Tautological bundles of matroids

Chris Eur (w/ Andrew Berget, Hunter Spink, Dennis Tseng)

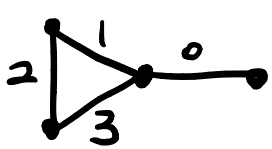
Defn A **matroid** M of rank r on $[n] = \{0, 1, \dots, n\}$ (ground set) is a collection $\mathcal{B} \subset \binom{[n]}{r}$ (bases) satisfying exchange axiom, i.e.

$P(M) = \mathrm{Conv}(e_B \mid B \in \mathcal{B}) \subset \mathbb{R}^{[n]}$ (base polytope) has all edges $/\!/ \; e_i - e_j \;\; \exists \, i, j \in [n]$.

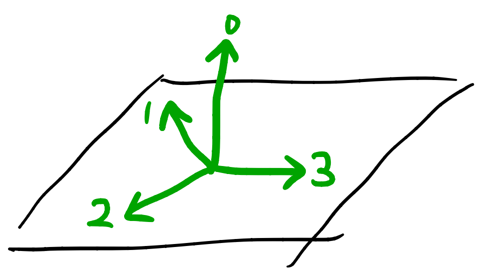
E.g. ① Graph $G \rightsquigarrow$ ground set $= \{\text{edges}\}$, bases $=$ spanning forests

② $L \subset \mathbb{C}^{n+1}$, $\dim L = r$ $\rightsquigarrow$ ground set $= \{\text{images of } e_i \text{ under } \mathbb{C}^{n+1} \twoheadrightarrow L^\vee\}$, bases $=$ bases of $L^\vee$ from the list.

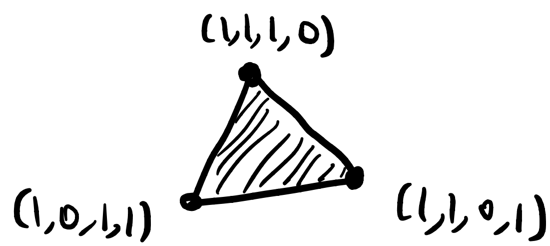
G



$\mathbb{C}^4 \twoheadrightarrow \mathbb{C}^3 = V^*$



$\mathcal{B} = \{012, 013, 023\}$



Tutte polynomial

$e \in [n]$ **loop**: e in no basis

coloop: e in every basis

deletion $M \backslash e$ $\qquad \mathcal{B}(M\backslash e) = \{B \mid e \notin B \in \mathcal{B}(M)\}$

(if e not coloop)

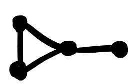
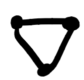
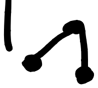
(ground set $[n] \backslash e$)

contraction M/e $\qquad \mathcal{B}(M/e) = \{B \backslash e \mid e \in B \in \mathcal{B}(M)\}$

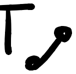
(if e not loop)

Tutte polynomial

$$T_M(x,y) = \begin{cases} x\, T_{M/e} & \text{if } e \text{ coloop} \\ y\, T_{M\backslash e} & \text{if } e \text{ loop} \\ T_{M\backslash e} + T_{M/e} & \text{if neither} \end{cases}$$

E.g. T  $= x\, T$  $= x\big(T$  $+ T$  $\big)$

$= x^2$

$= T$  $+ T$  $= x+y$

$$= x^3 + x^2 + xy$$

Characteristic polynomial

$$\chi_M(q) := (-1)^r\, T_M(1-q, 0) = (q-1)^2(q-2)$$

Geometric model ①: base polytope & K-thry

$L \subset \mathbb{C}^{n+1}$, i.e. $L \in Gr(r, n+1)$

$$\boxed{X_{P(M)}} \cong \boxed{\overline{T\cdot L}} \subset Gr(r,n+1) \quad \circlearrowleft T = (\mathbb{C}^*)^{n+1}$$

$$\begin{tikzcd} & Fl(1,r,n;n+1) \arrow[dl, "\pi_r"'] \arrow[dr, "\pi_{1n}"] & \\ Gr(r,n+1) & & \mathbb{P}^n \times \mathbb{P}^n \end{tikzcd}$$

Speyer '09, Fink-Speyer '12: $T_M(x,y)$ "=" $(\pi_{1n})_* \pi_r^* (\mathcal{O}_{\overline{T\cdot L}}(1))$

Cameron-Fink '18: $T_M(x,y)$ "=" # lattice pts of $P(M) + s\Delta_n + t\nabla_n$

Geometric model ②: Bergman fan & Chow rings

$$\begin{tikzcd} \text{wonderful cpt. } \boxed{X_{wnd}} \arrow[r, hook] \arrow[d] & X_{A_n} \text{ permutohedral variety} \arrow[d] \arrow[dr] & \\ \mathbb{P}L \arrow[r, hook] & \mathbb{P}^n \arrow[r, dashed, "crem"] & \mathbb{P}^n \end{tikzcd}$$

$\rightsquigarrow$ $\boxed{\Delta_M}$ Bergman fan $\subset \Sigma_{A_n}$ (= trop. lin. sp. (Ardila-Klivans '06))

$\quad \| $

$[X_{wnd}]$ homology class on X_{A_n}

Huh-Katz '12: image of Δ_M in $\mathbb{P}^n \times \mathbb{P}^n \rightsquigarrow \chi_M(q)$

Adiprasito-Huh-Katz '18: coeff. of $\chi_M(q)$ log-concave.
via Chow ring of X_{wnd}.

Geometric model ③: conormal fan (matroid duality) & Chow rings

$0 \to L \to \mathbb{C}^{n+1} \to L^\perp \to 0 \quad \overset{\text{dual}}{\underset{\text{mat.}}{\rightsquigarrow}} M^\perp$ from $\mathbb{C}^{n+1} \twoheadrightarrow L^\perp$

(bases of $M^\perp$: $\{[n] \setminus B \mid B \in \mathcal{B}(M)\}$)

$\boxed{X_{crit}} \rightsquigarrow \boxed{\Delta_{M,M^\perp}}$ conormal fan

$\downarrow$

$\mathbb{P}L \times \mathbb{P}L^\perp$

Lopez de Medrano – Rincon – Shaw '20 : CSM classes of matroids

$\rightsquigarrow$ coeff. $\frac{\chi_M(q+1)}{q}$ $\left(\approx T_M(q,0)\right)$

Ardila – Denham – Huh '20 : log-concavity of $T_M(q,0)$ coeff.

All-unifying model: tautological bdls of matroids

$$\begin{tikzcd} X_{A_n} \arrow[d] \arrow[dr, "\varphi"] & 0 \to \mathcal{S} \to \mathbb{C}^{n+1} \to \mathcal{Q} \to 0 \arrow[d, dash] \\ X_{P(M)} = \overline{T \cdot L} \arrow[r, hook] & Gr(r, n+1) \end{tikzcd}$$

Defn $\mathcal{S}_M := \varphi^* \mathcal{S}$, $\mathcal{Q}_M := \varphi^* \mathcal{Q}$ vec. bdls on X_{A_n}.

(T-equivariant) (K-class if not realizable)

N.B. (a) Chern roots of $\mathcal{S}_M$ & $\mathcal{Q}_M$ easy to describe combinatorlally

(i.e. $\mathcal{S}_M$ "$=$" $\bigoplus_i \mathcal{L}_i$)

(b) $c_1(\mathcal{Q}_M) = D_{P(M)}$, $c_{top}(\mathcal{Q}_M) = \Delta_M$

(c) crem $\mathcal{Q}_M^\vee = \mathcal{S}_{M^\perp}$

(d) $\mathrm{Biproj}(\mathcal{S}_M^\vee \oplus \mathcal{Q}_M) = \mathbb{P}(\mathcal{S}_M^\vee) \times_{X_{A_n}} \mathbb{P}(\mathcal{Q}_M)$

"$=$" $\Delta_M \times \Sigma_{A_n} \times \Delta_{M^\perp}$

Thm (Berget – E. – Spink – Tseng)

$$\begin{tikzcd} & X_{A_n} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\text{crem}"] & & \mathbb{P}^n \end{tikzcd}$$

$\alpha =$ hyperplane pullback via π_1
$\beta =$ —— " —— π_2

$$\sum \left(\int_{X_{A_n}} \alpha^i \beta^j c_k(S_M^\vee) c_\ell(Q_M) \right) x^i y^j z^k w^\ell$$

$$= (x+y)^{-1} (y+z)^r (x+w)^{n+1-r} T_M\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right)$$

Cor (a) All prev. formulas for T_M or χ_M

(b) Log-concavity.
(need borrow Kähler package)

E.g.
$\ell = n+1-r$
$x = 1$
$y = 0$
k free

$$[w^{n+1-r}]\, z^r (w+1)^{n+1-r} T_M\left(\frac{1}{z}, \frac{1}{w+1}\right)$$
$$= z^r T_M\left(\frac{1}{z}, 0\right)$$

KEY Tools

(1) $\sum_k \chi(\Lambda^k \mathcal{E})\, u^k = (u+1)^{\mathrm{rk}\,\mathcal{E}} \cdot \sum_k \left(\int_{X_{A_n}} \alpha^{n-k} c_k(\mathcal{E}) \right) \left(\frac{u}{u+1} \right)^k$

for special $\mathcal{E}$ on X_{A_n} (e.g. $\mathcal{S}_M^\vee$, $\mathcal{Q}_M$)

(2) T-equivariant localization.

(3) Valuativity

(4) Hopf monoid