Fano varieties from homogeneous vector bundles

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Definition

X (smooth, projective) is a *Fano* variety if $-K_X$ is ample.

Feature: *boundedness*. In every dimension there exists a finite number of deformation families of Fano varieties: they can be classified. An explicit bound for the number exists, but it is (hugely) not sharp. (In dimension 1, 3^{18} vs 1!)

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- dim 1: $\mathbb{P}^1$.
- dim 2: 10 families: $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{Bl}_p \mathbb{P}^2$, $|p| \leq 8$ (del Pezzo).
- dim 3: 105 (17+88 according to ρ) (Iskovskikh, Mori–Mukai).
- dim ≥ 4 ? (although many examples known, cf. Batyrev, Coates, Galkin, Kalashnikov, Kasprzyk, Küchle, Prince, Strangeway, etc)

There are many ongoing projects attempting to classify Fano 4-folds in dimension 4 (e.g. using Mirror Symmetry), but no final picture yet.

To refine the classification, we use the *index* ι_X of a Fano X (greatest integer for which $-K_X$ is divisible in $\mathrm{Pic}(X)$). If X has dimension n, ι_X is bounded by $n+1$ and

- $\iota_X = n+1 \iff X \cong \mathbb{P}^n$.
- $\iota_X = n \iff X \cong Q_2 \subset \mathbb{P}^{n+1}$.
- $\iota_X = n-1$: X is called *del Pezzo* manifold (classified).
- $\iota_X = n-2$ & $\rho_X = 1$: X is called *Mukai manifold* (classified).
- $\iota_X \geq \frac{n+1}{2}$ & $\rho_X > 1$ & $n \geq 5$: Wisniewski (classified).

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As we can see, the first outstanding case is for Fano 4-folds of index $\iota_X = 1$. This project has three main aims.

Aim

- *Produce many new examples in dimension 4, index 1.*
- *Establish a dictionary between the (biregular) list of examples and the (birational) Mori–Mukai-style language.*
- *(in higher dimension) Construct Fano with special Hodge-theoretical properties, and link with hyperkähler geometry.*

Our tool of choice to explore the landscape is to construct Fano varieties as zero loci of general sections of **homogeneous vector bundles over homogeneous varieties**.
The proper definition would require us to start with a (connected, simply connected, semisimple, complex) Lie group G, a parabolic subgroup P, and a vector bundle $\mathcal{E}$ on G/P isomorphic to $G \times_P E$, where E is a rational representation of P.

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The proper definition would require us to start with a (connected, simply connected, semisimple, complex) Lie group G, a parabolic subgroup P, and a vector bundle $\mathcal{E}$ on G/P isomorphic to $G \times_P E$, where E is a rational representation of P.
When $G/P = \mathrm{Gr}(k, n)$ a typical example is $\mathcal{E}$ completely reducible, that is $\mathcal{E} = \bigoplus \Sigma_\alpha \mathcal{U} \otimes \Sigma_\beta \mathcal{Q}$ on $\mathrm{Gr}(k, n)$, with $\mathcal{U}$ the rank k tautological, $\mathcal{Q}$ the rank $n - k$ quotient.
Advantages:

- Abundance of examples;
- Classification (of Fano of this type) can be reduced to a combinatorial problem;
- Easy to compute invariants (e.g. Hodge numbers) using Borel–Bott–Weil and Koszul complex.

Remark: All complete intersections are examples, but there are much more!

Inspiration: Mukai's classification of prime Fano threefolds.
There are 17 families of Fano threefolds with $\rho_X = 1$. Of 7 with $\iota_X > 1$ all except $\mathbb{V}_5$ are complete intersections in $w\mathbb{P}$. Of 10 with $\iota_X = 1$ $(g = 2, \ldots, 10, 12)$, $g = 2, 3, 4, 5$ are complete intersections in $w\mathbb{P}$.
The remaining Fano have a nice description as $X = (G, \mathcal{F})$ – that is (the general) X is the zero locus of a (general) section of $\mathcal{F}$ in G.

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- $\mathbb{V}_5$. $(\frac{-K_X}{2})^3 = 5$. $(\mathrm{Gr}(2,5), \mathcal{O}(1)^{\oplus 3})$;
- X_6, $-K_X^3 = 10$. $(\mathrm{Gr}(2,5), \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2))$.
- X_7, $-K_X^3 = 12$. $(\mathrm{Gr}(5,10), \mathrm{Sym}^2 \mathcal{U}^\vee \oplus \mathcal{O}(\frac{1}{2})^{\oplus 7})$ or $(\mathrm{Gr}(2,5), \mathcal{U}^\vee(1) \oplus \mathcal{O}(1))$.
- X_8, $-K_X^3 = 14$. $(\mathrm{Gr}(2,6), \mathcal{O}(1)^{\oplus 5})$.
- X_9, $-K_X^3 = 16$. $(\mathrm{Gr}(3,6), \bigwedge^2 \mathcal{U}^\vee \oplus \mathcal{O}(1)^{\oplus 3})$.
- X_{10}, $-K_X^3 = 18$. $(\mathrm{Gr}(2,7), \mathcal{Q}^\vee(1) \oplus \mathcal{O}(1)^{\oplus 2})$.
- X_{12}, $-K_X^3 = 22$. $(\mathrm{Gr}(3,7), (\bigwedge^2 \mathcal{U}^\vee)^{\oplus 3})$.

Question: Do we obtain all (classified) Fano varieties in this way?

Answer: Yes. Classical in dimension 1 and 2. In dimension 3, the case of non-prime Fano 3-folds ($\rho_X > 1$), there are 88 cases left. The original classification by Mori–Mukai was completed using Mori Theory. These Fano are in most cases described in terms of blow up of curves or point in other Fano with lower Picard rank. This description however is difficult to generalise in dimension > 3.

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Coates, Corti, Galkin and Kasprzyk (2013) rewrote Mori–Mukai classification. They were able to describe all Fano 3-folds as zero locus $(V /\!\!/ G, \mathcal{F})$, with the aim of computing the quantum periods. In many cases, their model of choice was a complete intersection in a toric variety.
As a first step, we decide to rewrite once again the Mori–Mukai classification. We wanted to check if we could describe the general element of each of 105 families taking as ambient variety only Grassmannians. This can be considered a test on the applicability of our tool in higher dimension.

Theorem (De Biase, –, Tanturri)

All 105 Fano threefolds can be described as $(\mathcal{F}, G)$, where $\mathcal{F}$ is homogeneous and $G = \prod \mathrm{Gr}(k_i, n_i)$ is a product of Grassmannians.

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Remark: In few cases we needed to use weighted projective spaces (as in the case of 3-folds). Also, in some cases we needed to use homogeneous bundles which are not completely reducible.

Example: Fano 3-fold 2–16.

- *Birational*: Blow up of the complete intersection of two quadrics in $\mathbb{P}^5$ in a conic C.
- *Biregular*: $(\mathbb{P}^2 \times \mathrm{Gr}(2,4), \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0) \oplus \mathcal{O}(0,2)))$.

Example: Fano 3-fold 4–4.

- *Birational*: Blow up of a three dimensional quadric in two points and in the proper transform of a conic through the points.
- *Biregular*: $(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^4, \mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,2) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,0,1))$.

Methodology

1. *Run a search to enumerate all the possible 3-folds that can be found as* $(G, \mathcal{F})$ *with* $c_1(-K_G \otimes \det(\mathcal{F})) < 0$.
2. *Compute the invariants* $(h^0(-K_X), -K_X^3, h^{p,q}(X))$ *to match each couple* $(G, \mathcal{F})$ *with a candidate in the MM list.*
3. *Prove a series of technical lemma that allow to translate biregular*$\leftrightsquigarrow$*birational (and viceversa).*

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3. *Prove a series of technical lemma that allow to translate biregular*$\leftrightsquigarrow$*birational (and viceversa).*

E.g. in 2–16 above one first check compute that $c_1(K_G \otimes \det(\mathcal{F})) = -h_1 - h_2$. Then we compute by Riemann-Roch and Koszul complex that the invariants of X satisfy $h^0(-K_X) = 14, -K_X^3 = 22, h^{1,1}(X) = h^{2,2}(X) = 2$. To match this candidate with the actual Fano we can use the following lemma:

Lemma

$\mathrm{Bl}_{\mathrm{Gr}(k-1,n-1)} \mathrm{Gr}(k,n) \cong (\mathrm{Gr}(k,n-1) \times \mathrm{Gr}(k,n), \mathcal{Q} \boxtimes \mathcal{U}^\vee)$ *where the centre of the blow up* $\mathrm{Gr}(k-1,n-1)$ *is identified with* $(\mathrm{Gr}(k,n), \mathcal{Q})$.

In particular $(\mathbb{P}^2 \times \mathrm{Gr}(2,4), \mathcal{U}^\vee_{\mathrm{Gr}(2,4)}(1,0))$ gives the blow up of $\mathrm{Gr}(2,4)$ in $\mathbb{P}^2$. The extra $\mathcal{O}(0,2)$ section cut the $\mathrm{Gr}(2,4)$ in an extra quadric, and $\mathbb{P}^2$ in a conic!

We now move our attention to the higher dimensional case. Some ongoing projects:

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1. Describe Kalashnikov's Fano 4-fold. In 2018 Kalashnikov found a list of 141 new families of Fano 4-folds of index 1, that can be described as zero loci of vector bundles in quiver flag varieties of dimension ≤ 8. With Kalashnikov–Tanturri we are describing them, both from an Hodge-theoretical and birational point of view. What we want to achieve is a MM-style classification.
Example: Fano with Period ID **689**. Can be described as $(\mathbb{P}^2 \times \mathbb{P}^5, \mathcal{O}(1,1) \oplus \mathcal{Q}_{\mathbb{P}^2}(0,2))$.
Invariants: $h^0(-K) = 17,\ (-K)^4 = 51,\ h^{2,2} = 2,\ h^{3,1} = 2,\ h^{2,2} = 41$.
Description: The blow up $\mathrm{Bl}_{S_8} X_3$, where X_3 is a cubic fourfold and S_8 is a (general) K3 surface of degree 8. X_3 is Hodge-special, but general in $\mathcal{C}_8$ (cubic 4-folds containing a plane), and therefore Fano 689 is not expected to be rational.

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2. Similarly, we are currently producing large list of examples of Fano 4-folds in product of Grassmannians, with similar associated description. Of course we will have to cross-check these data with the existing DB of Fano (e.g. Batyrev, Coates, Kalashnikov, Kasprzyk, Prince, etc.) to see how many new examples are there.

We are particularly interested in the case of Fano n-folds with high Picard rank. **Mukai's conjecture** predicts that for a smooth Fano of dimension n, one has $\rho_X(\iota_X - 1) \leq n$. It is of course interesting to look for examples (in any dimension) of Fano which are at the boundary of this conjecture.
However Mukai's conjecture gives no information on the index 1 case. In the 4-fold case the example with biggest Picard rank is $\rho(dP_1 \times dP_1) = 18$. However assuming that X is not a product, the *champion* has $\rho(X) = 9$ (Casagrande–Codogni–Fanelli). There is 1 example (each) with $\rho_X = 7, 8$ (Casagrande, Araujo–Casagrande), 6 (toric) with $\rho_X = 6$ and 20 (toric) with $\rho_X = 5$ (Batyrev).

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In progress (-, Tanturri)

From a preliminary search we found 1 Fano (non-toric, likely non-t.c.i) with $\rho_X = 6$ and 79 Fano with $\rho_X = 5$ which are not products (most of them likely t.c.i).

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Moving away from dimension 4 only, there is another class of Fano varieties we are particularly interested in!

Definition

For X smooth, projective, we say that $H^j(X,\mathbb{C}) \cong \bigoplus_{p+q=j} H^{p,q}(X)$ is of K3 type if

- it is of Hodge level 2.
- $h^{\frac{j+2}{2},\frac{j-2}{2}}(X) = 1$.

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Example

- A smooth cubic fourfold $X_3 \subset \mathbb{P}^5$ with Hodge diamond

$$\begin{array}{ccccccccc} 0 & & 1 & & 21 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

- A 20-fold hypersurface $W_\sigma \subset \mathrm{Gr}(3,10)$, where $\sigma \in \bigwedge^3 V_{10}^\vee$.

Why are these varieties interesting? Our main motivation lies in the study of IHS (Irreducible Holomorphic Symplectic) varieties (a.k.a. hyperkähler). Recall that these are varieties with $\pi_1(X) \cong \{*\}$ and such that $H^0(\Omega^2_X) \cong \mathbb{C} \cdot \sigma_X$, σ_X non-degenerate. Unlike Fano varieties, it is not clear if they are bounded or not, but examples are extremely rare (two classes by Beauville, two by O'Grady).

Idea

To a family of FK3 one expects to associate (many) examples of IHS, of different degree, dimension and even deformation type. But FK3 varieties are easier to hunt than IHS!

Example

- BD $X_3 \subset \mathbb{P}^5 \rightsquigarrow F_1(X_3) \subset \mathrm{Gr}(2,6)$. $F_1(X_3)$ is an IHS 4-fold, $\overset{def}{\sim} \mathit{Hilb}^2(S_8)$. One can describe $F_1(X_3) = (\mathrm{Gr}(2,6), \mathrm{Sym}^3\,\mathcal{U}^\vee)$.
- DV $W_\sigma \subset \mathrm{Gr}(3,10) \rightsquigarrow Z_\sigma \subset \mathrm{Gr}(6,10)$ as space of $\mathrm{Gr}(3,6) \subset W_\sigma$. $Z_\sigma \overset{def}{\sim} \mathit{Hilb}^2(S_{12})$. One can describe $Z = (\mathrm{Gr}(6,10), \bigwedge^3 \mathcal{U}^\vee)$.

Other FK3 in literature & Strategy

What about other FK3 varieties? A bunch of other 4-folds are known. When they are prime they have an actual or conjectural IHS linked to it.

- $(\mathrm{Gr}(2,5)\mathcal{O}(1) \oplus \mathcal{O}(2))$ (GM 4-fold). IHS via Lagrangian data [IM]-[DK].
- $(\mathrm{Gr}(3,7), \bigwedge^2 \mathcal{U}^\vee \oplus \mathcal{Q}^\vee(1) \oplus \mathcal{O}(1))$ (c5 4-fold). IHS conjectured.

There are several non-prime FK3 in dimension 4 that we found in our lists or in existing DB. However in most of these cases the K3 structure either comes from an actual K3 surfaces or from another FK3 blown up. We would therefore like to extend our search of FK3 varieties in higher dimension, in order to eliminate many of this type of examples. But how can we do this?

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Strategy: Translate the properties of *having a K3 structure* into relation between numerical invariants of Fano varieties. An example of such condition goes as follows:

Example

Let Y be a Fano $2t+1$-fold of index $\iota_Y = m$, such $t|m$. Assume that $lv(H^{2t+1}(Y)) \leq 1$ (+ some extra vanishing condition). Then a general element of $|-\frac{1}{t}K_Y|$ is a FK3.

It turns out that one can discover many more FK3! Below I recap the first result on this topic (more to come soon)

Theorem (–, Mongardi)

There exists 23 new families of FK3 varieties of dimension > 4. *They have dimension* $6 \leq n \leq 20$, $\rho_X \in \{1, 2, 3\}$ *and index* $\frac{n}{2} - 1 \leq \iota_X \leq \frac{n}{2}$.

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Let us focus on some of the most interesting FK3.

1. $X = (\mathrm{Gr}(3, 9), \bigwedge^2 \mathcal{U}^\vee \oplus \mathcal{O}(1))$. 14-fold of $\iota_X = 6$ with $1\times$ K3 structure in H^{14}. Given by $(\sigma, \omega) \in \bigwedge^3 V_9^\vee \oplus \bigwedge^2 V_9^\vee$. [IM]
2. $Y = (\mathrm{Gr}(3, 8), (\bigwedge^2 \mathcal{U}^\vee)^{\oplus 2} \oplus \mathcal{O}(1))$. 8-fold of $\iota_X = 3$ with $1\times$ K3 structure in H^8. Given by $(\sigma, \omega_1, \omega_2) \in \bigwedge^3 V_8^\vee \oplus \bigwedge^2 V_8^\vee \oplus \bigwedge^2 V_8^\vee$.
3. $T = (\mathrm{Gr}(2, 10), \mathcal{Q}^\vee(1))$. 8-fold $\iota_X = 3$ with $3\times$ K3 structure in $H^{6,8,10}$. Given by $\sigma \in \bigwedge^3 V_{10}^\vee$.
4. $HT(\mathrm{Gr}(2, 10), \mathcal{Q}^\vee(1) \oplus \mathcal{O}(1))$. 7-fold $\iota_X = 2$ with $2\times$ K3 structure in $H^{6,8}$, $1 \times 3CY$ in H^7.
5. $P \subset \mathbb{P}^9$, 6-fold $\iota_X = 3$ with $3\times$ K3 structure in $H^{4,6,8}$. Given by $\sigma \in \bigwedge^3 V_{10}^\vee$ (as degeneracy locus).

h^0		1		h^0		1		h^0		1					
h^2		1		h^2		1		h^2		1		h^0		1	
h^4		2		h^4		2		h^4		2		h^2		1	
h^6		6		h^6	1	22	1	h^6	1	22	1	h^4	1	22	1
h^8	1	26	1	h^8	1	23	1	h^7	1	44 44	1	h^6	1	22	1
		Y				T				HT				P	

Table: The nontrivial Hodge numbers of the above varieties

h^0		1	
h^2		1	
h^4		2	
h^6		6	
h^8	1	26	1
Y			

h^0		1	
h^2		1	
h^4		2	
h^6	1	22	1
h^8	1	23	1
T			

h^0		1	
h^2		1	
h^4		2	
h^6	1	22	1
h^7	1	44 44	1
HT			

h^0		1	
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h^4	1	22	1
h^6	1	22	1
P			

Table: The nontrivial Hodge numbers of the above varieties

Turns out that the DV hyperkähler 4-fold can be reconstructed from each of the above varieties! The latter can be obtained as the parameter space for $\mathrm{SGr}(3,6) \subset X$ [IM], $(\mathbb{P}^1)^3 \subset Y$ [FM], $C_1 \subset T$ [FM], $C_2 \subset P$ [H], where C_1 and C_2 are certain rational special fourfolds (resp. threefolds).

Question: Can we obtain the DV from the above Fano as $(G, \mathcal{F})$? (Yes for X, Y, not known for the others yet).

Turns out that the above FK3 are related not via the DV 4-fold. Although very different varieties, their Hodge structure is the one coming from the linear section $W_\sigma \subset \mathrm{Gr}(3,10)$, spread and multiplied to all these varieties through a very specific set of correspondences. Namely we have the following.

Theorem (Bernardara,–,Manivel)

Let $K := H^{20}_{van}(W_\sigma)$. Then $K \cong H^{14}_{van}(X) \cong H^{8}_{van}(Y)$ and $K \cong H^{i}_{prim}(T) \cong H^{j}_{prim}(P)$, $i = 6, 8, 10$, $j = 4, 6, 8$.

We have a similar statement for the derived category as well, with the Kuznetsov component A_W of $D^b(W)$ playing the role of the vanishing cohomology. We can predict the shapes of the semiorthogonal decomposition, as well as who should be the exceptional objects. However their number ($> 100!$) means that this is likely to be very hard to prove!

An idea of how this works, starting from $W = (\mathrm{Gr}(3,10), \mathcal{O}(1))$:

$$
\begin{tikzcd}
 & E \arrow[r, hook, "cdim7"] \arrow[d, "\mathbb{P}^7"] \arrow[dr, phantom, "(2)"] & q^* W \arrow[d, "\mathbb{P}^6"] \arrow[dr, "\mathbb{P}^2"] & & \mathrm{Bl}_S W \arrow[dl, "bu"'] \arrow[d, "\mathbb{P}^2"] \arrow[dr, phantom, "(3)"] & F_1 \arrow[l, hook', "cdim3"'] \arrow[d, "\mathbb{P}^3"] & \\
 & T \arrow[r, hook] & \mathrm{Gr}(2,10) & W & \mathrm{Gr}(3,9) & X \arrow[l, hook'] & \\
E' \arrow[r, hook, "exc.div."] \arrow[d, "\mathbb{P}^2"'] \arrow[dr, phantom, "(1)"] & q^* T \arrow[u, "\mathbb{P}^1"'] \arrow[d, "bu"] & & & & \mathrm{Bl}_{S_1} X \arrow[u, "bu"] \arrow[d, "\mathbb{P}^2"'] \arrow[dr, phantom, "(4)"] & F_2 \arrow[l, hook', "cdim3"'] \arrow[d, "\mathbb{P}^3"] \\
P \arrow[r, hook, "cdim3"'] & \mathbb{P}^9 & & & & S(3,8) & Y, \arrow[l, hook']
\end{tikzcd}
$$

Thanks for the attention!