Enumerative geometry in the Extended Tropical Vertex Group

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[arXiv:2012.05069][arXiv:1912.09956]

Overview

1. Mirror symmetry
2. Scattering diagrams in the extended tropical vertex group $\tilde{\mathbb{V}}$
3. Gromov–Witten invariants in $\tilde{\mathbb{V}}$

Mirror Symmetry

Mirror symmetry first appears in string theory as a *duality* between Calabi–Yau varieties

$$(X, \omega, J) \longleftrightarrow (\check{X}, \check{J}, \check{\omega})$$

P. Candelas, X. De La Ossa, P. S. Green and L. Parkes computed Gromov–Witten invariants for the quintic 3-fold, and it was the beginning of the interplay between enumerative geometry and mirror symmetry.

SYZ fibration

According to the Strominger–Yau–Zlasov (SYZ) conjecture

- (X_t, ω_t, J_t) and $(\check{X}_t, \check{J}_t, \check{\omega}_t)$ appears in *families*;
- as t approaches the *large complex structure limit* t^*, $X_t \to B$ and $\check{X}_t \to \check{B}$, where B and $\check{B}$ are integral affine manifolds;
- in a neighbourhood of t^*: (X_t, ω_t) admits a Lagrangian fibration over B, while $(\check{X}_t, \check{J}_t)$ admits a complex torus fibration over $\check{B}_0 \subset \check{B}$

$$\begin{array}{ccc} (X_t, \omega_t) & \qquad\qquad & (\check{X}_t, \check{J}_t) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \check{\pi}} \\ B & & \check{B}_0 \end{array}$$

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Unless $\dim_{\mathbb{C}} X = 1$, π and $\check{\pi}$ are singular:

- restricting on the smooth locus, π and $\check{\pi}$ are dual torus fibrations [see toy model]
- due to the presence of singularities, *quantum corrections* are needed in order to get a globally well–defined complex structure on the mirror.

Toy model

Let B_0 be a smooth integral affine manifold, $\Lambda \subset T^*B_0$ and $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z}) \subset TB_0$

$$(X = T^*B_0/\Lambda, \omega) \xrightarrow{\pi} B_0 \xleftarrow{\check{\pi}} (\check{X} = TB_0/\Lambda^*, \check{J})$$

ω is the canonical symplectic form on T^*B_0 (in local affine coordinates (x^i, y^i) it is $\omega = \sum_i dx^i \wedge dy^i$) and $\check{J}$ is a complex structure on TB_0 (that in local affine coordinates (x^i, y^i) reads $\check{J}(\frac{\partial}{\partial x^i}) = \sqrt{-1}\frac{\partial}{\partial y^i}$).
The discrete Legendre transform defines complex coordinates on X which are symplectic coordinates on $\check{X}$.

Quantum corrections

- Fukaya's approach [Fukaya,05]: the asymptotic behaviour of deformations of $(\check{X}, \check{J}_\hbar)$ as $\hbar \to 0$ encodes *enumerative geometric data* of $(X, \omega_\hbar = \hbar^{-1}\omega)$. In particular, the quantum corrections are encoded in counting *pseudoholomorphic disks* bounding the fibers $\pi^{-1}(b) = X_b$.

Relaying on the fact that B has integral affine structure, we expect to count tropical curves on B_0 underlying holomorphic curves on $(X, \omega_\hbar)$:

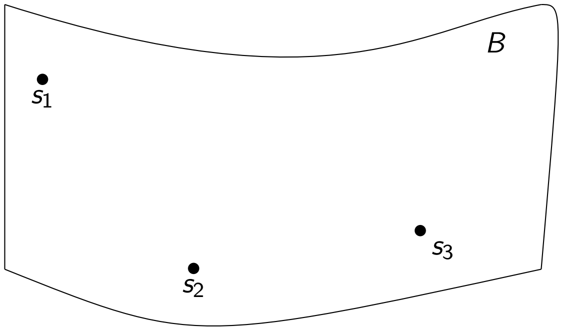
- Kontsevich–Soibelman [KS,06]

$\Rightarrow$ Scattering Diagrams

- Gross–Siebert program [GS,06][GS,10]

Scattering diagrams

Scattering diagrams are combinatorial objects: naively defined as a collections of lines of rational slope in B decorated with *automorphisms*



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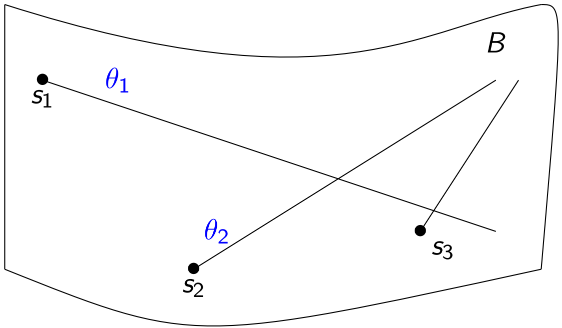

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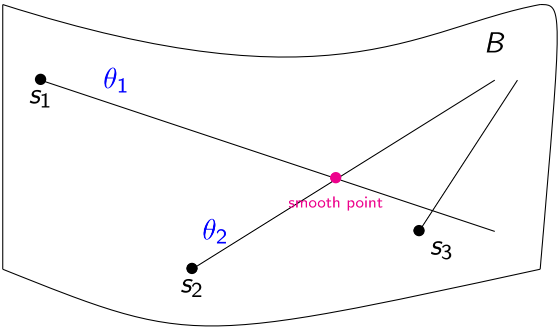

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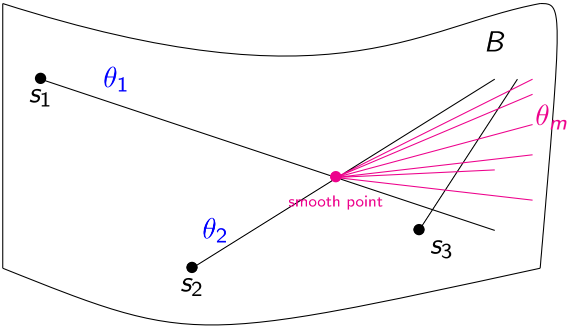

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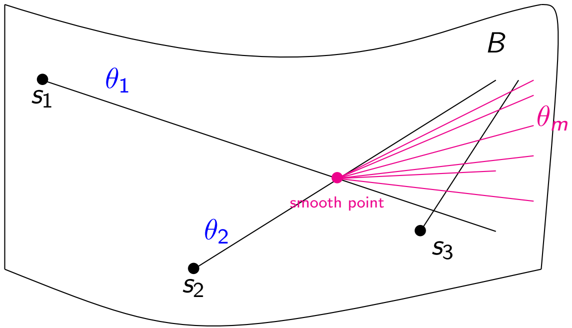


Locally, in a neighbourhood of a smooth point,

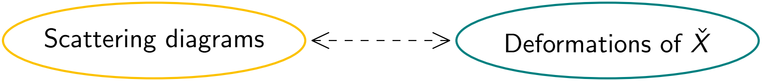
$$\mathfrak{D} = \{(\text{line}_j, \theta_j), j = 1, 2\} \rightsquigarrow \mathfrak{D}_\infty = \mathfrak{D} \cup \{(\text{ray}_m, \theta_m)\}$$

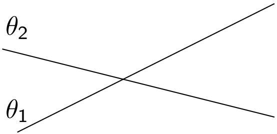
Examples

- non archimedean K3 [Kontsevich–Soibelman,06]
- $\mathbb{P}^2$ [Gross,09] mirror symmetry of $\mathbb{P}^2$ can be stated and proved via tropical geometry (i.e $J_{\mathbb{P}^2} = J^{\mathrm{trop}}_{\mathbb{P}^2}$, where J is Givental J-function).
- log Calabi–Yau surfaces $U := Y \setminus D$ where (Y, D) is a Looijenga pair [Gross–Hacking–Keel,15]; under certain condition on D, the mirror family is $\check{\mathcal{X}} \to S := \operatorname{Spec} \mathbb{C}[P]$, where $P := NE(Y)$ and $\check{\mathcal{X}} \subset \mathbb{A}^3_S$.
- cubic surface [Gross–Hacking–Keel–Siebert,19]
- non–toric del Pezzo [Barrot,19]
- quantum mirror of log Calabi–Yau surfaces [Bousseau,20]

Scattering diagrams and deformations

- K. Chan, N. Conan–Leung and N.Z. Ma [CLM20], according with Fukaya's approach to mirror symmetry, proved that the asymptotic behaviour (as $\hbar \to 0$) of the infinitesimal deformations of $(\check{X} = TB_0/\Lambda^*, \check{J}_\hbar)$ gives consistent scattering diagrams.
- K. Chan and N. Z. Ma [CM20] showed that the infinitesimal deformations encode the data of tropical disks.





solutions of Maurer–Cartan eq.
$d\psi + \frac{1}{2}[\psi, \psi] = 0$
up to gauge equivalence
locally, deformations are gauge equivalent to 0

θ_i represents a gauge group element

D–branes mirror symmetry

- in physics *open string*: what is the "complex D–brane" mirror of a "Lagrangian D–brane"? [Vafa,98][Hori–Iqbal–Vafa,00][Aganagic–Vafa,00];

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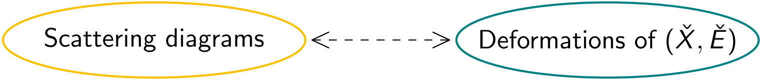
- Kontsevich Homological Mirror Symmetry [Kontsevich,94][KS,01]: equivalence of categories $\mathrm{Fuk}(X)$ " $\cong$ " $\mathrm{D}^b\,\mathrm{Coh}(\check{X})$.
- Fukaya's approach [Fukaya,05]: let $L_1, \dots, L_r \subset X$ be a special Lagrangian submanifolds. Then $L = L_1 \sqcup \dots \sqcup L_r \to B_0$ is a ramified cover over $B_0 = B_{00} \cup \{\text{ramification points}\}$.
  The holomorphic structure on the mirror rank r bundle $\check{E} \to \check{X}$ is defined including quantum corrections which encode counting of *pseudoholomrphic strips* which bounds L and the fiber $\pi^{-1}(b)$, $b \in B$.

Scattering diagrams and deformations of holomorphic pairs

A holomorphic pair $(\check{X}, \check{E})$ is the datum of a complex manifold $\check{X}$ together with a holomorphic vector bundle $\check{E} \to \check{X}$.
In [-,19], the author studied the relationship between scattering diagrams and deformations of $(\check{X}, \check{E})$



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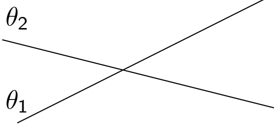

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The new feature is the extended tropical vertex group $\tilde{\mathbb{V}}$, where the scattering diagrams are defined.

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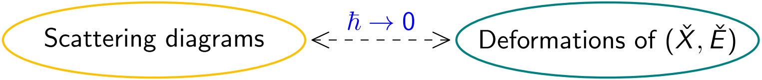

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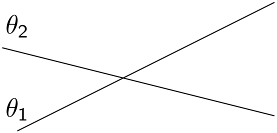


θ_i represents a gauge group element

The new feature is the extended tropical vertex group $\tilde{\mathbb{V}}$, where the scattering diagrams are defined.

The Extended tropical vertex group $\tilde{\mathbb{V}}$

$\tilde{\mathbb{V}}$ is a subgroup of the gauge group acting on solutions of the Maurer–Cartan equation for $(\check{X}, \check{E})$, in the limit $\hbar \to 0$.

Definition ([–,19])

The **extended tropical vertex group** $\tilde{\mathbb{V}} := \exp \tilde{\mathfrak{h}}$ where the Lie algebra $\tilde{\mathfrak{h}}$

$$\tilde{\mathfrak{h}} := \bigoplus_{m \in \Lambda \setminus \{0\}} \mathfrak{w}^m \, \underline{\mathbb{C}}(U_m, \mathfrak{gl}(r, \mathbb{C}) \oplus m^\perp)[\![t]\!]$$

$$[(A\mathfrak{w}^m, \mathfrak{w}^m \partial_n), (A'\mathfrak{w}^{m'}, \mathfrak{w}^{m'} \partial_{n'})]_{\tilde{\mathfrak{h}}} := \Big([A, A']_{\mathfrak{gl}} \mathfrak{w}^{m+m'} + \big(A'\langle n, m'\rangle - A\langle n', m\rangle\big) \mathfrak{w}^{m+m'}, \mathfrak{w}^{m+m'} \partial_{\langle n, m'\rangle n' - \langle n', m\rangle n} \Big).$$

$\tilde{\mathbb{V}}$ is a group with the Baker-Campbell-Hausdorff product.

Remark: why *extended*

M. Gross, R. Pandharipande and B. Siebert introduced the tropical vertex group $\mathbb{V}$:

Definition ([GPS,10])

The **tropical vertex group** $\mathbb{V} := \exp \mathfrak{h}$, where the Lie algebra $\mathfrak{h}$ is

$$\mathfrak{h} := \left(\bigoplus_{m \in \Lambda \smallsetminus \{0\}} \mathbb{C}\, \mathfrak{w}^m \cdot \boldsymbol{m}^\perp \right) \hat{\otimes}_{\mathbb{C}} \mathbb{C}[[t]] \subset \left(\mathbb{C}[\Lambda] \hat{\otimes}_{\mathbb{C}} \mathbb{C}[[t]] \right) \otimes_{\mathbb{Z}} \boldsymbol{\Lambda}^*,$$

$$[\mathfrak{w}^m \partial_n, \mathfrak{w}^{m'} \partial_{n'}]_{\mathfrak{h}} := \mathfrak{w}^{m+m'} \partial_{\langle n, m' \rangle n' - \langle n', m \rangle n}.$$

$\mathbb{V}$ is a group with the BCH product.

Scattering diagrams in $\tilde{\mathbb{V}}$

Definition

A $(2\text{-}\mathrm{dim})$ scattering diagram $\mathfrak{D}$ is a collection of *walls* $\mathsf{w}_i = (\mathfrak{d}_i, \overrightarrow{f}_i)$, where:

- $\mathfrak{d}_i$ can be either a *line* $\mathfrak{d}_i = m_i\mathbb{R}$ or a *ray* $\mathfrak{d}_i = \xi_0 - m_i\mathbb{R}_{\geq 0}$ through the point $\xi_0 \in \Lambda_{\mathbb{R}}$ in the direction of $m_i \in \Lambda$;
- $\overrightarrow{f}_i = (\mathrm{I}_r + At\mathfrak{w}^{\mathbf{m_i}}, 1 + t\mathfrak{w}^{\mathbf{m}_i} f)$ where $A \in \mathfrak{gl}(r, \mathbb{C}[\mathfrak{w}^{m_i}][\![t]\!])$, $f \in \mathbb{C}[\mathfrak{w}^{m_i}][\![t]\!]$.

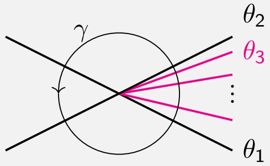
$$\theta_i := \exp\left(\log(\mathrm{I}_r + At\mathfrak{w}^{m_i}), \log(1 + t\mathfrak{w}^{m_i} f)\partial_{n_i}\right) \in \tilde{\mathbb{V}}$$

Moreover, for every $N > 0$ we assume there are only finitely many walls w_i such that $\theta_i \not\equiv 1 \mod t^N$.

Consistent scattering diagrams $\mathfrak{D}_\infty$

Theorem [Kontsevich–Soibelman,06]

Let $\mathfrak{D} = \{(m_1\mathbb{R}, \overrightarrow{f}_1); (m_2\mathbb{R}, \overrightarrow{f}_2)\}$. The *consistent* scattering diagram $\mathfrak{D}_\infty := \mathfrak{D} \cup \{(m_i\mathbb{R}_{\geq 0}, \overrightarrow{f}_i)\}_{i\geq 3}$ is the unique (up to equivalence) one such that $\Theta_{\gamma,\mathfrak{D}_\infty} = \mathrm{Id}_{\tilde{\mathbb{V}}}$ for every generic loop $\gamma\colon [0,1] \to \Lambda_\mathbb{R}$.



$$\Theta_{\gamma,\mathfrak{D}_\infty} := \theta_2^{-1} \cdot \theta_1 \cdot \theta_3 \ldots \cdot \theta_2 \cdot \theta_1^{-1}$$

Enumerative geometric interpretation

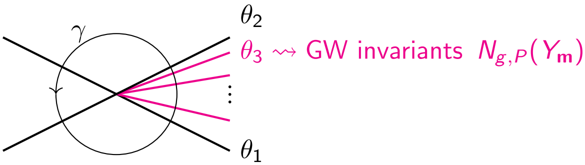
Q: Which invariants do we compute via scattering diagrams in $\tilde{\mathbb{V}}$?

Enumerative geometric interpretation

Q: Which invariants do we compute via scattering diagrams in $\tilde{\mathbb{V}}$?

- scattering diagrams compute log Gromov–Witten invariants for log Calabi–Yau surfaces:

$$\mathfrak{D} = \{(m_1\mathbb{R}, \overrightarrow{f}_1); (m_2\mathbb{R}, \overrightarrow{f}_2)\} \rightsquigarrow \mathfrak{D}_\infty$$

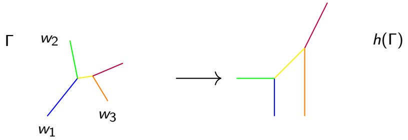


- the combinatorics behind consistent scattering diagrams encodes tropical curve counting.

Tropical curves

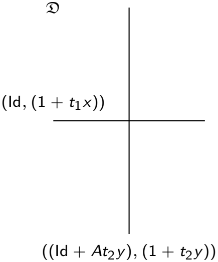
Tropical curves are equivalence classes of parametrized tropical curves (h, Γ)

- Γ is a weighted (each edges has a weight $w \in \mathbb{Z}_{>0}$), connected, finite graph without divalent and univalent vertices (with unbounded edges);
- $h\colon \Gamma \to B \cong \mathbb{R}^2$ is proper:
  - for every edge $E \in \Gamma^{[1]}$, $h(E) \subset \xi_j + m_j\mathbb{R}$ is contained in an affine line of rational slope;
  - for every vertex V, let $\{E_j\}_j$ be the set of edges adjacent to V, then $\sum_j w_j m_j = 0$.
- (h, Γ) and (h', Γ') are isomorphic if there exists $\Phi\colon \Gamma \to \Gamma'$ which preserves weights and $h' = \Phi \circ h$.



Scattering diagrams and tropical curves

Let $\mathfrak{D}$ be a scattering diagram, there is an algorithm to compute the consistent scattering diagram $\mathfrak{D}_\infty$



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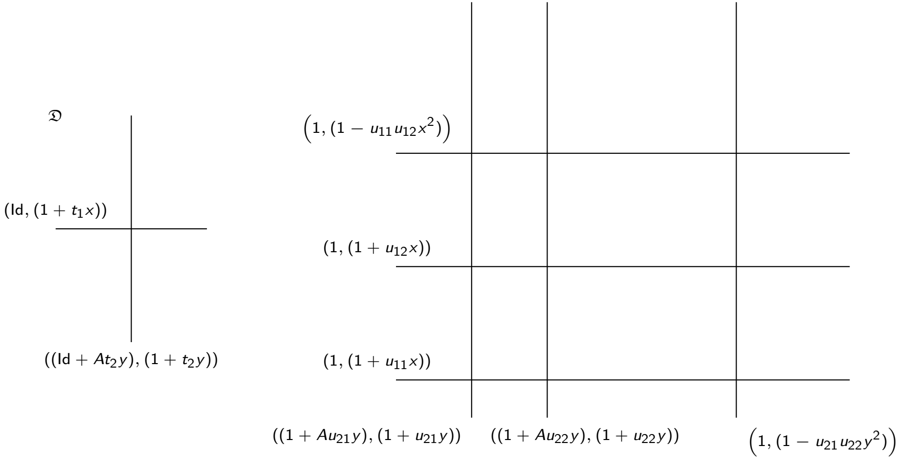

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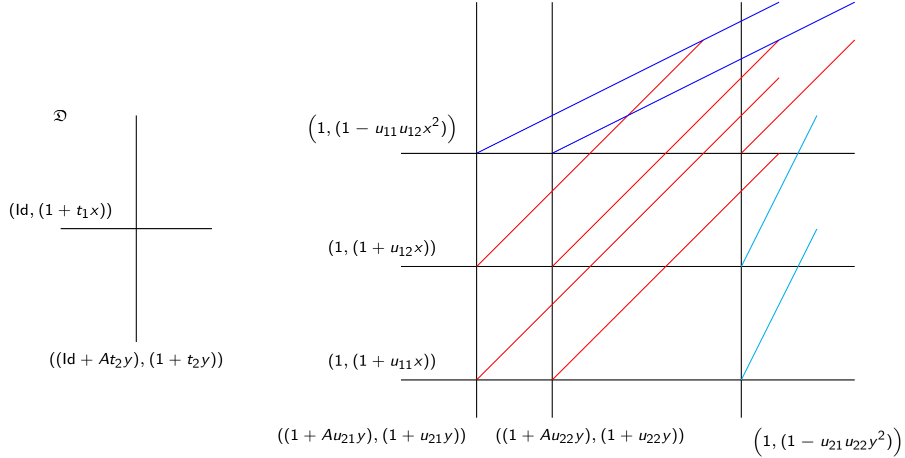

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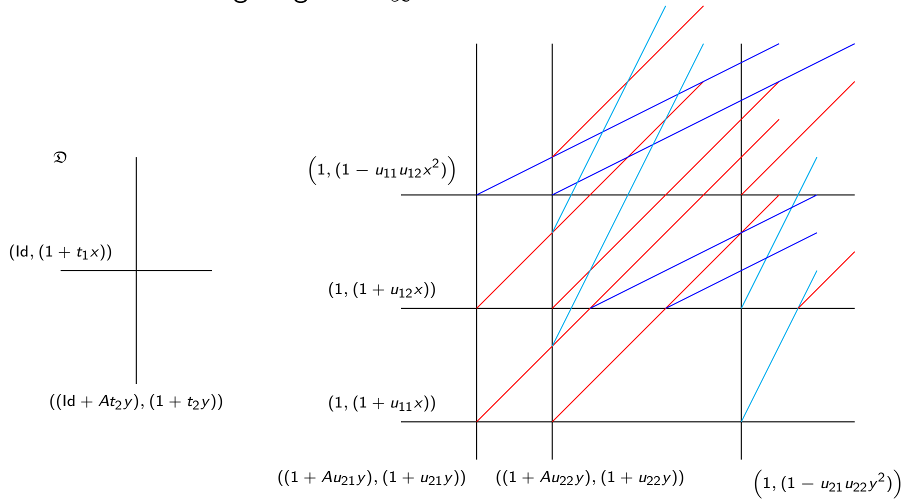

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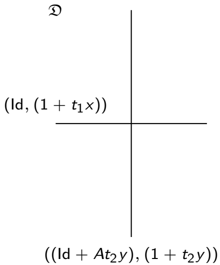

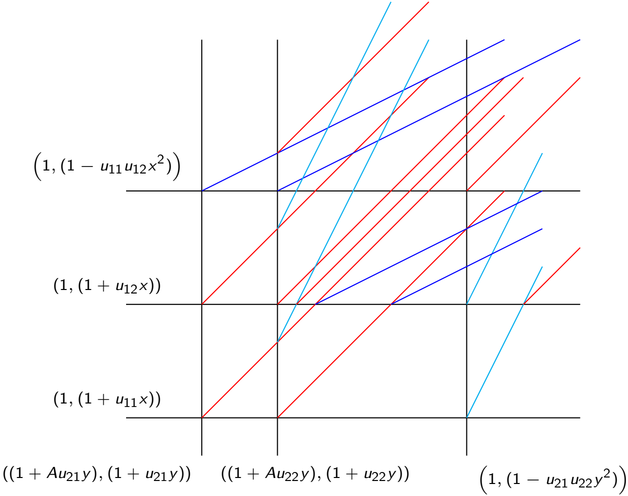

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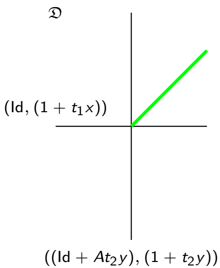
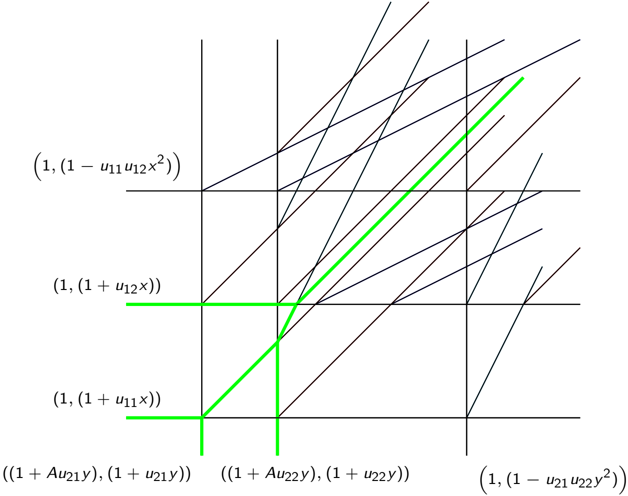
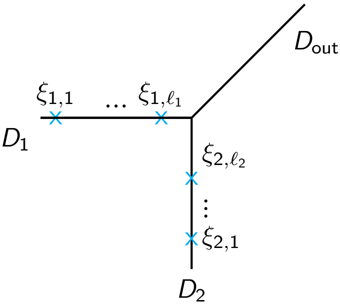




Log Calabi–Yau surfaces

Let $\mathbf{m} = (m_1, m_2, m_{\mathrm{out}}) \in \Lambda \setminus 0$ primitive, let $\{\xi_{ij}, j = 1, \ldots, \ell_i\}$ be generic points on $D_i := -m_i\mathbb{R}_{\geq 0}$ for $i = 1, 2$ and $D_{\mathrm{out}} = m_{\mathrm{out}}\mathbb{R}_{\geq 0}$.

Toric



$(\overline{Y}_{\mathbf{m}}, \partial\overline{Y}_{\mathbf{m}} := D_1 + D_2 + D_{\mathrm{out}})$

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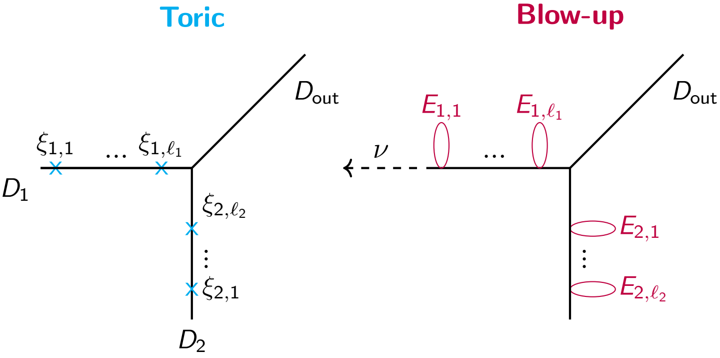

$(\overline{Y}_{\mathbf{m}}, \partial\overline{Y}_{\mathbf{m}} := D_1 + D_2 + D_{\text{out}})$ $(Y_{\mathbf{m}}, \partial Y_{\mathbf{m}} :=$ strict transform of $\partial\overline{Y}_{\mathbf{m}})$

Log Gromov–Witten invariants

Let $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ be a pair of weight vectors $\mathbf{w}_i = (w_{i1}, ..., w_{is_i})$, $i = 1, 2$ of lenght s_i such that $\sum_i |\mathbf{w}_i| = \lambda_{\mathbf{w}} m_{\text{out}}$. The curve class $\beta_{\mathbf{w}} \in H_2(\overline{Y}_{\mathbf{m}}, \mathbb{Z})$ is such that

$$D_i \cdot \beta_{\mathbf{w}} = |\mathbf{w}_i|, \quad D_{\text{out}} \cdot \beta_{\mathbf{w}} = \lambda_{\mathbf{w}} \text{ and } D \cdot \beta_{\mathbf{w}} = 0 \text{ if } D \neq \{D_1, D_2, D_{\text{out}}\}$$

$N_{0,\mathbf{w}}(\overline{Y}_{\mathbf{m}})$ is the *number* of (log stable) rational curves of class $\beta_{\mathbf{w}}$.

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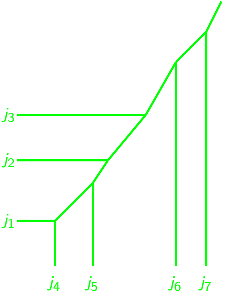
$N_{0,\mathbf{w}}(\overline{Y}_{\mathbf{m}})$ is the *number* of (log stable) rational curves of class $\beta_{\mathbf{w}}$.

Let $\mathbf{P} = (P_1, P_2)$ be a vector partitions such that $P_i = p_{i1} + ... + p_{i\ell_i}$, for $i = 1, 2$ and $\sum_i |P_i| m_i = \lambda_{\mathbf{P}} m_{\text{out}}$. Let $\beta \in H_2(\overline{Y}_{\mathbf{m}}, \mathbb{Z})$ be such that

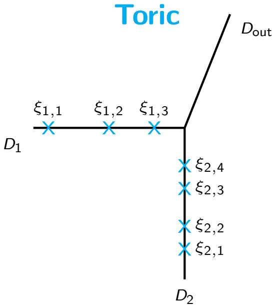
$$D_i \cdot \beta = |P_i|, \quad D_{\text{out}} \cdot \beta = \lambda_{\mathbf{P}} \text{ and } D \cdot \beta = 0 \text{ if } D \neq \{D_1, D_2, D_{\text{out}}\}$$

The curve class $\beta_{\mathbf{P}} \in H_2(Y_{\mathbf{m}}, \mathbb{Z})$ is $\beta_{\mathbf{P}} = \nu^*(\beta) - \sum_{i,j} p_{ij}[E_{ij}]$.
$N_{0,\mathbf{P}}(Y_{\mathbf{m}})$ is the *number* of (log stable) rational curves of class $\beta_{\mathbf{P}}$.

Tropical vs toric invariants



$\mathbf{w} = (j_1 m_1, \ldots, j_3 m_1, j_4 m_2, \ldots, j_7 m_2)$



$(\overline{Y}_{\mathbf{m}}, \partial\overline{Y}_{\mathbf{m}} := D_1 + D_2 + D_{\text{out}})$

$$N^{\text{trop}}_{0,\mathbf{w}} = N_{0,\mathbf{w}}(\overline{Y}_{\mathbf{m}}) \prod_{k\geq 1} j_k$$

[Bousseau,20][GPS,10]

Theorem 1 ([–,20])

Let m_1, m_2 be primitive non zero vectors in Λ. Set

$$\mathfrak{D} = \Big\{ \Big(\mathfrak{d}_i = m_1\mathbb{R}, \overrightarrow{f}_i = (\mathrm{I}_r + A_1 t_i \mathfrak{w}^{m_1}, 1 + t_i\mathfrak{w}^{m_1})\Big);$$
$$\Big(\mathfrak{d}_i = m_2\mathbb{R}, \overrightarrow{f}_j = (\mathrm{I}_r + A_2 s_j \mathfrak{w}^{m_2}, 1 + s_j\mathfrak{w}^{m_2})\Big) \Big| 1 \leq i \leq \ell_1,\ 1 \leq j \leq \ell_2 \Big\}$$

where $A_1, A_2 \in \mathfrak{gl}(r, \mathbb{C})$ and **assume** $[A_1, A_2] = 0$*. Then for every wall $(\mathfrak{d}_{\mathrm{out}} = m_{out}\,\mathbb{R}_{\geq 0}, \overrightarrow{f}_{\mathrm{out}}) \in \mathfrak{D}_\infty \setminus \mathfrak{D}$:*

$$\log \overrightarrow{f}_{\mathrm{out}} = \Big(\sum_{k\geq 1} \sum_{\mathbf{P}} \sum_{\mathbf{k}\vdash\mathbf{P}} N_{0,\mathbf{w}(\mathbf{k})}(\overline{Y}_{\mathbf{m}}) \left(C_1(\mathbf{k}_1)A_1 + C_2(\mathbf{k}_2)A_2\right) t^{P_1} s^{P_2} \mathfrak{w}^{k m_{\mathrm{out}}},$$
$$\sum_{k\geq 1} \sum_{\mathbf{P}=(P_1,P_2)} k N_{0,\mathbf{P}}(Y_{\mathbf{m}}) t^{P_1} s^{P_2} \mathfrak{w}^{k m_{\mathrm{out}}} \Big)$$

where the sum is over all $\mathbf{P} = (P_1, P_2)$ such that $\sum_{i=1}^{2} |P_i| m_i = k m_{\mathrm{out}}$ and $C_i(\mathbf{k}_i)$ are explicit constants which depend on partitions $\mathbf{k}_i$ of P_i.

The result for $N_{0,\mathbf{P}}(Y_{\mathbf{m}})$ is analogous to the result of [GPS,2010], and it can be rephrased by saying that $\log f_{\text{out}}$ is a generating series of $N_{0,\mathbf{P}}(Y_{\mathbf{m}})$.

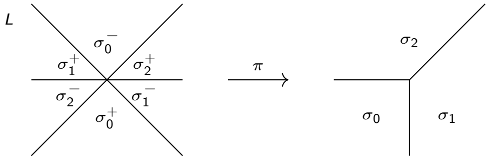
Idea of the proof:

- Scattering diagrams $\leftrightsquigarrow$ tropical curve
- tropical curves counting $\leftrightsquigarrow$ log (toric) Gromov–Witten invariants
- Degeneration formula

  $N_{0,\mathbf{P}}(Y_{\mathbf{m}}) = \sum_{\mathbf{k}\vdash\mathbf{P}} N_{0,\mathbf{w}(\mathbf{k})}(\overline{Y}_{\mathbf{m}}) \prod_{j=1}^{2}\prod_{l=1}^{l_j} \frac{l^{k_{jl}}}{k_{jl}!}(R_l)^{k_{jl}}.$

Remarks: Lagrangian multisections

- Lagrangian multisections $\mathbb{L} = (L, \pi, \mathcal{P}_\pi, \varphi')$ allow to reconstruct the mirror sheaf of a Lagrangian $\pi\colon L \to B$. [Chan–Ma–Suen,21]
  - ∗ reconstruct $T_{\mathbb{P}^2}$ from the data of $\mathbb{L} = (L, \pi\colon L \xrightarrow{2:1} \mathbb{R}^2, \varphi)$ together with local automorphisms $\Theta_{10}, \Theta_{21}, \Theta_{02}$ which encode the *quantum corrections*.[Suen,21]



$$\varphi_k^\pm : \sigma_k^\pm \to \mathbb{R}$$

Q:are the quantum corrections Θ_{ij} elements of $\tilde{\mathbb{V}}$?

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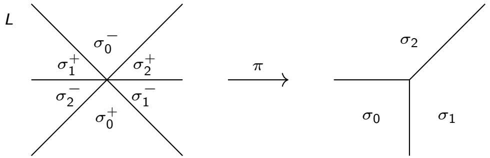

$$\mathcal{E}_i = \mathcal{O}_{\sigma_i^-} \oplus \mathcal{O}_{\sigma_i^+} \to U_{\sigma_i}$$

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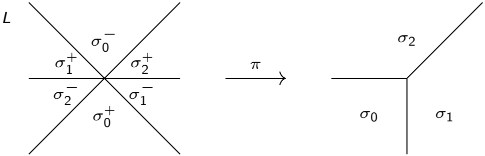

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$$\Theta_{10} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \in \mathrm{Aut}(\mathcal{E}_0|_{U_{10}})$$

$$\Theta_{21} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{Aut}(\mathcal{E}_2|_{U_{02}})$$

$$\Theta_{02} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \in \mathrm{Aut}(\mathcal{E}_1|_{U_{21}})$$

Q:are the quantum corrections Θ_{ij} elements of $\tilde{\mathbb{V}}$?

Thank you for your attention!

Deformations of $(\check{X}, E)$

$*$ $\pi\colon \mathcal{X} \to B_\epsilon(0)$ holomorphic, proper

$\mathcal{X}_0 := (\check{X}, \bar{\partial}_{\check{X}})$

$\pi^{-1}(t) =: \mathcal{X}_t$ is a complex manifold

$\mathcal{X}_t = (\check{X}, \bar{\partial}_{\check{X}} + \varphi_t \lrcorner \partial)$,

$\varphi_t \in \Omega^{0,1}(\check{X}, T^{1,0}\check{X})$

$+$ integrability

$*$ $\varpi\colon \mathcal{E} \to \mathcal{X}$ holomorphic

$\mathcal{E}_0 := (E, \bar{\partial}_E)$

$\mathcal{E}|_{\pi^{-1}(t)} =: \mathcal{E}_t$

$(\mathcal{E}_t, \mathcal{X}_t)$ is a holomorphic pair

$\mathcal{E}_t = (E, \bar{\partial}_E + A_t + \varphi_t \lrcorner \nabla^E)$,

$A_t \in \Omega^{0,1}(\check{X}, \mathrm{End}\, E)$

$+$ integrability

Infinitesimal deformations of $(\check{X}, E)$

Let $(E, \bar{\partial}_E)$ be a rank r holomorphically trivial vector bundle on $\check{X}$. Then define $\mathbf{A}(E) := \operatorname{End} E \oplus T^{1,0}\check{X}$ and $(\mathbf{A}(E), \bar{\partial}_{\mathbf{A}(E)})$ is a holomorphic bundle on $\check{X}$.

$$\mathrm{KS}(\check{X}, E) := (\Omega^{0,\bullet}(\check{X}, \mathbf{A}(E)), \bar{\partial}_{\mathbf{A}(E)}, [-,-]_{\mathrm{KS}}),$$

where $[-,-]_{\mathrm{KS}}$ is the Lie bracket.

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where $[-,-]_{\mathsf{KS}}$ is the Lie bracket.
Infinitesimal deformations of $(\check{X}, E)$ are elements in degree one $(A, \varphi) \in \Omega^{0,1}(\check{X}, \mathbf{A}(E))[\![t]\!]$, which are solutions of the **Maurer-Cartan equation**

$$\bar{\partial}_{\mathbf{A}(E)}(A, \varphi) + \frac{1}{2}[(A, \varphi), (A, \varphi)]_{\mathsf{KS}} = 0 \tag{1}$$

up to **gauge equivalence**. The action of the gauge group is defined by $h \in \Omega^0(\check{X}, \mathbf{A}(E))[\![t]\!]$ via $\exp(h) * (A, \varphi)$.