Bernstein-Kouchnirenko-Khovanskii with a symmetry

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Nottingham AG seminar
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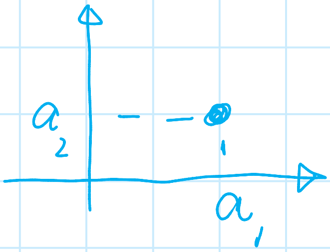
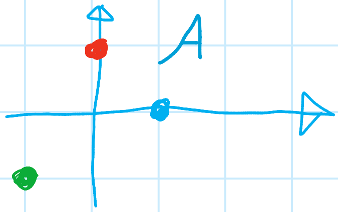
Prequel: the classics



$\longmapsto \; x_1^{a_1}\, x_2^{a_2} =: x^a$

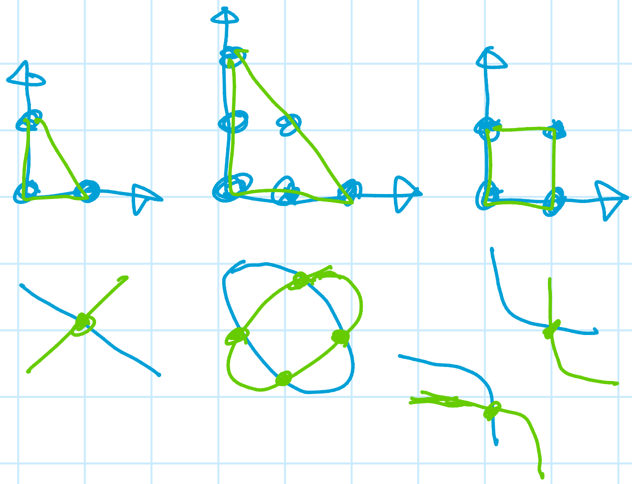


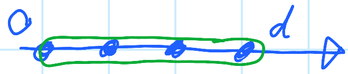
$\longmapsto \; \alpha \cdot x_1 + \beta \cdot x_2 + \gamma \cdot x_1^{-1} x_2^{-1} = 0$

a polynomial supported at A

$A, B \subset \mathbb{Z}^n \;\longmapsto\;$ systems of equations $f = g = 0$ supported at A, B

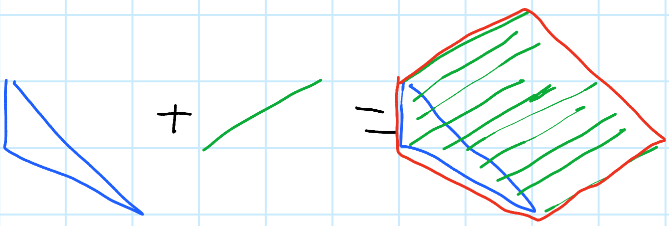
Kouchnirenko-Bernstein formula:
$f_1 = \cdots = f_n = 0$ is a generic system supported at $A_1, \ldots, A_n \subset \mathbb{Z}^n \quad \Rightarrow$
the number of its solutions in $(\mathbb{C}\setminus 0)^{\mathrm{n}}$ equals the mixed volume of $A_1, \ldots, A_n$





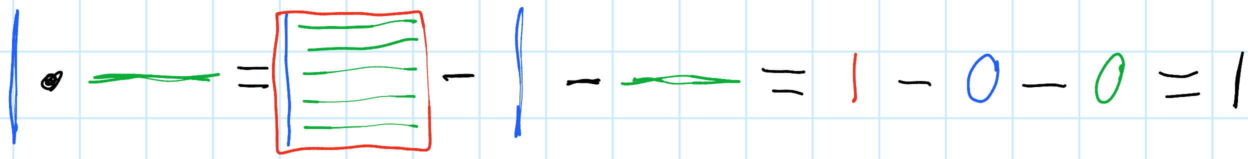
Minkowski sum

$A + B := \{a + b \mid a \in A, b \in B\}$



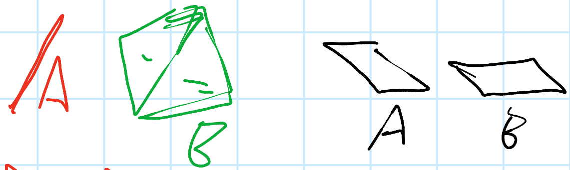
Mixed volume

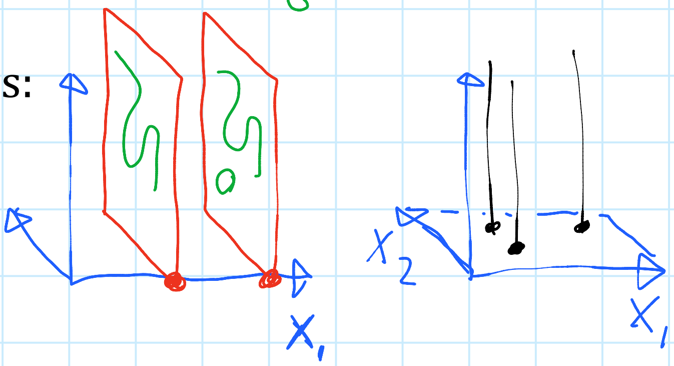
$A_1 \cdot \ldots \cdot A_n := \sum_{i_1 < \cdots < i_q} (-1)^{n-q}\, Vol(A_{i_1} + \cdots + A_{i_q})$



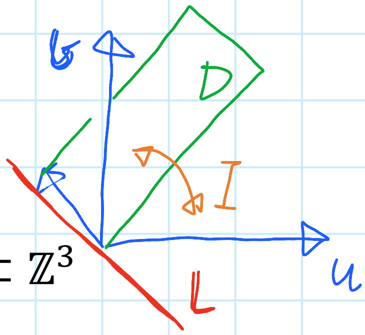
BKK toolkit: $f = g = 0$ is a generic system supported at $A, B \subset \mathbb{Z}^3$ $\Rightarrow$

1) $f = g = 0$ defines a smooth curve in $(\mathbb{C}\backslash 0)^3$
2) $e(f = g = 0) = -(A + B) \cdot A \cdot B$
3) The genus, the tropical fan, $\cdots$
4) The curve is irreducible unless:
   a. $f(x_1) = g(x_1, x_2, x_3) = 0$
   b. $f(x_1, x_2) = g(x_1, x_2) = 0$





Chapter 1: the symmetry

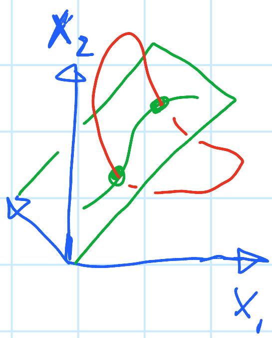


The involution $I: \mathbb{Z}^3 \to \mathbb{Z}^3$, $I(u, v, w) \coloneqq (v, u, w)$

The diagonal plane $D \subset \mathbb{Z}^3$ and the fixed line $L \subset \mathbb{Z}^3$

Same in the algebraic torus: $I: (\mathbb{C}\backslash 0)^3 \to (\mathbb{C}\backslash 0)^3 \supset D$

The symmetric curve C:
$f(x) = f(Ix) = 0$ for generic f supported at $A \subset \mathbb{Z}^3$

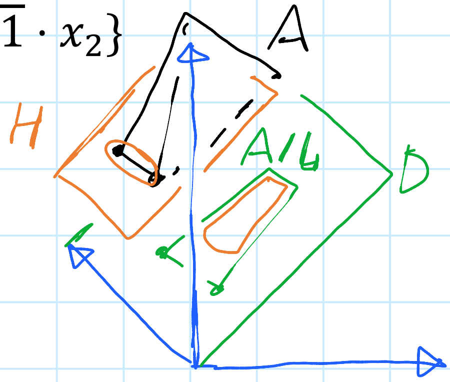


- BKK Toolkit for it?
- Higher dimensions & symmetries?
- What for?

C is never irreducible: it has a diagonal compopent
$C \cap D = \{f(x_1, x_2, x_3) = f(x_2, x_1, x_3) = 0, x_1 = x_2\}$

This is planar: lies in $D \simeq (\mathbb{C}\backslash 0)^2$, so covered by the classical BKK.

Other diagonal components: $\{f = 0, x_1 = \sqrt[d]{1} \cdot x_2\}$
where $d = |\mathbb{Z}^3/(D + A + IA)|$



The rest of C is its proper part C_P

Theorem: 1. C_P is smooth.

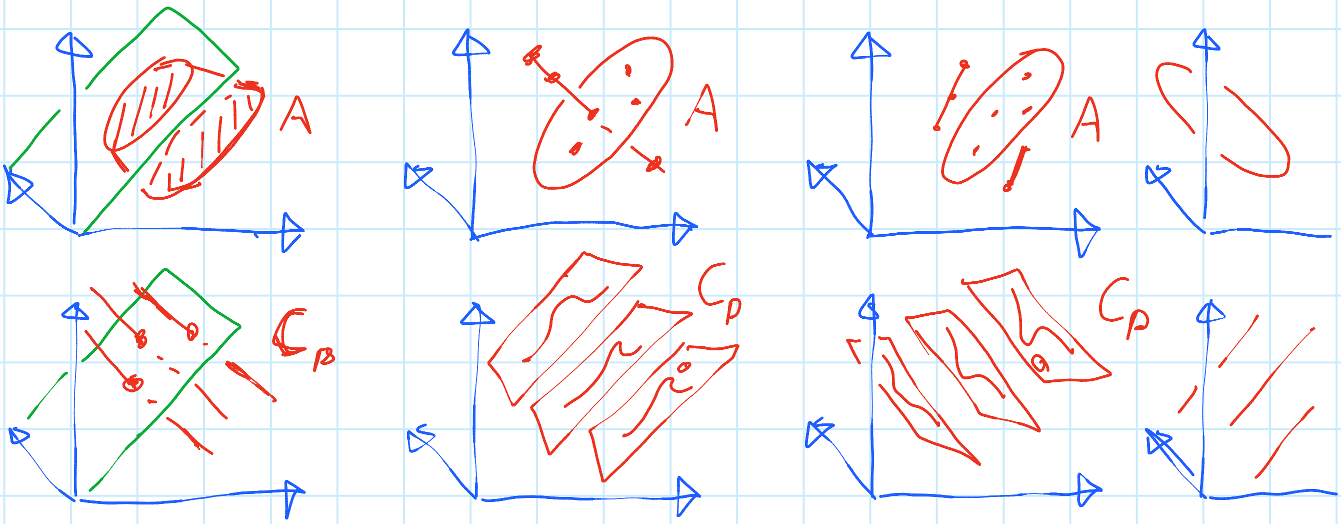
2. It intersects transversally every diagonal component at
$A/L \cdot A(/L - \sum_{H||D}(A/L - (A\backslash H)/L))$ points.

3. $e(C_P) = \# - (A + IA) \cdot A(\cdot IA - \sum_{H||D} A(\cdot IA - (A\backslash H) \cdot I(A\backslash H)))$

4. The genus, the tropical fan, ...

- Have you seen expressions like this?
- Simple to write, difficult to count
- No blinders - no sums

5. C_P is irreducible except for the following A:



Example: $f(x_1, x_2, x_3) = g(x_1 \cdot x_2, x_3) + x_1 \cdot h(x_1 \cdot x_2, x_3)$
$f(x_1, x_2, x_3) = f(x_2, x_1, x_3) = 0 \Leftrightarrow (x_1 - x_2) \cdot g = (x_1 - x_2) \cdot h = 0$

Proof: difficult
Conjecture: easy

The proper part has more than one component

The proper part is locally planar

Chapter 2: generalities and applications

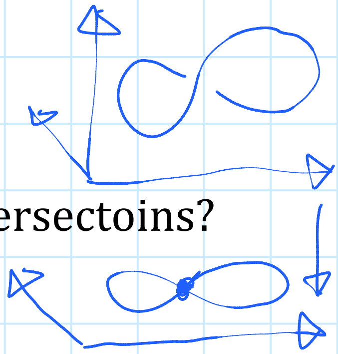
A finite group G acts on $\mathbb{Z}^n$, $(\mathbb{C}\backslash 0)^n$, $\{\pm 1\}$ and $\{1,\dots,k\}$.

Finite sets $A_1,\dots,A_k \in \mathbb{Z}^n$ satisfy $A_{gi} = gA_i$ for $g \in G$.

Polynomials f_i supported at A_i & generic modulo $f_{gi} = (-1)^g f_i \circ g$.

Study the complete intersection $f_1 = \cdots = f_k = 0$ in $(\mathbb{C}\backslash 0)^n$.

Interesting special cases



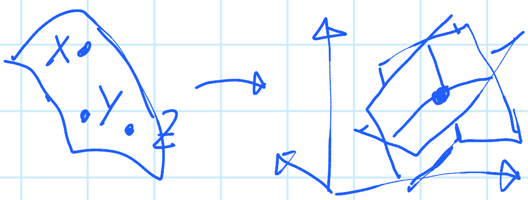
Self-intersections of an algebraic knot link

$\{f_1 = f_2 = 0\} \subset (\mathbb{C}\backslash 0)^3 \to (\mathbb{C}\backslash 0)^2$, how many self-intersectoins?
$f_1(x,y,z) = f_1(x,y,z') = f_2(x,y,z) = f_2(x,y,z') = 0$

- Voorhaar'19
- c.f. classical multiple point formulas

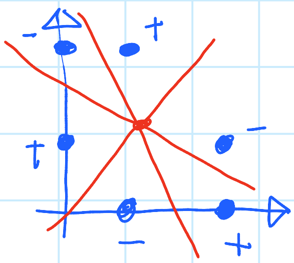
Affine multiple point formulas

$f = (f_1, f_2, f_3)\colon (\mathbb{C}\backslash 0)^2 \to \mathbb{C}^3$, how many 3-points $f(x) = f(y) = f(z)$?



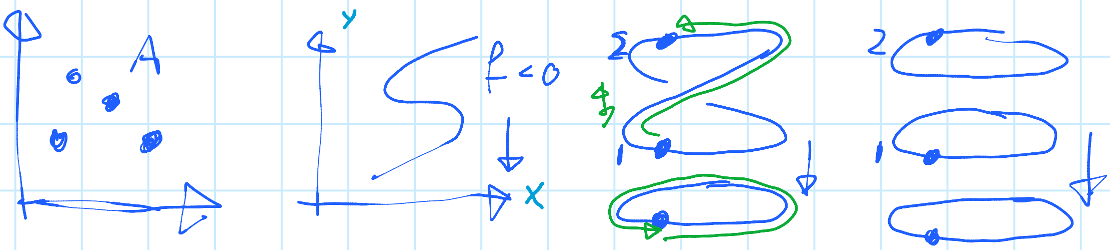
Irreducibility of Schur polynomials

$$\begin{vmatrix} x^a & x^b & x^c \\ y^a & y^b & y^c \\ 1 & 1 & 1 \end{vmatrix}$$



- Dvornicich, Zannier'09
- Applications in representation theory (unique factorization of representations of GL)

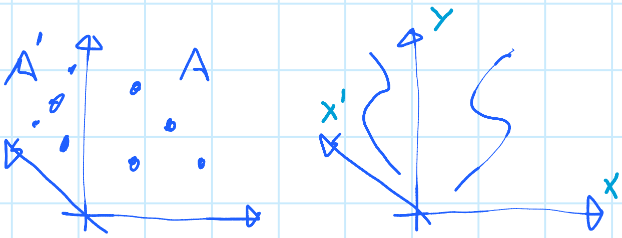
Transitivity of monodromy: $\{f = 0\} \subset (\mathbb{C}\backslash 0)^2$ is irreducible $\Leftrightarrow$ monodromy of $\{f = 0\} \to (\mathbb{C}\backslash 0)^1, (x,y) \mapsto x$, is transitive

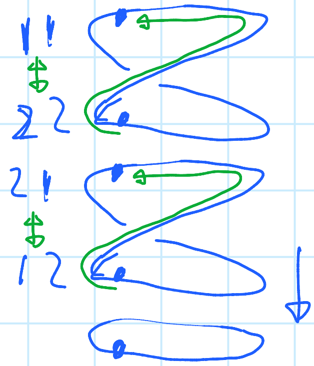


A group $G \subset S_k$ is 2-transitive if $\forall (i,j)$ is sent to $\forall (i',j')$ with $g \in G$.

monodromy of $\{f = 0\} \to (\mathbb{C}\backslash 0)^1, (x,y) \mapsto x$, is 2-transitive $\Leftrightarrow$

the fiber square $\{f(x,y) = f(x',y) = 0\} \subset (\mathbb{C}\backslash 0)^3$ is irreducible





Chapter 3: Sparse Galois theory

$c_1 x + c_0 = 0 \Longrightarrow x = -c_0/c_1$

$$c_2 x^2 + c_1 x + c_0 = 0 \Longrightarrow x = \frac{-c_1 \pm \sqrt{c_1^2 - 4c_0c_2}}{c_2}$$

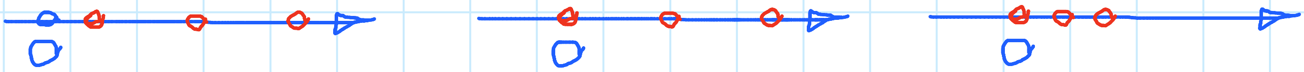
$c_3 x^3 + \cdots = 0 \Longrightarrow$

$$x = \sqrt[3]{\left(\frac{bc}{6a^2} - \frac{d}{2a} - \frac{b^3}{27a^3}\right) + \sqrt{\left(\frac{bc}{6a^2} - \frac{d}{2a} - \frac{b^3}{27a^3}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} + \sqrt[3]{\left(\frac{bc}{6a^2} - \frac{d}{2a} - \frac{b^3}{27a^3}\right) - \sqrt{\left(\frac{bc}{6a^2} - \frac{d}{2a} - \frac{b^3}{27a^3}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

$c_4 x^4 + \cdots = 0 \Longrightarrow \cdots$

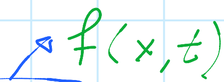
$c_5 x^5 + \cdots = 0 \Longrightarrow$ NO formula by radicals. But:

$$px^2 + qx^{12} + rx^{22} \quad \mapsto \quad px^0 + qx^{10} + rx^{20} \quad \mapsto \quad p + qy + ry^2$$

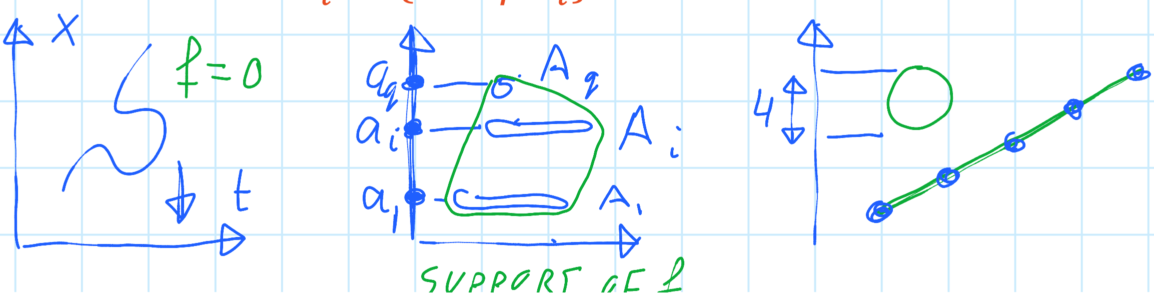


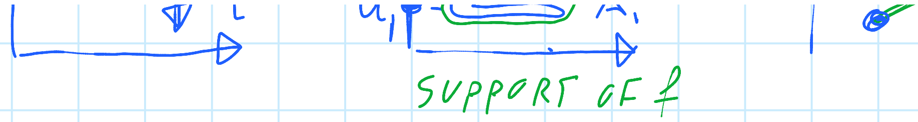
Theorem: given several monomials $A \subset \mathbb{Z}$,
assume wlog that A starts at 0 and generates $\mathbb{Z}$.
Then the general equation supported at A is solvable iff $\max A \leq 4$:

$$c_0 + c_1 x^{a_1} + \cdots + c_q x^{a_q} = 0$$



Specialization: consider $c_0(t) + c_1(t)x^{a_1} + \cdots + c_q(t)x^{a_q} = 0$,
where $c_i(t)$ is a generic polynomial supported at $A_i \subset \mathbb{Z}$.
Its solutions $x = x(t)$ can be expressed by radicals
Iff $\max A \leq 4$ OR $A_i = \{\alpha + \beta a_i\}$:





Why: the Galois group of the covering $(x, t) \mapsto t$ is full symmetric,
UNLESS $A_i = \{\alpha + \beta a_i\}$

Why: it contains a transposition and is 2-transitive
UNLESS $A_i = \{\alpha + \beta a_i\}$ $\Rightarrow$ contains all transpositions

Thank you!