmatrix} y_3 & x_1 & x_4 & y_2 \\ & y_4 & x_2 & x_5 \\ & & y_5 & x_3 \\ & & & y_1 \end{pmatrix}$$

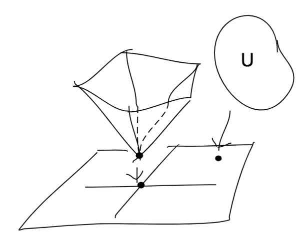
to get a homogeneous recurrence relation

$$x_{i-1}x_{i+1} = x_i y_i + y_{i-2}y_{i+2} \qquad i = 1, \dots, 5$$

The resulting variety $\mathcal{U} \subset \mathbb{A}^{10}_{x_i,y_i}$ is the affine cone over the Grassmannian $\mathrm{Gr}(2,5)$.

Consider the projection $\pi\colon \mathcal{U} \to \mathbb{A}^5_{y_1,\ldots,y_5}$, which is a fibration of affine del Pezzo surfaces.

Clearly we have $U = \pi^{-1}(1,\ldots,1)$, but in fact all of the fibres of π over $(\mathbb{C}^\times)^5 \subset \mathbb{A}^5$ are isomorphic. They start to degenerate over the coordinate strata, with the 'worst' fibre being $\pi^{-1}(0,\ldots,0)$ a cycle of five coordinate planes.



Consider the invariant Laurent polynomial

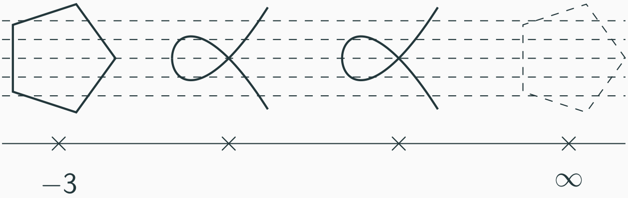
$$\begin{aligned} w &= x_1 + x_2 + x_3 + x_4 + x_5 \\ &= \frac{(1+x_1)(1+x_1+x_2)(1+x_2)}{x_1 x_2} - 3 \end{aligned}$$

and the corresponding fibration $w\colon U \to \mathbb{A}^1$. We see that the two complementary anticanonical pentagons in Y appear as fibres $w^{-1}(-3) = E$ and $w^{-1}(\infty) = D$.

Extending to the compactified variety $w\colon Y \dashrightarrow \mathbb{P}^1$, the map w has five basepoints which are given by the five points of $D \cap E$.

3. Mirror symmetry for dP_5

Blowing these five points up gives an elliptic fibration with four singular fibres. The dashed locus is everything not contained in U.



Thus the fibres of $w\colon U \to \mathbb{A}^1$ are elliptic curves with five points deleted. This fibration is the Landau–Ginzburg model which is mirror to dP_5 with an anticanonical section $s \in H^0(dP_5, -K_{dP_5})$ such that $\operatorname{div} s$ is a pentagon of (-1)-curves. The five deleted sections correspond to the five nodes of $\operatorname{div} s$.

Dimension 3: the Fano 3-fold V_{12}

The 3-dimensional Lyness recurrence

Recall the eight functions of the 3-dimensional recurrence

$$x_1,\ x_2,\ x_3,\ x_4 = \frac{1+x_2+x_3}{x_1},\ x_5 = \frac{1+x_1+x_2+x_3+x_1x_3}{x_1x_2},\ \text{etc.}$$

What is the 3-fold $U = \mathrm{Spec}(\mathbb{C}[x_1,\dots,x_8]) \subset \mathbb{A}^8$?

Just as the affine del Pezzo surface turned out to be a 'dehomogenisation' of the Grassmannian $\mathrm{Gr}(2,5)$, it turns out that there is another type of Grassmannian lurking in the background here.

The orthogonal Grassmannian $\mathrm{OGr}(4,9)$

The *orthogonal Grassmannian* $\mathrm{OGr}(4,9)$ parameterises 4-planes in $\mathbb{C}^9$ which are isotropic with respect to a given quadratic form (or equivalently, one of the two isomorphic connected components of $\mathrm{OGr}(5,10)$).

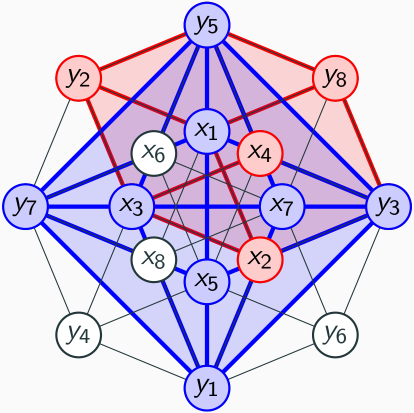
The relevant Lie group is $SO(9)$ of type B_4, with 16-dimensional spin representation $S = \bigoplus_{i=0}^{4} \bigwedge^i \mathbb{C}^4 \cong \mathbb{C}^{16}$.

The weight polytope in the weight lattice for B_4 is a 4-dimensional hypercube C with vertices $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$.

The Weyl group $W(B_4)$ is the symmetry group of C and the Coxeter element is a rotation of C of order 8.

The equations of $\mathrm{OGr}(4,9)$

To define $\mathrm{OGr}(4,9) \subset \mathbb{P}^{16}$ call the 16 variables $x_1, \ldots, x_8$ and $y_1, \ldots, y_8$ according to the following labelling of the vertices of C.



Now $\mathrm{OGr}(4,9)$ has eight equations corresponding to the 3-cube faces of C:

$x_1x_4 = x_2y_5 + x_3y_8 + y_2y_3$, etc.

and two equations corresponding to bipartite decompositions:

$x_1x_5 - x_3x_7 = y_1y_5 - y_3y_7$, etc.

Summary of the equations

Let $\mathcal{U} \subset \mathbb{A}^{16}$ be the affine cone over OGr$(4, 9)$. Then summarising the last slide, $\mathcal{U}$ is defined by the ten equations

$$x_1x_4 = x_2y_5 + x_3y_8 + y_2y_3 \qquad x_5x_8 = x_6y_1 + x_7y_4 + y_6y_7$$
$$x_2x_5 = x_3y_6 + x_4y_1 + y_3y_4 \qquad x_6x_1 = x_7y_2 + x_8y_5 + y_7y_8$$
$$x_3x_6 = x_4y_7 + x_5y_2 + y_4y_5 \qquad x_7x_2 = x_8y_3 + x_1y_6 + y_8y_1$$
$$x_4x_7 = x_5y_8 + x_6y_3 + y_5y_6 \qquad x_8x_3 = x_1y_4 + x_2y_7 + y_1y_2$$

$$x_1x_5 - x_3x_7 = y_1y_5 - y_3y_7 \qquad x_2x_6 - x_4x_8 = y_2y_6 - y_4y_8$$

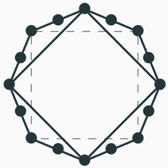
Note that this a nice homogenisation of the 3-dimensional Lyness recurrence.

A fibration of affine Fano 3-folds

Consider the projection $\pi\colon \mathcal{U} \to \mathbb{A}^8_{y_1,\dots y_8}$. This is a flat family of affine Fano 3-folds of type V_{12} (an intersection of $\mathrm{OGr}(4,9)$ with seven hyperplane sections). However these are *special* V_{12}s, since the projective closure of each of fibre is *very singular*.

Proposition. For all $\lambda, \mu \in \mathbb{C}^\times$, the fibres of π are all isomorphic over $\{\frac{y_1y_5}{y_3y_7} = \lambda, \frac{y_2y_6}{y_4y_8} = \mu\} \subset (\mathbb{C}^\times)^8_{y_i}$. Call this fibre $U_{\lambda,\mu} \subset \mathbb{A}^8_{x_i}$.

The projective closure $\overline{U}_{\lambda,\mu} \subset \mathbb{P}^{16}$ is a (non-$\mathbb{Q}$-factorial) Fano 3-fold of type V_{12}. It has boundary divisor with ten components meeting as follows:



$8 \times \mathbb{P}^2$

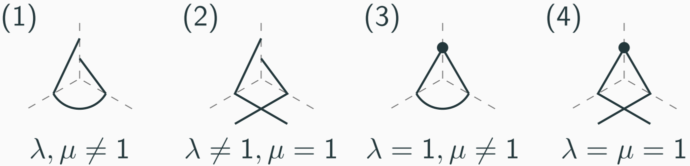
$2 \times \mathbb{P}^1 \times \mathbb{P}^1$

$16 \times \bullet$ nodes

$U_{\lambda,\mu}$ as a cluster variety

By considering the projection $p\colon U_{\lambda,\mu} \to \mathbb{A}^3_{x_1,x_2,x_3}$ we see that for all $\lambda, \mu \neq 1$ the 3-fold $U_{\lambda,\mu}$ is given by blowing up two lines $L_1 \subset \{x_1 = 0\}$, $L_3 \subset \{x_3 = 0\}$ and a conic $C_2 \subset \{x_2 = 0\}$ in $\mathbb{A}^3$ and deleting the strict transform of the coordinate axes.

If $\mu = 1$ then the conic C_2 splits into two lines. If $\lambda = 1$ then the two lines L_1, L_3 touch in the x_2-axis, and p also *blows up an embedded point* at $L_1 \cap L_3$.



Mutations in $U_{\lambda,\mu}$

Consider the generic case $\lambda, \mu \neq 1$. Then we have three exceptional divisors $\{E_1, E_2, E_3\}$ in the projection $U \to \mathbb{A}^3_{x_1,x_2,x_3}$, which dominate L_1, C_2, L_3 respectively. The homogenised Lyness map

$$(x_1, x_2, x_3) \mapsto \left(\frac{x_2 y_5 + x_3 y_8 + y_2 y_3}{x_1}, x_2, x_3 \right)$$

is the mutation at E_1 and similarly for the mutation at L_3.

But what about the mutation of E_2? It is given by

$$(x_1, x_2, x_3) \mapsto \left(x_1, \frac{x_1 x_3 y_6 + x_1 y_3 y_4 + x_3 y_8 y_1 + y_1 y_2 y_3}{x_2}, x_3 \right)$$

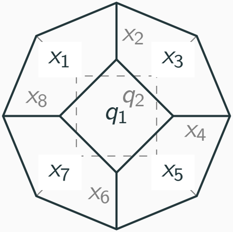
where the numerator is the equation defining the conic C_2.

Mutations in $U_{\lambda,\mu}$

Amazingly, the new cluster variable

$$q_1 := \frac{x_1 x_3 y_6 + x_1 y_3 y_4 + x_3 y_8 y_1 + y_1 y_2 y_3}{x_2}$$

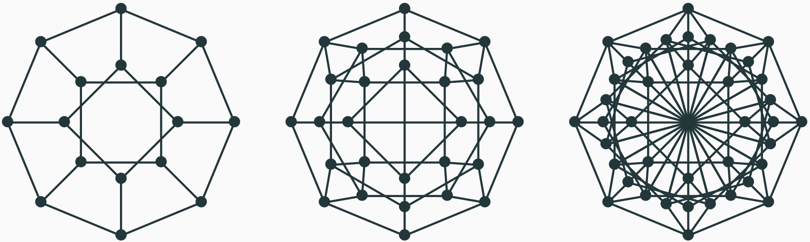
turns out to be equivalent to $q_1 = x_1 x_5 - y_1 y_5 = x_3 x_7 - y_3 y_7$. By similarly adding in $q_2 = x_2 x_6 - y_2 y_6 = x_4 x_8 - y_4 y_8$ we get a cluster variety with a closed and finite system of torus charts related by mutations.



Finite number of cluster torus charts

In fact this also holds in the other cases where one or both of $\lambda, \mu = 1$ (even though there are more divisors to mutate).

Proposition. $U_{\lambda,\mu}$ is a rank 3 (resp. 4, 5) cluster variety with 16 (resp. 28, 48) different torus charts if $\lambda, \mu \neq 1$ (resp. one of $\lambda = 1$ or $\mu = 1$, both $\lambda = \mu = 1$). The exchange graphs in each case are given by



A mirror K3 fibration for V_{12}

From now on concentrate on the special fibre $U := U_{1,1}$, for which the two new cluster variables are

$$q_1 = \frac{(1+x_1)(1+x_3)}{x_2} \qquad q_2 = \frac{(1+x_2)(1+x_1+x_2+x_3)}{x_1 x_3}$$

The Lyness-invariant Laurent polynomial

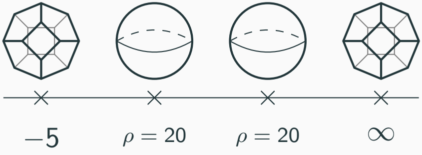
$$\begin{aligned} w &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + q_1 + q_2 \\ &= \frac{(1+x_1+x_2)(1+x_1+x_2+x_3+x_1x_3)(1+x_2+x_3)}{x_1x_2x_3} - 5 \end{aligned}$$

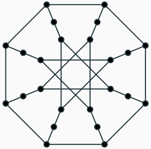
should give a fibration of K3 surfaces on $w\colon U \to \mathbb{A}^1$ which is mirror to the Fano 3-fold V_{12}.

(This is because the classical period of w agrees with the regularised quantum period of V_{12}.)

A mirror K3 fibration for V_{12}

Indeed we can use the explicit equations and the geometry to see this fibration. Resolving w we get a symmetric pencil of K3 surfaces $w\colon U \to \mathbb{A}^1$ with two type III fibres and two fibres of Picard rank 20.





The fibres $w^{-1}(t) \subset U$ have 24 (-2)-curves deleted. The classes of these 24 curves span the lattice $NS(\overline{U}_t)$ of rank 19, such that $H^{1,1}(\overline{U}_t, \mathbb{Z}) = NS(\overline{U}_t) \oplus \langle 12 \rangle$ as expected (since a general hyperplane section of V_{12} is a K3 surface with Néron–Severi lattice $\langle 12 \rangle$).

Mirrors for other Fano 3-folds

Interestingly, $x_1, \ldots, x_8, q_1, q_2$ can be used as building blocks to construct other interesting potentials on U.

Proposition. Consider $w = \sum_{i=1}^{8} \varepsilon_i x_i + \varepsilon_9 q_1 + \varepsilon_{10} q_2$ with coefficients $\varepsilon_i \in \{0, 1\}$. The 1024 possibilities for w give rise to 46 distinct non-degenerate period sequences, of which 20 are period sequences for smooth Fano 3-folds.

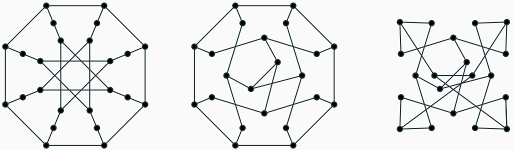
Fano 3-fold	Mirror Laurent polynomial w
V_{12}	$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + q_1 + q_2$
V_{14}	$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + q_1$
V_{16}	$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8$
V_{18}	$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$
V_{22}	$x_1 + x_2 + x_3 + x_4 + x_5 + x_6$
MM_{2-9}	$x_1 + x_2 + x_3 + x_6 + q_1 + q_2$
MM_{2-12}	$x_1 + x_2 + x_3 + x_5 + x_6 + x_7$
$\vdots$	$\vdots$

This can be used to study how the geometry of these fibrations changes, e.g.

$$\begin{array}{ccccccc} \overline{U} \subset \mathbb{P}^{10} & \overset{\widehat{q}_2}{\dashrightarrow} & \overline{U} \subset \mathbb{P}^{9} & \overset{\widehat{q}_1}{\dashrightarrow} & \overline{U} \subset \mathbb{P}^{8} & \overset{\widehat{x}_8}{\dashrightarrow} & \cdots \\ w_{V_{12}} \downarrow & & w_{V_{14}} \downarrow & & w_{V_{16}} \downarrow & & \\ \mathbb{A}^1 & & \mathbb{A}^1 & & \mathbb{A}^1 & & \end{array}$$

We can see how the extra (-2)-curves needed to complete the fibres of the K3 fibration changes.

V_{12} $\quad$ V_{14} $\quad$ V_{16} $\quad$ etc...



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