Quartic surfaces up to volume preserving equivalence

Nottingham algebraic geometry seminar

Tom Ducat

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Durham University

Log Calabi–Yau geometry

Log Calabi–Yau pairs

Definition

A **log Calabi–Yau pair** (X, Δ_X) is a log canonical pair consisting of a proper $\mathbb{Q}$-factorial variety X over $\mathbb{C}$ and a reduced effective integral Weil divisor Δ_X such that $K_X + \Delta_X \sim 0$.

The interior $U = X \setminus \Delta_X$ is called a **log Calabi–Yau variety** and behaves like a non-compact analogue of a Calabi–Yau variety.

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A global section Ω of $H^0(X, K_X + \Delta_X) \cong \mathbb{C}$ defines a holomorphic volume form $\Omega_U := \Omega|_U$ on U, which is uniquely determined up to a scalar and which extends to a volume form on X with simple poles along Δ_X.

An example

Example

Pairs of the form $(\mathbb{P}^2, \Delta)$ where Δ is a cubic curve. The cubic curve Δ can be smooth or singular, but it can only have **at worse nodal singularities** or else the pair won't be log canonical.

As we will see shortly, the behaviour of $U = \mathbb{P}^2 \smallsetminus \Delta$ depends crucially on which of these two cases we are in.

Log canonical centres

If $\pi\colon Z \to X$ is a resolution of singularities, and $E \subset Z$ is a divisor, then the **discrepancy** $a_E(X, \Delta_X) \in \mathbb{Q}$ of E over (X, Δ_X) satisfies

$$K_Z \sim_{\mathbb{Q}} \pi^*(K_X + \Delta_X) + \sum_E a_E(X, \Delta_X)E.$$

A **log canonical centre** (lcc) of (X, Δ_X) is the image $\pi(E) \subset X$ of a divisor E over X with discrepancy $a_E(X, \Delta_X) = -1$.

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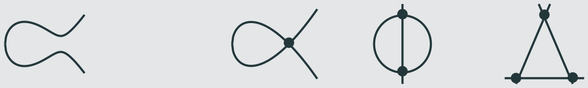
Example

If X is smooth and Δ_X is snc, then the lccs of (X, Δ_X) are given by the strata of Δ_X. So, for our previous example, $(\mathbb{P}^2, \Delta)$ has more lccs if Δ is nodal.



Volume preserving maps

The natural notion of birational equivalence between LCY pairs is **volume preserving** (vp) equivalence.

Definition

A proper birational morphism $f\colon (Z, \Delta_Z) \to (X, \Delta_X)$ is **vp** if $f^*(K_X + \Delta_X) \sim K_Z + \Delta_Z$ and $f_*(\Delta_Z) = \Delta_X$.

A birational map $\varphi\colon (Y, \Delta_Y) \dashrightarrow (X, \Delta_X)$ is **vp** if it admits a resolution by vp morphisms f and g.

Properties of vp maps

For any vp map $\varphi\colon (Y, \Delta_Y) \dashrightarrow (X, \Delta_X)$ we have the following.

1. If $U = X \smallsetminus \Delta_X$ and $V = Y \smallsetminus \Delta_Y$ then φ preserves the volume form, i.e. $\Omega_V = \varphi^* \Omega_U$, for (appropriate scalings of) the naturally defined volume forms on each side.
2. φ preserves discrepancies, i.e. for a divisor E over both X and Y we have $a_E(X, \Delta_X) = a_E(Y, \Delta_Y)$.
3. A composition of vp maps is vp.

The coregularity

Since vp maps preserve discrepancies, they must also send lccs onto lccs. This leads to a fundamental vp invariant of (X, Δ_X).

Definition

The **coregularity** $\mathrm{coreg}(X, \Delta_X) := \dim Z$ is the dimension of a minimal lcc $Z \subset \widetilde{X}$, where $\pi\colon (\widetilde{X}, \Delta_{\widetilde{X}}) \to (X, \Delta_X)$ is a vp dlt modification of (X, Δ_X).

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Example

If (X, Δ_X) is a smooth variety with an snc boundary divisor then $\operatorname{coreg}(X, \Delta_X)$ is the dimension of the smallest stratum of Δ_X.

Maximal pairs

The coregularity $c = \operatorname{coreg}(X, \Delta_X)$ always satisfies $0 \le c \le \dim X$, and $c = \dim X$ iff X is a Calabi–Yau variety and $\Delta_X = 0$.

At the opposite end of the spectrum, if $c = 0$ then we say that the pair (X, Δ_X) is **maximal**.

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Example

Toric pairs (X, Δ_X) consisting of a toric variety X with its torus-invariant boundary divisor Δ_X are always maximal.

Maximal LCY pairs are interesting due to their role in the Gross–Siebert program, and for the fantastic properties they are expected to have from mirror symmetry (which generalise some of the fantastic properties of toric varieties).

Toric models

Definition

We say that (X, Δ_X) has a **toric model** if it is vp equivalent to a toric pair.

It is an interesting (but difficult) open problem to give some characterisation of exactly when a maximal LCY pair (X, Δ_X) has a toric model.

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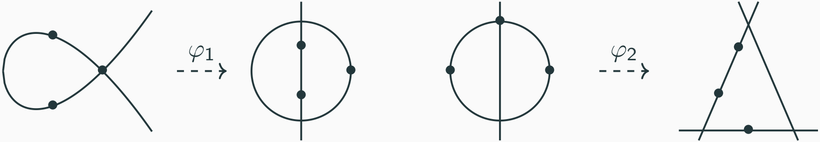
Theorem (Gross–Hacking–Keel)

Every two-dimensional maximal LCY pair (X, Δ_X) has a toric model.

Example revisited

Returning to our example of the pairs $(\mathbb{P}^2, \Delta)$ we see that the coregularity $c = 1$ if Δ is smooth and $c = 0$ if Δ is nodal.

Moreover, we can find explicit vp **Cremona transformations** $\varphi\colon (\mathbb{P}^2, \Delta) \dashrightarrow (\mathbb{P}^2, \Delta')$ between the three maximal cases.



The condition to be volume preserving in this case is that the basepoints of φ **must lie in the boundary**.

Quartic surfaces

Quartic surfaces

We aim to prove the analogous result to the last example for log canonical pairs of the form $(\mathbb{P}^3, \Delta)$, where Δ is a quartic surface. The coregularity can be $c = 0, 1$ or 2.

1. $c = 2$ if and only if $(\mathbb{P}^3, \Delta)$ is **canonical**, i.e. Δ is an irreducible quartic K3 surface with at worst Du Val singularities.
2. For the remaining cases, in which $c \leq 1$, the pair $(\mathbb{P}^3, \Delta)$ must have a **strictly log canonical** singularity.

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Remark

The trichotomy $c = 2, 1$ or 0 exactly corresponds to the cases in which a general pencil of quartic K3 surfaces passing through Δ is a type I, II or III degeneration of K3 surfaces.

Quartic surfaces of coregularity 2

Araujo–Corti–Massarenti study pairs $(\mathbb{P}^3, \Delta)$ with $c = 2$. In particular they completely describe $\mathrm{Bir}^{vp}(\mathbb{P}^3, \Delta)$ in the case in which Δ is a very general smooth quartic, or very general with one ordinary double point.

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However, explicitly classifying all pairs $(\mathbb{P}^3, \Delta)$ with $c = 2$ up to vp equivalence is likely to be **far too hard** to do in general!

Example (Oguiso)

There exist smooth isomorphic quartic surfaces $\Delta_1, \Delta_2 \subset \mathbb{P}^3$ for which there is no birational automorphism φ (let alone a vp one) of $\mathbb{P}^3$ with $\Delta_2 = \varphi(\Delta_1)$.

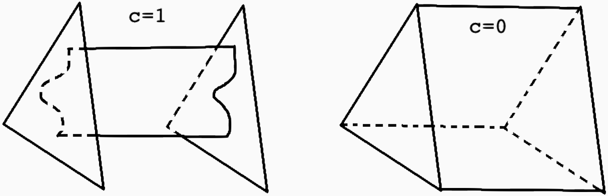
Main result

Theorem (D.)

Suppose that $(\mathbb{P}^3, \Delta)$ is a LCY pair with $c \leq 1$. Then it is vp equivalent to the pair

$$\left(\mathbb{P}^1 \times \mathbb{P}^2,\ \{0\} \times \mathbb{P}^2) + (\mathbb{P}^1 \times E) + (\{\infty\} \times \mathbb{P}^2)\right)$$

where E is a smooth cubic curve if $c = 1$, or the triangle of coordinate lines if $c = 0$. In particular, if $c = 0$ then $(\mathbb{P}^3, \Delta)$ admits a **toric model**.



More general conjecture

This result is a special case of the following more general conjecture, which grew out of work of **Shokurov**.

Conjecture

If X is a rational 3-fold and (X, Δ_X) is a LCY pair of coregularity 0, then (X, Δ_X) admits a vp map onto a toric pair.

More general conjecture

This result is a special case of the following more general conjecture, which grew out of work of **Shokurov**.

Conjecture

If X is a rational 3-fold and (X, Δ_X) is a LCY pair of coregularity 0, then (X, Δ_X) admits a vp map onto a toric pair.

As we saw, the conjecture holds for two dimensional pairs (X, Δ_X) and, in fact, having coregularity 0 forces X to be rational.

There are examples of non-rational 3-fold pairs (X, Δ_X) of coregularity 0 due to **Kaloghiros** and **Svaldi**. These can be used to construct counterexamples to the conjecture in dimension 4.

Relationship to Mella's work

Mella (2020) has shown that any rational quartic surface $\Delta \subset \mathbb{P}^3$ can be mapped onto a plane by a birational map $\varphi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$.

Our theorem strengthens this result (for semi-log canonical surfaces Δ, at least) by showing the same result holds with **vp maps**.

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Our theorem strengthens this result (for semi-log canonical surfaces Δ, at least) by showing the same result holds with **vp maps**.

Most of the maps that **Mella** constructs do not extend to vp maps. Roughly speaking, in order for φ to be vp we need to ensure that

1. any curve in $\mathrm{Bs}(\varphi)$ is contained in a component of Δ,
2. any point in $\mathrm{Bs}(\varphi)$ is contained in a 1-stratum of Δ.

(Or in other words, all centres of $\mathrm{Bs}(\varphi)$ have discrepancy ≤ 0.)

Overview of the proof

Any pair $(\mathbb{P}^3, \Delta)$ with $c \leq 1$ must have a strictly (semi-)log canonical singularity $p \in \Delta$. We first sort such pairs into 11 deformation families depending on the types of possible singularities of Δ.

- **(A.1-4)** irreducible normal quartic surfaces with a simple elliptic (or cusp) singularity,
- **(B.1-3)** irreducible non-normal quartic surfaces,
- **(C.1-4)** reducible quartic surfaces.

We then construct 10 explicit vp maps which link these 11 different families together, according to the following flowchart.

An arrow $F \to G$ in the diagram below means that we construct a vp map $\varphi\colon (\mathbb{P}^3, \Delta_F) \dashrightarrow (\mathbb{P}^3, \Delta_G)$ between the general member of family F and a member of family G.

$$
\begin{array}{ccccccccccc}
\text{(A.3)} & \longrightarrow & \text{(A.2)} & & \text{(B.2)} & & \text{(B.3)} & & & & \text{(A.1)} \\
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Flowchart

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Thus we can eventually reduce the proof of our main theorem to the case (C.4), which by definition consists of pairs for which Δ is the union of a plane and the cone over a cubic curve.

The eleven families

Semi-log canonical surface singularities

Two-dimensional slc hypersurface singularities have been classified.
The pair $(\mathbb{P}^3, \Delta)$ has coregularity 0 iff Δ has a (degenerate) cusp.

Type	Name	Normal form for f	Condition
Simple elliptic	$\widetilde{E}_6$	$\lambda xyz = x^3 + y^3 + z^3$	$\lambda^3 \neq 27$
	$\widetilde{E}_7$	$\lambda xyz = x^2 + y^4 + z^4$	$\lambda^4 \neq 64$
	$\widetilde{E}_8$	$\lambda xyz = x^2 + y^3 + z^6$	$\lambda^6 \neq 432$
Cusp	T_{pqr}	$xyz = x^p + y^q + z^r$	$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$
Normal crossing	A_∞	$xy = 0$	
Pinch point	D_∞	$x^2 + y^2 z = 0$	
Degenerate cusp	$T_{\infty\infty\infty}$	$xyz = 0$	
	$T_{p\infty\infty}$	$xyz = x^p$	$p \geq 2$
	$T_{pq\infty}$	$xyz = x^p + y^q$	$\frac{1}{p} + \frac{1}{q} < 1$

Irreducible singular quartic surfaces

If $\Delta \subset \mathbb{P}^3$ is irreducible then $(\Delta, 0)$ is a LCY pair, by adjunction for $(\mathbb{P}^3, \Delta)$. Moreover $\operatorname{coreg}(\mathbb{P}^3, \Delta) = \operatorname{coreg}(\Delta, 0) = c$.

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1. $\widetilde{\Delta}_0$ is a ruled elliptic surface and E_0 is two disjoint sections,
2. $\widetilde{\Delta}_0$ is a rational surface and E_0 is a smooth elliptic curve,

and if $c = 0$ then

3. $\widetilde{\Delta}_0$ is a rational surface and E_0 is an anticanonical cycle.

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Now blowdown (-1)-curves $f_i\colon (\widetilde{\Delta}_{i-1}, \widetilde{D}_{i-1}) \to (\widetilde{\Delta}_i, \widetilde{D}_i)$, until we reach a minimal LCY pair $(\widetilde{\Delta}_n, \widetilde{D}_n)$.

Rational quartic surfaces

If Δ is rational then, by blowing up more if necessary, we can assume that $(\widetilde{\Delta}_n, E_n) = (\mathbb{P}^2, E)$, giving a vp rational parameterisation of Δ.

$$\begin{array}{ccccc} & & (\widetilde{\Delta}_0, E_0) & & \\ & \overset{\pi}{\swarrow} & & \overset{f}{\searrow} & \\ (\Delta, 0) & & \overset{\mu}{\dashleftarrow\dashleftarrow\dashleftarrow} & & (\mathbb{P}^2, E) \end{array}$$

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This parameterisation is determined by a linear system $\mathcal{L} = |\pi^* \mathcal{O}_\Delta(1)|$. All possible such linear systems are classified into

1. four cases by **Noether** (1889) when Δ has isolated singularities, and
2. four more cases by **Urabe** (1986) when Δ is nonnormal.

Table of rational quartic surfaces

The map $f\colon(\widetilde{\Delta}_0, E_0) \to (\mathbb{P}^2, E)$ is a vp blowup of points $p_1, \ldots, p_n$ lying on a cubic curve $E \subset \mathbb{P}^2$. Let $h = f^*\mathcal{O}_{\mathbb{P}^2}(1)$ and let e_i be the exceptional line over p_i.

Case	Singularity	$\widetilde{\Delta}_0$	$\mathcal{L} = \lvert\mu^*\mathcal{O}_\Delta(1)\rvert$
(A.1)	$\widetilde{E}_6$	$Bl_{12}\mathbb{P}^2$	$\left\lvert 4h - \sum_{i=1}^{12} e_i \right\rvert$
(A.2)	$\widetilde{E}_7$	$Bl_{11}\mathbb{P}^2$	$\left\lvert 6h - \sum_{i=1}^{7} 2e_i - \sum_{i=8}^{11} e_i \right\rvert$
(A.3)	$\widetilde{E}_8$	$Bl_{10}\mathbb{P}^2$	$\left\lvert 9h - \sum_{i=1}^{8} 3e_i - 2e_9 - e_{10} \right\rvert$
(A.4)	$\widetilde{E}_8$	$Bl_{10}\mathbb{P}^2$	$\left\lvert 7h - 3e_1 - \sum_{i=2}^{10} 2e_i \right\rvert$
(B.1)	Line	$Bl_9\mathbb{P}^2$	$\left\lvert 4h - 2e_1 - \sum_{i=2}^{9} e_i \right\rvert$
(B.2)	Conic	$Bl_5\mathbb{P}^2$	$\mathcal{L} \subset \left\lvert 3h - \sum_{i=1}^{5} e_i \right\rvert$
(B.3)	Twisted cubic	$Bl_2\mathbb{P}^2$	$\mathcal{L} \subset \lvert 3h - 2e_1 - e_2 \rvert$
(B.4)	Three lines	$\mathbb{P}^2$	$\mathcal{L} \subset \lvert 2h \rvert$

Remaining cases

(A.1) is the case in which Δ has a triple point.

(B.4) is the case of Steiner's Roman quartic surface. It necessarily has a triple point, so we treat it as a special case of (A.1).

The case of a ruled elliptic surface $\widetilde{\Delta}$ has a similar, shorter classification due to **Umezu** (1984) and **Urabe**. They occur as degenerations of the rational cases.

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The only other cases we need to consider are **reducible quartics**:

- **(C.1)** The union of a plane and (smooth) cubic surface.
- **(C.2)** The union of two quadrics.
- **(C.3)** The union of a plane and singular cubic surface.
- **(C.4)** The union of a plane and the cone over a cubic curve.

3-dimensional Cremona transformations

Maps of low bidegree

To construct the links between our eleven families we use Cremona transformations of low bidegree.

Definition

The **bidegree** of a birational map $\varphi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is given by

$$\operatorname{bideg}(\varphi) = \left(\deg \varphi, \deg \varphi^{-1}\right).$$

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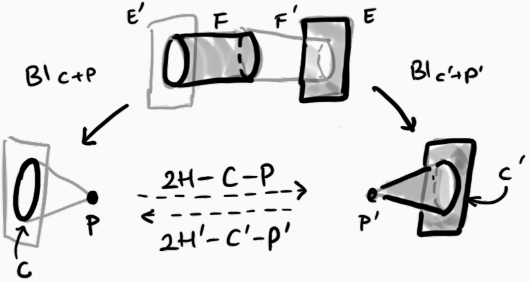
Pan–Ronga–Vust classified maps of bidegree $(2, k)$.

Deserti–Han treat maps of bidegree $(3, k)$ for $k \leq 5$.

We manage, almost exclusively, to get away with using maps of bidegree $(2, 2)$, $(2, 3)$ and $(3, 3)$.

Maps of bidegree $(2,2)$

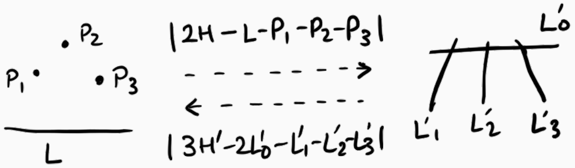
A map φ of bidegree $(2,2)$ are obtained by picking a point $P \in \mathbb{P}^3$ and a plane conic $C \subset \mathbb{P}^3$ and looking at quadrics through P and C.



The choice of P and C is allowed to be degenerate (e.g. C is reducible and/or $P \in C$). The inverse φ^{-1} is a map of the same form.

Maps of bidegree $(2,3)$

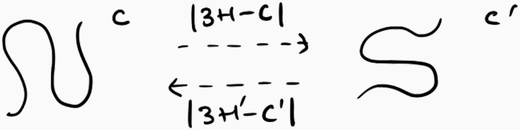
Maps of bidegree $(2,3)$ are obtained by picking three points $P_1, P_2, P_3 \in \mathbb{P}^3$ and a line $L \subset \mathbb{P}^3$ and looking at quadrics through P_1, P_2, P_3, L.



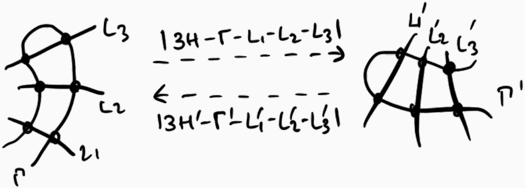
The inverse φ^{-1} is a cubic map with baselocus $2L'_0 + L'_1 + L'_2 + L'_3$ for three skew lines L'_1, L'_2, L'_3 meeting a common line L'_0.

Maps of bidegree $(3, 3)$

The generic map is the **cubo-cubic Cremona transformation**. It blows up a curve C of genus 3 and degree 6.



We make particular use of the case in which C degenerates into the union $\Gamma \cup L_1 \cup L_2 \cup L_3$ of a twisted cubic and three secant lines.



The ten maps

Recall that we are trying to build links between our families according to the flowchart.

$$
\begin{array}{ccccccccccc}
(A.3) & \longrightarrow & (A.2) & & (B.2) & & (B.3) & & & & (A.1) \\
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I will just explain how to construct some of the more interesting cases.

The map (A.1)→(C.4)

By far the easiest case is the map (A.1)→(C.4) in which Δ has a triple point.

$$\Delta = \mathbb{V}\left(tf_3(x,y,z) + g_4(x,y,z)\right) \subset \mathbb{P}^3_{t,x,y,z}$$

Then the map $\varphi(t,x,y,z) = \left(t + f_3^{-1}g_4,\ x,\ y,\ z\right)$ is a vp map $\varphi\colon(\mathbb{P}^3,\Delta) \dashrightarrow (\mathbb{P}^3,\Delta')$ where $\Delta' = \mathbb{V}(tf_3)$ is the union of the plane $\mathbb{V}(t)$ and the cone over a cubic curve $\mathbb{V}(f_3(x,y,z))$.

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To see this, pull back a volume form with a pole on Δ' to get a volume form with a pole along Δ.

$$\varphi^*\left(\frac{dt \wedge dx \wedge dy}{tf_3}\right) = \frac{d(t + f_3^{-1}g_4) \wedge dx \wedge dy}{(t + f_3^{-1}g_4)f_3} = \frac{dt \wedge dx \wedge dy}{tf_3 + g_4}$$

Linking families (A.2), (A.3) and (C.1)

The general member of (A.2) and (A.3) have singularities of type $\widetilde{E}_7$ ($\lambda xyz = x^2 + y^4 + z^4$) or $\widetilde{E}_8$ ($\lambda xyz = x^2 + y^3 + z^6$). Following the Sarkisov links that begin with the $(2,1,1)$ or $(3,2,1)$ weighted blowups of these points gives a diagram of vp maps

$$
\begin{array}{ccccc}
(\mathbb{P}^3, \Delta_{(\mathrm{A.3})}) & & (\mathbb{P}^3, \Delta_{(\mathrm{A.2})}) & & \\
\varphi_1 \downarrow & & \downarrow \varphi_2 & & \\
(\mathbb{P}(1,1,2,3), D_1) & \overset{\varphi_3}{\dashrightarrow} & (\mathbb{P}(1,1,1,2), D_2) & \overset{\varphi_4}{\dashrightarrow} & (\mathbb{P}^3, D_3)
\end{array}
$$

where $D_d = A + B_d$ is the union of a plane A and a dP surface B_d of degree d. In particular $(\mathbb{P}^3, D_3)$ is in family (C.1). The maps φ_3, φ_4 on the bottom row are given by contracting a line in B_d.

The map (B.1)→(C.1)

One of the hardest cases to deal was (B.1), in which Δ is singular along a line $L \subset \Delta$.

The planes through L define a pencil of conics in Δ, which generically has 8 reducible members. We can pick three skew lines $L_1, L_2, L_3 \subset \Delta$ meeting L from three of these eight planes and consider the map $\varphi = |3H - 2L - L_1 - L_2 - L_3|$ of bidegree $(3, 2)$.

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This defines a vp map from Δ onto $\Delta' = D_1 + D_2$, the union of a plane D_1 and cubic surface D_2. The map φ^{-1} blows up three points in $D_1 \cap D_2$ and a line in D_2.

The hard part is dealing with the degenerate cases.

The map (B.3)→(C.2)

This is the only case (other than (A.1)) in which **Mella**'s construction already produces a vp map.

We consider Δ singular along a twisted cubic $\Gamma \subset \Delta$. One can prove that Δ contains an infinite number of secant lines to Γ. Pick three of them $L_1, L_2, L_3 \subset \Delta$ and consider the degenerate cubo-cubic Cremona transformation $\varphi = |3H - \Gamma - L_1 - L_2 - L_3|$.

The map (B.3)→(C.2)

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This defines a vp map $\varphi\colon (\mathbb{P}^3, \Delta) \dashrightarrow (\mathbb{P}^2, \Delta')$, where $\Delta' = Q_1 + Q_2$ the sum of two quadrics; $Q_1 = \varphi(\Delta)$ and Q_2 is the exceptional divisor over Γ. The inverse map φ^{-1} blows up a twisted cubic $\Gamma' \subset Q_1$ and three secant lines to Γ' lying in Q_2.

Final link

We eventually reduce to the case $\Delta = \mathbb{V}(tf_3(x,y,z))$ of the sum of a plane $\mathbb{V}(t)$ and the cone over a (possibly degenerate) cubic $\mathbb{V}(f_3(x,y,z))$.

Now the final conclusion of the theorem is easy. Without loss of generality we may assume that z does not divide f_3. Then

$$\varphi\colon (\mathbb{P}^3,\Delta) \dashrightarrow (\mathbb{P}^1\times\mathbb{P}^2,\Delta'), \qquad \varphi(t,x,y,z) = (t,z)\times(x,y,z)$$

is a vp map, where

$$\Delta' = (\{0\}\times\mathbb{P}^2) + (\mathbb{P}^1\times E) + (\{\infty\}\times\mathbb{P}^2)$$

for a cubic curve $E\subset\mathbb{P}^2$. The coregularity of $(\mathbb{P}^3,\Delta)$ is $c=1$ iif E is smooth. If $c=0$ then E is nodal and we apply the 2-dimensional result to $E\subset\mathbb{P}^2$ to reduce to the case in which E is a triangle of lines.

The end