Gauss Manin Connection in Disguise and Mirror Symmetry

Modular-type functions for open-string Mirror Symmetry on the quintic

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Outline

What is GMCD?

GMCD or *Gauss Manin Connection in Disguise* is a program which has the goal to generalize the concept of modular forms.

Basic idea:

- Consider the (quasi-affine) moduli space of n-dimensional varieties enhanced with **a basis** for its n-th algebraic de Rham cohomology and take coordinates for it;
- Compute the Gauss Manin connection of this family w.r.t this basis;
- Find the *modular vector field*, which gives differential relations among the coordinates. This vector field, when integrated, will also give rise to a *modular domain*.

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Example: Elliptic Curves

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What is a Mirror Quintic?

Consider the family of quintics in $\mathbb{P}^4$ given by:

$$X_\psi : x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0, \qquad \psi^5 \neq 1$$

Let G be the group

$$G = \left\{ (a_0, \ldots, a_4) \in \mathbb{Z}_5^5 : \sum_i a_i \equiv 0 \bmod 5 \right\} \Big/ \mathbb{Z}_5 \ , \qquad (1)$$

which acts on $\mathbb{P}^4$ by

$$(a_0, \ldots, a_4) \bullet [x_0, \ldots, x_4] \mapsto [\mu^{a_0} x_0 : \ldots \mu^{a_4} x_4],$$

The mirror quintic family is the family of resolutions of the singularities of each quotient X_ψ / G.

Moduli Space

An enhanced Mirror Quintic is a pair $(X, [\alpha_1, \alpha_2, \alpha_3, \alpha_4])$, where the α's are a basis for $H^3_{dR}(X)$ satisfying

$$\alpha_i \in F^{4-i}/F^{5-i}, \quad i \in \{1, 2, 3, 4\};$$
$$[\langle \alpha_i, \alpha_j \rangle] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

In the above, F represents the Hodge filtration.

Moduli Space

Theorem (Movasati, 2015)

Enhanced mirror quintics can be parametrized by the affine open set

$$T \cong \left\{ (t_0, t_1, t_2, t_3, t_4, t_5, t_6) \in \mathbb{C}^7 \mid t_5 t_4 \left(t_4 - t_0^5 \right) \neq 0 \right\}.$$

Moduli Space

Write the equation of X as

$$-t_4x_0^5 - x_1^5 - \cdots - x_4^5 + 5t_0x_0 \ldots x_4 = 0$$

and fix ω_1 as a holomorphic form (notice that pairs (X, ω)) form a dimension two space.

Consider a basis $\Omega := [\omega_1, \omega_2, \omega_3, \omega_4]$ of the de Rham cohomology, where $\omega_i = \frac{\partial}{\partial t_0}\omega_{i-1}$ and ω_1 is a holomorphic 3-form.

Any basis satisfying the two properties can be obtained from Ω via multiplication by a matrix. The independent coefficients of this matrix are the coordinates t_i associated to this basis.

Picard Fuchs Equation

Let $z := \psi^{-5} = \frac{t_4}{t_0^5}$. Candelas et al showed that the solutions of the *Picard-Fuchs* differential equation

$$\mathcal{L}\varpi_0 = \left[\frac{d^4}{dz^4} - \frac{2(4z-3)}{z(1-z)}\frac{d^3}{dz^3} - \frac{(72z-35)}{5z^2(1-z)}\frac{d^2}{dz^2} - \frac{(24z-5)}{5z^3(1-z)}\frac{d}{dz} - \frac{24}{625z^3(1-z)}\right]\varpi_0 = 0,$$

are periods of the mirror quintic (integrals of ω_1).
This equation will help us to compute the Gauss Manin connection in the basis $[\omega_1, \omega_2, \omega_3, \omega_4]$.

Ramanujan Vector Field

Theorem (Movasati, 2015)

There is a unique vector field $\mathbf{R}$ *in* T *such the Gauss-Manin connection composed with the vector field* $\mathbf{R}$ *satisfies*

$$\begin{aligned} \nabla_{\mathbf{R}}(\alpha_1) &= \alpha_2; \\ \nabla_{\mathbf{R}}(\alpha_2) &= Y\alpha_3; \\ \nabla_{\mathbf{R}}(\alpha_3) &= -\alpha_4; \\ \nabla_{\mathbf{R}}(\alpha_4) &= 0; \end{aligned}$$

for some regular function Y *in* T.

Ramanujan Vector Field

The vector field is given, as a differential equation, by

$$\mathbf{R}: \begin{cases} \dot{t}_0 = \frac{1}{t_5}\left(6\cdot 5^4 t_0^5 + t_0 t_3 - 5^4 t_4\right) \\ \dot{t}_1 = \frac{1}{t_5}\left(-5^8 t_0^6 + 5^5 t_0^4 t_1 + 5^8 t_0 t_4 + t_1 t_3\right) \\ \dot{t}_2 = \frac{1}{t_5}\left(-3\cdot 5^9 t_0^7 - 5^4 t_0^5 t_1 + 2\cdot 5^5 t_0^4 t_2 + 3\cdot 5^9 t_0^2 t_4 + 5^4 t_1 t_4 + 2 t_2 t_3\right) \\ \dot{t}_3 = \frac{1}{t_5}\left(-5^{10} t_0^8 - 5^4 t_0^5 t_2 + 3\cdot 5^5 t_0^4 t_3 + 5^{10} t_0^3 t_4 + 5^4 t_2 t_4 + 3 t_3^2\right) \\ \dot{t}_4 = \frac{1}{t_5}\left(5^6 t_0^4 t_4 + 5 t_3 t_4\right) \\ \dot{t}_5 = \frac{1}{t_5}\left(-5^4 t_0^5 t_6 + 3\cdot 5^5 t_0^4 t_5 + 2 t_3 t_5 + 5^4 t_4 t_6\right) \\ \dot{t}_6 = \frac{1}{t_5}\left(3\cdot 5^5 t_0^4 t_6 - 5^5 t_0^3 t_5 - 2 t_2 t_5 + 3 t_3 t_6\right) \end{cases}$$

and the regular function Y is given by

$$Y = \frac{5^8 (t_0^5 - t_4)^2}{t_5^3}.$$

Gromov Witten invariants and Periods

We can solve the equation considering a coordinate q and a derivation $5q\frac{d}{dq}$, we can find functions that work as modular forms.

If we compute such a q-expansion for Y, we get, **up to a constant**, the number n_d of rational curves of degree d on a generic quintic threefold, as computed by Candelas et al.

$$\left(5 + 2875\frac{q}{1-q} + 609250 \cdot 2^3\frac{q^2}{1-q^2} + \cdots + n_d d^3\frac{q^d}{1-q^d} + \cdots\right)$$

This relationship is better explained by periods, as in the Elliptic Curve case. We will come back to this in the end of the lecture.

What is different in the open case?

Instead of counting rational curves on the quintic, we want to count disks with boundary on a Lagrangian in the quintic. For this, we need to consider a pair of conics in the mirror, as below:

$$C_{\pm} = \left\{ x_0 + x_1 = 0, x_2 + x_3 = 0, x_4^2 \pm \sqrt{5\psi} x_1 x_3 = 0 \right\} \subset X_{\psi}$$

After the quotient by the action of the group G and the resolution of singularities, this curves may be considered as curves on the mirror quintic.

Relative Algebraic de Rham cohomology

In this context, we need to deal not with the absolute algebraic de Rham cohomology $H^3_{dR}(X)$ but with the relative algebraic de Rham cohomology $H^3_{dR}(X, C_+ \cup C_-)$.

Mixed Hodge Structure and Gauss Manin Connection

Instead of a usual Hodge structure as we have on the absolute cohomology, we have a **mixed Hodge structure**. Also, we have to define a relative version of the Gauss Manin connection.

Mixed Hodge Structure and Gauss Manin Connection

Relatively Enhanced Mirror Quintics

An relatively enhanced Mirror Quintic is a triple

$$(X, C_\pm, [\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4]),$$

where the α's are a basis for $H^3_{dR}(X, C_\pm)$ satisfying

1. $\alpha_i \in F^{4-i} \setminus F^{5-i}, \quad i = 1, 2, 3, 4$
2. $\alpha_i \in W_3 \setminus W_2, \quad i = 1, 2, 3, 4.$
3. $\alpha_0 \in F^1 \setminus F^2$
4. $\alpha_0 \in W_2$
5. $[\langle \alpha_i, \alpha_j \rangle] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$
6. $\int_{\delta_0} \omega_0 = 1$

Moduli Space

Theorem (F.E.)

Relatively enhanced mirror quintics can be parametrized by the affine open set

$$S \cong \left\{ (s_0, s_1, \dots, s_7, s_8) \in \mathbb{C}^9 \mid s_0 s_5 s_4 (s_4^{10} - s_0^{10}) \neq 0 \right\}.$$

Picard-Fuchs Equation

Walcher and his collaborators showed that the extra period that appears in our case is a solution of the differential equation

$$\mathcal{L}\varpi = \frac{-15\sqrt{z}}{8z^4(z-1)}$$

We can repeat the process of the absolute cohomology to compute the Gauss Manin connection.

Ramanujan Vector Field

Theorem (F.E.)

There is a unique vector field **R** *in S such the Gauss-Manin connection composed with the vector field* **R** *satisfies*

$$\begin{aligned}
\nabla_{\mathbf{R}}(\alpha_0) &= 0;\\
\nabla_{\mathbf{R}}(\alpha_1) &= \alpha_2;\\
\nabla_{\mathbf{R}}(\alpha_2) &= F\alpha_0 + Y\alpha_3;\\
\nabla_{\mathbf{R}}(\alpha_3) &= -\alpha_4;\\
\nabla_{\mathbf{R}}(\alpha_4) &= 0;
\end{aligned}$$

for some regular functions F and Y in S.

Ramanujan Vector Field

The vector field is given, as a differential equation, by

$$\mathbf{R}: \begin{cases} \dot{s}_0 = \frac{1}{2s_0 s_5}\left(6\cdot 5^4 s_0^{10} + s_0^2 s_3 - 5^4 s_4^{10}\right) \\ \dot{s}_1 = \frac{1}{s_5}\left(-5^8 s_0^{12} + 5^5 s_0^8 s_1 + 5^8 s_0^2 s_4^{10} + s_1 s_3\right) \\ \dot{s}_2 = \frac{1}{s_5}\left(-3\cdot 5^9 s_0^{14} - 5^4 s_0^{10} s_1 + 2\cdot 5^5 s_0^8 s_2 + 3\cdot 5^9 s_0^4 s_4^{10} + 5^4 s_1 s_4^{10} + 2 s_2 s_3\right) \\ \dot{s}_3 = \frac{1}{s_5}\left(-5^{10} s_0^{16} - 5^4 s_0^{10} s_2 + 3\cdot 5^5 s_0^8 s_3 + 5^{10} s_0^3 s_4^{10} + 5^4 s_2 s_4^{10} + 3 s_3^2\right) \\ \dot{s}_4 = \frac{1}{10 s_5}\left(5^6 s_0^8 s_4 + 5 s_3 s_4\right) \\ \dot{s}_5 = \frac{1}{s_5}\left(-5^4 s_0^{10} s_6 + 3\cdot 5^5 s_0^8 s_5 + 2 s_3 s_5 + 5^4 s_4^{10} s_6\right) \\ \dot{s}_6 = \frac{1}{s_5}\left(3\cdot 5^5 s_0^8 s_6 - 5^5 s_0^6 s_5 - 2 s_2 s_5 + 3 s_3 s_6\right) \\ \dot{s}_7 = -s_8 \\ \dot{s}_8 = -\frac{5^{12}\left(s_0^{10} - s_4^{10}\right)}{s_5} \cdot \frac{15}{8}\left(\frac{s_4}{s_0}\right)^5 \frac{1}{25\sqrt{5}} \end{cases}$$

$$Y = \frac{5^8\left(s_4^{10} - s_0^{10}\right)^2}{s_5^3}, \quad F = -s_7 Y,$$

Disk counts

After solving the differential equation considering the same coordinate q and derivation $5q\frac{d}{dq}$ as in the absolute case, we get:

$$\frac{-4}{5^3}F(q) := 30q^{1/2} + 13800q^{3/2} + 27206280q^{5/2} + ... =$$

$$= \sum_{d \text{ odd}} n_d^{\text{disk}} d^2 \frac{q^{d/2}}{1-q^d}$$

We will now try to answer the question: Why are these numbers appearing?

Group Action

$$G_4 := \left\{ g = [g_{ij}]_{4\times 4} \in \mathrm{GL}(4,\mathbb{C}) \mid g_{ij} = 0,\ \text{for } j < i \text{ and } g^{\mathrm{t}}\Phi g = \Phi \right\}$$

$$G_5 := \left\{ g = [g_{ij}]_{5\times 5} \in \mathrm{GL}(5,\mathbb{C}) \mid g_{ij} = 0,\ \text{for } j < i \text{ and } g^{\mathrm{t}}\Phi g = \Phi \right\}$$

They act, respectively, on T and S by changing the basis α.

Period Matrix

- The period matrix is the matrix given by

$$P = \left[\int_{\delta_i} \alpha_j \right],$$

  where δ_0 is the homology connecting the two conics and the other δ_i are a symplectic basis for the homology.
- Using the properties of the basis α, δ and the Poincaré duality, we can derive restrictions for the entries of P.
- The groups G_4 and G_5 act by multiplication form the right. This action is compatible with the period map.
- There is also a left action on the space matrix via multiplication. This is simply changing the basis of the homology! This groups are the "modular" groups.

τ-locus

After some computations, one can prove that the matrices modulo the action of the groups are of the form:

$$\tau = \begin{pmatrix} \tau_0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \tau_1 & \tau_3 & 1 & 0 \\ \tau_2 & -\tau_0\tau_3 + \tau_1 & -\tau_0 & 1 \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} 1 & \tau_4 & \tau_5 & 0 & 0 \\ 0 & \tau_0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \tau_1 & \tau_3 & 1 & 0 \\ 0 & \tau_2 & -\tau_0\tau_3 + \tau_1 & -\tau_0 & 1 \end{pmatrix}$$

Mirror Map and Periods

The τ_i are quotients of periods and derivatives of quotients of periods. Those are the main ingredients in the changes of coordinates that happen in Mirror Symmetry.

Let L denote the fundamental domains above. L depends only on τ_0 and the vector field $\frac{\partial}{\partial \tau_0}$ correspond to the modular vector field **R** after pulling back to T or S!

This means that the coordinate q is actually the exponential of τ_0: the Physics' mirror map.

Open Questions

1. Make the same process to general Calabi-Yau varieties or even more general spaces. What kind of generating functions should we get? Is the Moduli Space quasi-affine?
2. Consider a moving family inside X, not only a fixed family (e.g a family of divisors). This should give us a generalization of Jacobi forms.
3. Consider other cases in which we have a mixed Hodge structure, for example, singular projective varieties.

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