Newton–Okounkov bodies arising from cluster structures

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Online Algebraic Geometry Seminar
Nottingham University
November 19, 2020

Toric degenerations

- Z: an irreducible normal projective variety over $\mathbb{C}$ ($m := \dim_{\mathbb{C}}(Z)$),
- $\mathcal{L}$: an ample line bundle on Z.

Definition

A **toric degeneration** of $(Z, \mathcal{L})$ is a flat morphism $\pi\colon \mathfrak{X} = \mathrm{Proj}(\mathcal{R}) \to \mathbb{C}$ such that $(\pi^{-1}(t), \mathcal{O}_{\mathfrak{X}}(1)|_{\pi^{-1}(t)}) \simeq (Z, \mathcal{L})$ for all $t \in \mathbb{C}^\times$, and $Z_0 := \pi^{-1}(0)$ is an irreducible normal projective toric variety.

Theorem (Harada–Kaveh 2015)

If Z is smooth, $\mathcal{L}$ is very ample, and there exists a toric degeneration of $(Z, \mathcal{L})$ satisfying some "good conditions", then

- *there exists a surjective continuous map $Z \twoheadrightarrow Z_0$ which induces a symplectomorphism from an open dense subset $U \subseteq Z$;*
- *there exists a completely integrable system on Z whose image coincides with a moment polytope of Z_0.*

Toric degenerations

There exists a systematic way to construct toric degenerations of $(Z, \mathcal{L})$, called a **Rees-type construction**, which is roughly as follows:

- construct a "good" $\mathbb{Z} \times \mathbb{Z}^m$-filtration on the section ring
$$R := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^0(Z, \mathcal{L}^{\otimes k}),$$
where the first $\mathbb{Z}$-filtration is given by the natural $\mathbb{Z}_{\geq 0}$-grading on R;
- there exists a linear projection $\mathbb{Z} \times \mathbb{Z}^m \to \mathbb{Z}$ which induces a $\mathbb{Z}_{\geq 0}$-filtration $R_{\leq k} \subseteq R$ whose associated graded is $\mathrm{gr}(R)$;
- the Rees algebra $\mathcal{R} := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} R_{\leq k} t^k$ gives a toric degeneration $\mathfrak{X} := \mathrm{Proj}(\mathcal{R}) \to \mathbb{C}$.

There exist several ways to construct such $\mathbb{Z} \times \mathbb{Z}^m$-filtration, including

- representation theory (Caldero 2002, Alexeev–Brion 2004),
- Newton–Okounkov bodies (Anderson 2013),
- cluster algebras (Gross–Hacking–Keel–Kontsevich 2018).

Newton–Okounkov bodies

Assume that Z is rational, and fix an identification

$$\mathbb{C}(Z) \simeq \mathbb{C}(t_1, \ldots, t_m).$$

Let

- $\leq$: a total order on $\mathbb{Z}^m$, respecting the addition,
- $\tau \in H^0(Z, \mathcal{L})$: a nonzero section.

The **lowest term valuation** $v_{\leq}^{\mathrm{low}} \colon \mathbb{C}(Z) \setminus \{0\} \to \mathbb{Z}^m$ is defined as follows:

$$v_{\leq}^{\mathrm{low}}(f/g) \coloneqq v_{\leq}^{\mathrm{low}}(f) - v_{\leq}^{\mathrm{low}}(g), \text{ and}$$
$$v_{\leq}^{\mathrm{low}}(f) \coloneqq (a_1, \ldots, a_m) \Leftrightarrow f = c t_1^{a_1} \cdots t_m^{a_m} + (\text{higher terms w.r.t. } \leq)$$

for $f, g \in \mathbb{C}[t_1, \ldots, t_m] \setminus \{0\}$, where $c \in \mathbb{C}^\times$.

Newton–Okounkov bodies

Define a semigroup $S(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau) \subseteq \mathbb{Z}_{>0} \times \mathbb{Z}^m$, a real closed convex cone $C(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau) \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^m$, and a convex set $\Delta(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau) \subseteq \mathbb{R}^m$ by

$$S(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau) := \{(k, v_{\leq}^{\mathrm{low}}(\sigma/\tau^k)) \mid k \in \mathbb{Z}_{>0},\ \sigma \in H^0(Z, \mathcal{L}^{\otimes k}) \setminus \{0\}\},$$

$C(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau)$: the smallest real closed cone containing $S(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau)$,

$$\Delta(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau) := \{\mathbf{a} \in \mathbb{R}^m \mid (1, \mathbf{a}) \in C(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau)\}.$$

Definition (Lazarsfeld–Mustata 2009, Kaveh–Khovanskii 2012)

The convex set $\Delta(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau)$ is called a **Newton–Okounkov body**.

Theorem (Anderson 2013)

If the semigroup $S(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau)$ is finitely generated and saturated, then there exists a toric degeneration of $(Z, \mathcal{L})$ to the normal projective toric variety corresponding to $\Delta(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau)$.

Newton–Okounkov bodies

Example (Toric variety case)

If Z is toric with the open dense torus $(\mathbb{C}^\times)^m = \mathrm{Spec}(\mathbb{C}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}])$ and $\leq$ is the lexicographic order on $\mathbb{Z}^m$, then the Newton–Okounkov body $\Delta(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau)$ coincides with the moment polytope of $(Z, \mathcal{L})$.

Example (Flag variety case)

If Z is a flag variety, then the Newton–Okounkov bodies $\Delta(Z, \mathcal{L}, v_{\leq}^{\mathrm{low}}, \tau)$ realize the following representation-theoretic polytopes:

- string polytopes (Kaveh 2015, F.–Oya 2017),
- Nakashima–Zelevinsky polytopes (F.–Naito 2017, F.–Oya 2017),
- Feigin–Fourier–Littelmann–Vinberg polytopes (Feigin–Fourier–Littelmann 2017, Kiritchenko 2017).

Cluster varieties

Following Fock–Goncharov (2009), let us consider an $\mathcal{A}$-cluster variety

$$\mathcal{A} = \bigcup_{\mathbf{s}} \mathcal{A}_{\mathbf{s}} = \bigcup_{\mathbf{s}} \mathrm{Spec}(\mathbb{C}[A_{j;\mathbf{s}}^{\pm 1} \mid j \in \{1, \ldots, m\} = J_{\mathrm{uf}} \sqcup J_{\mathrm{fr}}]),$$

where $\mathbf{s}$ runs over the set of seeds which are mutually mutation equivalent, and the tori are glued via the following birational cluster mutations:

$$\mu_k^*(A_{i;\mathbf{s}'}) = \begin{cases} A_{i;\mathbf{s}} & (i \neq k), \\ A_{k;\mathbf{s}}^{-1}(\prod_{\varepsilon_{k,j}>0} A_{j;\mathbf{s}}^{\varepsilon_{k,j}} + \prod_{\varepsilon_{k,j}<0} A_{j;\mathbf{s}}^{-\varepsilon_{k,j}}) & (i = k) \end{cases}$$

if $\mathbf{s}' = \mu_k(\mathbf{s})$, where $\varepsilon = (\varepsilon_{i,j})_{i,j}$ is the exchange matrix of $\mathbf{s}$.

Definition (Berenstein–Fomin–Zelevinsky 2005)

The ring $\mathbb{C}[\mathcal{A}]$ of regular functions is called an **upper cluster algebra**.

Assumption

The exchange matrix ε is of full rank for all seeds $\mathbf{s}$.

Cluster varieties

Example

$m = 2$, $J_{uf} = \{1, 2\}$, $J_{fr} = \phi$, $s = ((x_1, x_2), \varepsilon)$

$$\mathcal{A}_s = (\mathbb{C}^\times)^2 \ni (x_1, x_2)$$

$$x_1 = A_{1;s}, \quad x_2 = A_{2;s}$$

$$x_3 = \frac{x_2 + 1}{x_1} \qquad \mu_1 \qquad \mu_2 \qquad x_4 = \frac{x_1 + 1}{x_2}$$

$$(x_3, x_2) \in (\mathbb{C}^\times)^2 \qquad (\mathbb{C}^\times)^2 \ni (x_1, x_4)$$

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$x_5 = \frac{x_1 + x_2 + 1}{x_1 x_2} \qquad \mu_2 \qquad \mu_1$$

$$(x_3, x_5) \in (\mathbb{C}^\times)^2 \qquad (\mathbb{C}^\times)^2 \ni (x_5, x_4)$$

$$\mu_1 \qquad \cong$$

$$(x_4, x_5) \in (\mathbb{C}^\times)^2$$

$$\rightsquigarrow \mathcal{A} = \bigcup_{i=1}^{5} (\mathbb{C}^\times)^2,$$

$$\mathbb{C}[\mathcal{A}] = \mathbb{C}[x_1, \ldots, x_5] \subseteq \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$$

Tropicalized cluster mutations

Denoting the dual torus of $\mathcal{A}_{\mathbf{s}}$ by $\mathcal{A}_{\mathbf{s}}^{\vee}$, we have

$$\mathcal{A} = \bigcup_{\mathbf{s}} \mathcal{A}_{\mathbf{s}} \quad \xleftrightarrow{\text{“mirror”}} \mathcal{A}^{\vee} = \bigcup_{\mathbf{s}} \mathcal{A}_{\mathbf{s}}^{\vee}.$$

The space $\mathcal{A}^{\vee}$ is called the **Fock–Goncharov dual**, and defined to be the Langlands dual of the $\mathcal{X}$-cluster variety. Since the gluing maps of $\mathcal{A}^{\vee}$ are given by subtraction-free rational functions, we obtain the set $\mathcal{A}^{\vee}(\mathbb{R}^T)$ of $\mathbb{R}^T$-valued points, where $\mathbb{R}^T$ is a semifield $(\mathbb{R}, \max, +)$. More precisely, $\mathcal{A}^{\vee}(\mathbb{R}^T)$ is defined by gluing $\mathcal{A}_{\mathbf{s}}^{\vee}(\mathbb{R}^T) = \mathbb{R}^m$ via the following tropicalized cluster mutations:

$$\mu_k^T \colon \mathcal{A}_{\mathbf{s}}^{\vee}(\mathbb{R}^T) \to \mathcal{A}_{\mathbf{s}'}^{\vee}(\mathbb{R}^T),\ (g_1, \ldots, g_m) \mapsto (g_1', \ldots, g_m'),$$

where

$$g_i' = \begin{cases} g_i + \max\{\varepsilon_{k,i}, 0\} g_k - \varepsilon_{k,i} \min\{g_k, 0\} & (i \neq k), \\ -g_k & (i = k). \end{cases}$$

Extended g-vectors

Theorem (Fomin–Zelevinsky 2007, Derksen–Weyman–Zelevinsky 2010, Gross–Hacking–Keel–Kontsevich 2018)

For all $\mathbf{s}, \mathbf{s}'$ *and* $1 \leq i \leq m$, *the variable* $A_{i;\mathbf{s}'}$ *is pointed for* $\mathbf{s}$, *that is,*

$$A_{i;\mathbf{s}'} \in A_{1;\mathbf{s}}^{g_1} \cdots A_{m;\mathbf{s}}^{g_m} \left(1 + \sum_{0 \neq (a_j)_{j \in J_{\mathrm{uf}}} \in \mathbb{Z}_{\geq 0}^{J_{\mathrm{uf}}}} \mathbb{Z} \prod_{j \in J_{\mathrm{uf}}} (A_{1;\mathbf{s}}^{\varepsilon_{j,1}} \cdots A_{m;\mathbf{s}}^{\varepsilon_{j,m}})^{a_j} \right)$$

for some $g_{\mathbf{s}}(A_{i;\mathbf{s}'}) = (g_1, \ldots, g_m) \in \mathbb{Z}^m$ *(the* ***extended*** g***-vector*** *of* $A_{i;\mathbf{s}'}$*).*

Definition (Qin 2017)

For each seed $\mathbf{s} = ((A_{j;\mathbf{s}})_j, \varepsilon)$, define a partial order $\preceq_{\mathbf{s}}$ on $\mathbb{Z}^m$ by

$$g' \preceq_{\mathbf{s}} g \Leftrightarrow g' - g \in \sum_{j \in J_{\mathrm{uf}}} \mathbb{Z}_{\geq 0}(\varepsilon_{j,1}, \ldots, \varepsilon_{j,m}).$$

This $\preceq_{\mathbf{s}}$ is called the **dominance order** associated with $\mathbf{s}$.

Extended g-vectors as higher rank valuations

Fix a total order $\leq_{\mathbf{s}}$ on $\mathbb{Z}^m$ refining the opposite dominance order $\preceq_{\mathbf{s}}^{\mathrm{op}}$.

Definition (F.–Oya)

For each seed $\mathbf{s}$, define a valuation $v_{\mathbf{s}}$ on $\mathbb{C}(\mathcal{A}) = \mathbb{C}(A_{1;\mathbf{s}}, \ldots, A_{m;\mathbf{s}})$ to be the lowest term valuation $v^{\mathrm{low}}_{\leq_{\mathbf{s}}}$.

Proposition

For all $\mathbf{s}, \mathbf{s}'$ *and* $1 \leq i \leq m$, *the equality* $v_{\mathbf{s}}(A_{i;\mathbf{s}'}) = g_{\mathbf{s}}(A_{i;\mathbf{s}'})$ *holds.*

Let Z be a compactification of $\mathcal{A}$. Then $\Delta(Z, \mathcal{L}, v_{\mathbf{s}}, \tau)$ does not depend on the choice of a refinement $\leq_{\mathbf{s}}$ of $\preceq_{\mathbf{s}}^{\mathrm{op}}$ if for each $k \in \mathbb{Z}_{>0}$,

$$\{\sigma/\tau^k \mid \sigma \in H^0(Z, \mathcal{L}^{\otimes k})\} \subseteq \mathbb{C}(Z) \simeq \mathbb{C}(\mathcal{A})$$

is compatible with a specific cluster-theoretic $\mathbb{C}$-basis such as

- a theta function basis (Gross–Hacking–Keel–Kontsevich 2018),
- a common triangular basis (Qin 2017).

Toric degenerations arising from cluster structures

Let

$$\mathcal{A}_{\mathrm{prin}} = \bigcup_{\mathbf{s}} \mathcal{A}_{\mathrm{prin},\mathbf{s}}$$

be the $\mathcal{A}$-cluster variety with principal coefficients.

- There naturally exists a morphism

$$\pi \colon \mathcal{A}_{\mathrm{prin}} \to T_M = \mathrm{Spec}(\mathbb{C}[N]) = (\mathbb{C}^\times)^m$$

  such that $\pi^{-1}(e) \simeq \mathcal{A}$.
- The morphism π induces a $\mathbb{C}[N]$-algebra structure on $\mathbb{C}[\mathcal{A}_{\mathrm{prin}}]$.
- There exists a canonical surjective map

$$\rho^T \colon \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T) \to \mathcal{A}^\vee(\mathbb{R}^T).$$

- $\mathcal{A}^\vee(\mathbb{R}^T) \simeq \mathcal{A}^\vee_{\mathbf{s}}(\mathbb{R}^T) = \mathbb{R}^m$ for each seed $\mathbf{s}$.

For $q \in \mathcal{A}^\vee(\mathbb{R}^T)$ and $\Xi \subseteq \mathcal{A}^\vee(\mathbb{R}^T)$, let $q_{\mathbf{s}}$ and $\Xi_{\mathbf{s}}$ denote their images in $\mathcal{A}^\vee_{\mathbf{s}}(\mathbb{R}^T) = \mathbb{R}^m$, respectively.

Toric degenerations arising from cluster structures

Assume that $\mathcal{A}$ satisfies some "good conditions". For instance, we assume that

$$\mathbb{C}[\mathcal{A}_{\mathrm{prin}}] = \sum_{q \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)} \mathbb{C}\vartheta_q \quad \text{and} \quad \mathbb{C}[\mathcal{A}] = \sum_{q \in \mathcal{A}^\vee(\mathbb{Z}^T)} \mathbb{C}\vartheta_q,$$

where $\{\vartheta_q \mid q \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)\}$ and $\{\vartheta_q \mid q \in \mathcal{A}^\vee(\mathbb{Z}^T)\}$ are the theta function bases. Then we have

$$\mathbb{C}[\mathcal{A}] = \mathbb{C}[\mathcal{A}_{\mathrm{prin}}] \otimes_{\mathbb{C}[N]} \mathbb{C} \quad \text{and} \quad \vartheta_{\rho^T(q)} = \vartheta_q \otimes 1$$

for $q \in \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{Z}^T)$, where $\mathbb{C}[N] \to \mathbb{C}$ is given by $e \in T_M$.

Proposition

For all seeds $\mathbf{s}$ *and* $q \in \mathcal{A}^\vee(\mathbb{Z}^T)$*, the equality* $v_{\mathbf{s}}(\vartheta_q) = q_{\mathbf{s}}$ *holds.*

Toric degenerations arising from cluster structures

Let $\Xi \subseteq \mathcal{A}_{\mathrm{prin}}^{\vee}(\mathbb{R}^T)$ be a full-dimensional bounded rationally-defined positive convex polytope. We set $\widetilde{\Xi} := \Xi + (N \otimes_{\mathbb{Z}} \mathbb{R})$, and define $\widetilde{S}_{\widetilde{\Xi}} \subseteq \mathbb{C}[\mathcal{A}_{\mathrm{prin}}][x]$ by

$$\widetilde{S}_{\widetilde{\Xi}} := \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \bigoplus_{q \in d\widetilde{\Xi}(\mathbb{Z})} \mathbb{C}\vartheta_q x^d,$$

where x is an indeterminate and $d\widetilde{\Xi}(\mathbb{Z})$ is the set of $q \in \mathcal{A}_{\mathrm{prin}}^{\vee}(\mathbb{Z}^T)$ such that

$$(d, q) \in \overline{\{(r, p) \mid r \in \mathbb{R}_{\geq 0},\ p \in r\widetilde{\Xi}\}}.$$

The inclusion of $\mathbb{C}[N] = \mathbb{C}[N]\vartheta_0$ in the degree 0 part of $\widetilde{S}_{\widetilde{\Xi}}$ induces a flat morphism

$$\mathfrak{X}' := \mathrm{Proj}(\widetilde{S}_{\widetilde{\Xi}}) \to \mathrm{Spec}(\mathbb{C}[N]) = T_M.$$

Toric degenerations arising from cluster structures

Theorem (Gross–Hacking–Keel–Kontsevich 2018)

Under some "good conditions" on $\mathcal{A}$, the following hold.

(1) *For $z \in T_M$, the fiber $\mathfrak{X}_z$ of the family $\mathfrak{X}' \to T_M$ is a normal projective variety containing $\mathcal{A}$ as an open subscheme.*

(2) *For each seed $\mathbf{s}$, the flat family $\mathfrak{X}' \to T_M = (\mathbb{C}^\times)^m$ extends to a flat family*

$$\mathfrak{X} = \mathrm{Proj}(\widetilde{S}_{\Xi^+}) \to \mathbb{C}^m$$

such that the central fiber $\mathfrak{X}_0$ is the normal projective toric variety corresponding to the rational convex polytope $\rho^T(\Xi)_\mathbf{s}$.

Theorem (F.–Oya)

For each seed $\mathbf{s}$, the Newton–Okounkov body $\Delta(\mathfrak{X}_e, \mathcal{L}, v_\mathbf{s}, x)$ coincides with the rational convex polytope $\rho^T(\Xi)_\mathbf{s}$, where $\mathcal{L}$ is the restriction of $\mathcal{O}_\mathfrak{X}(1)$.

Flag varieties

- G: a connected, simply-connected semisimple algebraic group over $\mathbb{C}$,
- $B \subseteq G$: a Borel subgroup,
- P_+: the set of dominant integral weights.

Definition

The quotient variety G/B is called the **full flag variety**.

Example

If $G = SL_n(\mathbb{C})$, then we can take B to be the subgroup of upper triangular matrices. In this case, we have

$$G/B \xrightarrow{\sim} \{(\{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}}(V_i) = i,\ 1 \leq i \leq n\},$$
$$gB \mapsto (\{0\} \subsetneq \langle g\mathbf{e}_1 \rangle_{\mathbb{C}} \subsetneq \cdots \subsetneq \langle g\mathbf{e}_1, \ldots, g\mathbf{e}_n \rangle_{\mathbb{C}} = \mathbb{C}^n),$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denotes the standard basis of $\mathbb{C}^n$.

Flag varieties

Theorem (Borel–Weil theory)

There exists a natural bijective map

$$P_+ \xrightarrow{\sim} \{\textit{globally generated line bundles on } G/B\}, \quad \lambda \mapsto \mathcal{L}_\lambda,$$

such that $H^0(G/B, \mathcal{L}_\lambda)^*$ is the irreducible highest weight G-module with highest weight λ.

The anti-canonical bundle of G/B is isomorphic to $\mathcal{L}_{2\rho}$, where $\rho \in P_+$ is the half sum of positive roots.

e.g. $G = SL_n(\mathbb{C})$, $B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$

$\lambda \in P_+ \rightsquigarrow B \to \mathbb{C}^\times$, $\begin{pmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \mapsto d_1^{\lambda_1} d_2^{\lambda_2} \cdots d_n^{\lambda_n}$

$(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0)$

$\lambda = 2\rho \rightsquigarrow (\lambda_1, \ldots, \lambda_n) = (2n-2, 2n-4, \ldots, 4, 2, 0)$

Cluster structures on unipotent cells

Let U^- be the unipotent radical of the opposite Borel subgroup B^-, and

$$G/B = \bigsqcup_{w \in W} BwB/B$$

the Bruhat decomposition of G/B, where W is the Weyl group.

Definition

For $w \in W$, the **unipotent cell** U_w^- is defined by

$$U_w^- := BwB \cap U^- \subseteq G.$$

Theorem (Berenstein–Fomin–Zelevinsky 2005)

The coordinate ring $\mathbb{C}[U_w^-]$ admits an upper cluster algebra structure.

There exists $w_0 \in W$, called the **longest element**, such that the natural projection $G \twoheadrightarrow G/B$ induces an open embedding $U_{w_0}^- \hookrightarrow G/B$.

Cluster structures on unipotent cells

e.g. $G = SL_n(\mathbb{C}) \rightsquigarrow U^- = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}$

$U^-_{w_0} = \{ u \in U^- \mid \Delta_{n,1}(u), \Delta_{n-1\,n,12}(u), \dots, \Delta_{23\cdots n, 12\cdots n-1}(u) \neq 0 \}$

e.g. $G = SL_3(\mathbb{C}) \rightsquigarrow U^- = \left\{ u = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \right\}$

$\Delta_{3,1}(u) = b, \quad \Delta_{23,12}(u) = ac - b$

$\rightsquigarrow U^-_{w_0} = \{ u \in U^- \mid b \neq 0, ac - b \neq 0 \}$

$\rightsquigarrow \mathbb{C}[U^-_{w_0}] = \mathbb{C}[a, c, b^{\pm 1}, (ac-b)^{\pm 1}]$

Mutation $S_1 \leftrightarrow S_2$

$c = \dfrac{b + (ac - b)}{a}$

There are only two seeds S_1 and S_2:

$S_1 = ((a, b, ac - b), (0\ 1\ -1)), \quad S_2 = ((c, b, ac - b), (0, -1, 1))$

Cluster structures on unipotent cells

Let $R(w)$ denote the set of reduced words for $w \in W$. For each reduced word $\boldsymbol{i} = (i_1, \ldots, i_m) \in R(w)$, we obtain a seed $\mathbf{s}_{\boldsymbol{i}} = ((A_{j;\mathbf{s}_{\boldsymbol{i}}})_j, \varepsilon^{\boldsymbol{i}})$ for U_w^- given by

- $A_{j;\mathbf{s}_{\boldsymbol{i}}} \in \mathbb{C}[U_w^-]$ is the restriction of the generalized minor $\Delta_{s_{i_1} \cdots s_{i_j} \varpi_{i_j}, \varpi_{i_j}} \in \mathbb{C}[G]$ for $1 \leq j \leq m$;
- if we write $\varepsilon^{\boldsymbol{i}} = (\varepsilon_{s,t})_{s,t}$, then

$$\varepsilon_{s,t} = \begin{cases} 1 & \text{if } s^+ = t, \\ -1 & \text{if } s = t^+, \\ \langle \alpha_{i_s}, h_{i_t} \rangle & \text{if } s < t < s^+ < t^+, \\ -\langle \alpha_{i_s}, h_{i_t} \rangle & \text{if } t < s < t^+ < s^+, \\ 0 & \text{otherwise,} \end{cases}$$

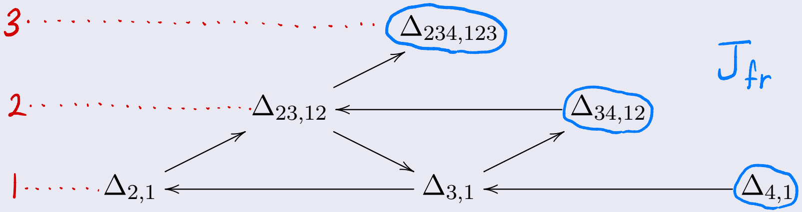
  where

$$k^+ := \min(\{k+1 \leq j \leq m \mid i_j = i_k\} \cup \{m+1\}).$$

Cluster structures on unipotent cells

Example

Let $G = SL_4(\mathbb{C})$, and $\mathbf{i} = (1, 2, 1, 3, 2, 1) \in R(w_0)$. Then the seed $\mathbf{s}_{\boldsymbol{i}} = ((A_{j;\mathbf{s}_{\boldsymbol{i}}})_j, \varepsilon^{\boldsymbol{i}})$ for $U^-_{w_0}$ is given as follows (there exists $s \to t$ if and only if $\varepsilon_{t,s} = 1$ or $\varepsilon_{s,t} = -1$):



Cluster structures on unipotent cells

Theorem (Kashiwara–Kim 2019, Qin preprint 2020)

For $w \in W$, the upper global basis $\mathbf{B}_w^{\mathrm{up}} \subseteq \mathbb{C}[U_w^-]$ is (the specialization at $q = 1$ of) a common triangular basis. In particular, the following hold.

(1) *Each element $b \in \mathbf{B}_w^{\mathrm{up}}$ is pointed for all $\mathbf{s}$, that is,*

$$b \in A_{1;\mathbf{s}}^{g_1} \cdots A_{m;\mathbf{s}}^{g_m} \left(1 + \sum_{0 \neq (a_j)_{j \in J_{\mathrm{uf}}} \in \mathbb{Z}_{\geq 0}^{J_{\mathrm{uf}}}} \mathbb{Z} \prod_{j \in J_{\mathrm{uf}}} (A_{1;\mathbf{s}}^{\varepsilon_{j,1}} \cdots A_{m;\mathbf{s}}^{\varepsilon_{j,m}})^{a_j} \right)$$

for some $g_{\mathbf{s}}(b) = (g_1, \ldots, g_m) \in \mathbb{Z}^m$ (the extended g-vector of b).

(2) *If $\mathbf{s}' = \mu_k(\mathbf{s})$, then $g_{\mathbf{s}'}(b) = \mu_k^T(g_{\mathbf{s}}(b))$ for all $b \in \mathbf{B}_w^{\mathrm{up}}$.*

Corollary

For all $b \in \mathbf{B}_w^{\mathrm{up}}$ and $\mathbf{s}$, the equality $v_{\mathbf{s}}(b) = g_{\mathbf{s}}(b)$ holds.

Associated Newton–Okounkov bodies

Let $\tau_\lambda \in H^0(G/B, \mathcal{L}_\lambda)$ be a lowest weight vector.

Theorem (F.–Oya)

Let $\mathbf{s}$ *be a seed for* $U^-_{w_0}$, $\lambda \in P_+$, *and* $\boldsymbol{i} \in R(w_0)$.

(1) $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}}, \tau_\lambda)$ *does not depend on the choice of a refinement* $\leq_{\mathbf{s}}$ *of the opposite dominance order* $\preceq^{\mathrm{op}}_{\mathbf{s}}$.

(2) $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}}, \tau_\lambda)$ *is a rational convex polytope.*

(3) $S(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}}, \tau_\lambda)$ *is finitely generated and saturated.*

(4) *If* $\mathbf{s}' = \mu_k(\mathbf{s})$, *then* $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}'}, \tau_\lambda) = \mu_k^T(\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}}, \tau_\lambda))$.

(5) $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}_{\boldsymbol{i}}}, \tau_\lambda)$ *is unimodularly equivalent to the string polytope* $\Delta_{\boldsymbol{i}}(\lambda)$ *by an explicit unimodular transformation.*

(6) *There is a seed* $\mathbf{s}^{\mathrm{mut}}_{\boldsymbol{i}}$ *such that* $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}^{\mathrm{mut}}_{\boldsymbol{i}}}, \tau_\lambda)$ *is unimodularly equivalent to the Nakashima–Zelevinsky polytope* $\widetilde{\Delta}_{\boldsymbol{i}}(\lambda)$.

Combinatorial mutations

- $N \simeq \mathbb{Z}^m$: a $\mathbb{Z}$-lattice of rank m,
- $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$,
- $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$,
- $H_{w,h} := \{v \in N_{\mathbb{R}} \mid \langle w, v \rangle = h\}$ for $w \in M$ and $h \in \mathbb{Z}$,
- $P \subseteq N_{\mathbb{R}}$: an integral convex polytope with the vertex set $V(P) \subseteq N$,
- $w \in M$: a primitive vector,
- $F \subseteq H_{w,0}$: an integral convex polytope.

Assumption

For every $h \in \mathbb{Z}_{\leq -1}$, there exists a possibly-empty integral convex polytope $G_h \subseteq N_{\mathbb{R}}$ such that

$$V(P) \cap H_{w,h} \subseteq G_h + |h|F \subseteq P \cap H_{w,h}.$$

If this assumption holds, then we say that the combinatorial mutation $\mathrm{mut}_w(P, F)$ of P is well-defined.

Combinatorial mutations

Definition (Akhtar–Coates–Galkin–Kasprzyk 2012)

The **combinatorial mutation** $\mathrm{mut}_w(P, F)$ of P is defined as follows:

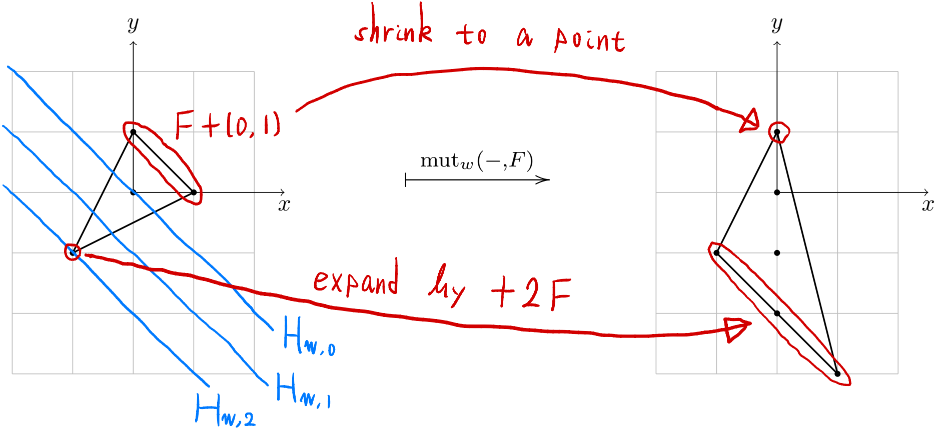
$$\mathrm{mut}_w(P, F) := \mathrm{conv}\left(\bigcup_{h \leq -1} G_h \cup \bigcup_{h \geq 0} ((P \cap H_{w,h}) + hF)\right) \subseteq N_{\mathbb{R}}.$$

Properties

- $\mathrm{mut}_w(P, F)$ *is an integral convex polytope.*
- $\mathrm{mut}_w(P, F)$ *is independent of the choice of* $\{G_h\}_{h \leq -1}$.
- *If* $Q = \mathrm{mut}_w(P, F)$*, then we have* $P = \mathrm{mut}_{-w}(Q, F)$.

Example

For $w = (-1,-1) \in M$ and $F = \mathrm{conv}\{(0,0),(1,-1)\}$, we have



Dual operations

- $P \subseteq N_{\mathbb{R}}$: an integral convex polytope containing the origin in its interior,
- $w \in M$: a primitive vector,
- $F \subseteq H_{w,0}$: an integral convex polytope.

The **polar dual** P^* of P is a rational convex polytope defined by

$$P^* := \{\boldsymbol{u} \in M_{\mathbb{R}} \mid \langle \boldsymbol{u}, \boldsymbol{u}' \rangle \geq -1 \text{ for all } \boldsymbol{u}' \in P\}.$$

Define a map $\varphi_{w,F} \colon M_{\mathbb{R}} \to M_{\mathbb{R}}$ by

$$\varphi_{w,F}(u) := u - u_{\min} w$$

for $u \in M_{\mathbb{R}}$, where $u_{\min} := \min\{\langle u, v \rangle \mid v \in F\}$.

Proposition (Akhtar–Coates–Galkin–Kasprzyk 2012)

If $\mathrm{mut}_w(P, F)$ is well-defined, then it holds that

$$\varphi_{w,F}(P^*) = \mathrm{mut}_w(P, F)^*.$$

Interior lattice points

Theorem (Steinert preprint 2019)

If the semigroup $S(G/B, \mathcal{L}_{2\rho}, v^{\mathrm{low}}_{\leq}, \tau_{2\rho})$ is finitely generated and saturated, then $\Delta := \Delta(G/B, \mathcal{L}_{2\rho}, v^{\mathrm{low}}_{\leq}, \tau_{2\rho})$ contains exactly one lattice point $\boldsymbol{a}$ in its interior, and the dual polytope

$$\Delta^\vee := (\Delta - \boldsymbol{a})^*$$

is an integral convex polytope.

Corollary (F.–Higashitani)

The unique interior lattice point $\boldsymbol{a}_{\mathbf{s}} = (a_j)_{1 \leq j \leq m}$ of $\Delta(G/B, \mathcal{L}_{2\rho}, v_{\mathbf{s}}, \tau_{2\rho})$ is given by

$$a_j = \begin{cases} 0 & (\text{if } j \in J_{\mathrm{uf}}), \\ 1 & (\text{if } j \in J_{\mathrm{fr}}). \end{cases}$$

Relation with combinatorial mutations

It has been known that the tropicalized cluster mutation μ_k^T can be realized as $\varphi_{w,F}$ for some w and F. Using the computation of $\boldsymbol{a}_{\mathrm{s}}$, we obtain the following as the polar dual of this fact.

Theorem (F.–Higashitani)

(1) *The dual polytopes* $\Delta(G/B, \mathcal{L}_{2\rho}, v_{\mathrm{s}}, \tau_{2\rho})^\vee$ *for seeds* s *are all related by sequences of combinatorial mutations up to unimodular equivalence.*

(2) *In particular, the dual polytopes* $\Delta_{\boldsymbol{i}}(2\rho)^\vee$ *and* $\widetilde{\Delta}_{\boldsymbol{i}}(2\rho)^\vee$ *of string polytopes and Nakashima–Zelevinsky polytopes for reduced words* $\boldsymbol{i}$ *are all related by sequences of combinatorial mutations up to unimodular equivalence.*

Future directions

- Describe $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}}, \tau_\lambda)$ for various seeds $\mathbf{s}$ explicitly.
- For various cluster varieties $\mathcal{A}$, compute Newton–Okounkov bodies $\Delta(\overline{\mathcal{A}}, \mathcal{L}, v_{\mathbf{s}}, \tau)$ of their compactifications $\overline{\mathcal{A}}$.
- Relate mirror-symmetric properties of the dual polytopes $\Delta(G/B, \mathcal{L}_{2\rho}, v_{\mathbf{s}}, \tau_{2\rho})^\vee$.
  For $G = SL_{n+1}(\mathbb{C})$, Rusinko (2008) proved that certain mirror families $F_{\boldsymbol{i}}$, $\boldsymbol{i} \in R(w_0)$, which are subfamilies of $|\mathcal{O}_{\Delta_{\boldsymbol{i}}(2\rho)^\vee}(1)|$, are birational.
- Classify integral convex polytopes which are related with $\Delta(G/B, \mathcal{L}_{2\rho}, v_{\mathbf{s}}, \tau_{2\rho})^\vee$ by a sequence of combinatorial mutations.
  For $G = SL_{n+1}(\mathbb{C})$ or $G = Sp_{2n}(\mathbb{C})$, the dual polytope $FFLV(2\rho)^\vee$ of the Feigin–Fourier–Littelmann–Vinberg polytope $FFLV(2\rho)$ is contained in this class (F.–Higashitani).

Thank you for your attention!