# Tropical correspondence for smooth del Pezzo log Calabi-Yau pairs

Online Algebraic Geometry Seminar, Nottingham University

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# Overview

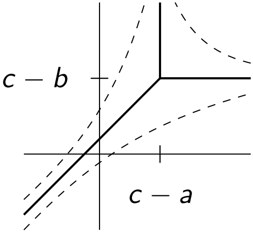
# Tropical geometry

Tropical geometry = piecewise linear geometry $\rightsquigarrow$ combinatorics

- tropical semiring $(\mathbb{R}^n, \min, +)$, tropical variety $V^{\mathrm{trop}}(f)$ = corner locus
- limit of amoebas $\mathbb{A}^n_{\mathbb{C}\{\{t\}\}} \to \mathbb{R}^n, (x_i)_i \mapsto (-log_{t\to 0}|x_i|)_i$
- parametrized tropical curves $h : \Gamma \to \mathbb{R}^n \rightsquigarrow$ enumerative geometry

**Example (Tropical line in $\mathbb{R}^2$)**

$V^{\mathrm{trop}}(\min\{a+x, b+y, c\})$



# Toric varieties

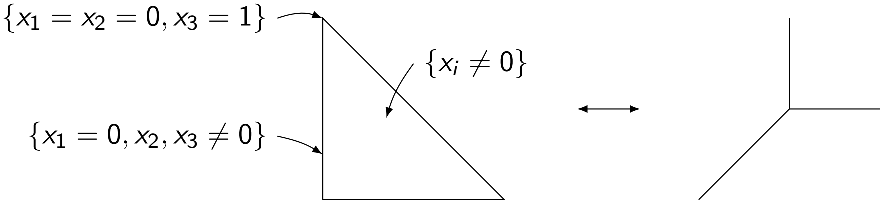
## Definition

A *toric variety* is an algebraic variety $X$ containing $(\mathbb{C}^\star)^n$ as a dense open subset such that the action of $(\mathbb{C}^\star)^n$ on itself extends to $X$.

given by polytope $\Delta$ or fan $\Sigma$, components $\leftrightarrow$ orbits of $(\mathbb{C}^\star)^n$-action
$X_\Delta = \text{Spec } \mathbb{C}[C(\Delta) \cap \mathbb{Z}^{n+1}]$

## Example ($\mathbb{P}^2$)

$$\mathbb{P}^2 = \{(x_0, x_1, x_2) \mid (x_0, x_1, x_2) = (\lambda x_0, \lambda x_1, \lambda x_2), \lambda \in \mathbb{C}^\star = \mathbb{C} \setminus \{0\}\}$$



# Tropical curves

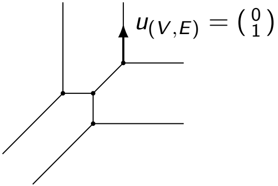
## Definition (Tropical curve)

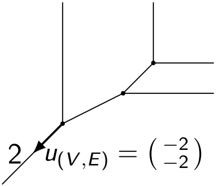
$h : \Gamma \to \mathbb{R}^2$, $\Gamma$ weighted graph with legs, $h$ continuous, piecewise linear, balancing $\forall V$:

$$\sum_{E \ni V} u_{(V,E)} = 0 \quad \text{for weight vectors } u_{(V,E)}.$$

- in $X$ if the legs point in the directions of the fan of $X$
- degree/class given by number and direction of legs
- genus $=$ # cycles

## Example (Tropical conics in $\mathbb{P}^2$)



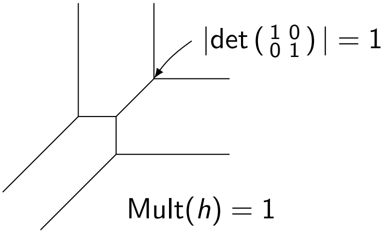



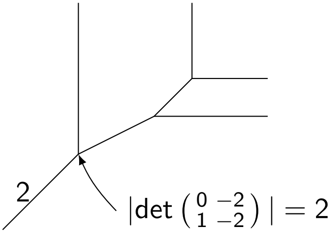
# Tropical curves

## Definition (Multiplicity)

$$\mathrm{Mult}(h) = \prod_V \prod_{l=2}^{k-1} \left| u_l \wedge \sum_{j=1}^{l} u_j \right|, \quad u_1, \ldots, u_k \text{ weight vectors at } V$$
$$= \prod_V |u_1 \wedge u_2| = \prod_V |u_2 \wedge u_3| = \prod_V |u_3 \wedge u_1| \quad \text{if trivalent}$$

## Example (Tropical conics in $\mathbb{P}^2$)



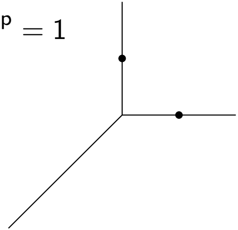



# Tropical correspondence

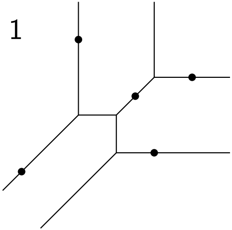
## Observation

$N_d^{\text{trop}} = \#$ trop. curves in $\mathbb{P}^2$ of genus 0 (no cycles) and degree $d$ through $3d-1$ general points, counted with multiplicity

$N_1^{\text{trop}} = 1$



$N_2^{\text{trop}} = 1$



$N_3^{\text{trop}} = 12$

...

## Theorem (Mikhalin, Nishinou-Siebert)

$$N_d = N_d^{\text{trop}}$$

# Tropical correspondence

**Theorem (General form of a tropical correspondence theorem)**

$$N_d = N_d^{\text{trop}}$$

**History**

- Mikhalkin '03: genus 0 curves on toric surfaces through general points
- Nishinou-Siebert '04: higher dimension (toric degenerations)
- Mandel-Ruddat '16: descendant invariants ($\psi$-classes)
- Bousseau '17: higher genus: generating functions of $q$-refined inv. (degeneration formula, vanishing of $\lambda$-classes)
- Gräfnitz '20: non-toric cases (resolution of log singularities)

# Main tool: log(arithmic) geometry

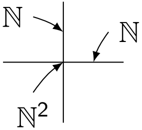
## Definition (Log structure)

Morphism of sheaves of monoids $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ with $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times$

## Example (Divisorial log structure by $j : D \hookrightarrow X$)

$$\mathcal{M}_{(X,D)} := (j^\star \mathcal{O}_{X\setminus D}^\times) \cap \mathcal{O}_X \overset{\alpha_{(X,D)}}{\hookrightarrow} \mathcal{O}_X$$

$\overline{\mathcal{M}}_{(X,D)} := \mathcal{M}_{(X,D)}/\mathcal{O}_{(X,D)}^\times$ captures vanishing order along $D$




## Magic Powder (K. Kato)

Allow functions to vanish along $D$

$\rightsquigarrow$ can treat some varieties that are singular along $D$ as being smooth!

(e.g. toric varieties are log smooth wrt. toric log structure $\mathcal{M}_{(X,\partial X)}$)

$\rightsquigarrow$ applications in degeneration situations

# Log geometry

### Example (Standard log point $\mathrm{pt}_{\mathbb{N}}$)

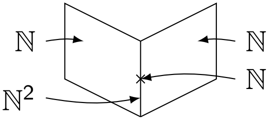
Pull $\mathcal{M}_{(\mathbb{A}^1,\{0\})}$ back along $\{0\} \hookrightarrow \mathbb{A}^1 \rightsquigarrow$ point with $\overline{\mathcal{M}}_{\mathrm{pt}} = \mathbb{N}$.

### Definition

A log structure is fine (saturated) if it is étale locally given by a chart $\underline{P} \to \mathcal{O}_X$ for a finitely generated (and saturated) monoid $P$.
Having a chart means the log structure is locally the toric one.

### Example

$\mathcal{M}_{(X,D)}$ for $D = \{t = 0\} \subset X = \{xy = tw\} \subset \mathbb{A}^4$ has no chart at 0:



# Log geometry

## Definition

$f : X \to Y$ is log smooth if $X, Y$ are fine and $f$ is locally of finite presentation and formally smooth in the category of fine log schemes.

## Example

$X = \text{Spec } \mathbb{C}[P] \to \mathbb{A}^1 = \text{Spec } \mathbb{C}[\mathbb{N}]$, with toric log structures, map induced by $\mathbb{N} \to P, 1 \mapsto \rho \neq 0$, is log smooth, and so is $X_0 \to \text{pt}_{\mathbb{N}}$.

## Example

$\pi : \mathfrak{X} \to \mathbb{A}^1$ semistable degeneration, i.e., proper map from smooth variety $\mathfrak{X}$ with $X_0 = \pi^{-1}(0)$ a normal crossings divisor and $\pi|_{\mathfrak{X} \setminus X_0}$ smooth. Then $\mathfrak{X} \to \mathbb{A}^1$ is log smooth for divisorial log structures by $X_0$ and $\{0\}$. Indeed, locally $\pi$ is projection to $t$-coordinate, with toric log structures,

$$\text{Spec } \mathbb{C}[t, x_1, \ldots, x_n]/(x_1 \cdot \ldots \cdot x_n - t^l) \to \mathbb{A}^1.$$

# Smooth del Pezzo log Calabi-Yau pairs

## Definition

Smooth del Pezzo log Calabi-Yau pair: $(X, D)$

- $X$ smooth projective surface with very ample anticanonical class (smooth del Pezzo surface of degree $\geq 3$)
- $D$ smooth anticanonical divisor.

## Example

There are exactly 8 such pairs:

- $(\mathbb{P}^2, E)$, $E$ elliptic curve;
- $X = \mathbb{P}^1 \times \mathbb{P}^1$, $D$ smooth bidegree $(2, 2)$-curve;
- $X = Bl_P^k \mathbb{P}^2$, $k = 1, \ldots, 6$.

## Remark

This talk: only $(\mathbb{P}^2, E)$. Note: $E$ is non-toric!

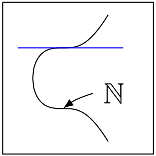
# Logarithmic Gromov-Witten invariants

Stable log map: log version of a stable map $\rightsquigarrow$ can specify tangency

**Definition**

$\beta$ class of stable log maps $f : C \to (\mathbb{P}^2, \mathcal{M}_{(\mathbb{P}^2,E)})$:

- genus 0;
- degree $d$;
- 1 marked point $p$ with full tangency $3d$ at $E$.




**Proposition (GS '11)**

The moduli space of basic stable log maps $\mathscr{M}(X, \beta)$ is a proper algebraic stack admitting a virtual fundamental class $[\![\mathscr{M}(X, \beta)]\!]$.

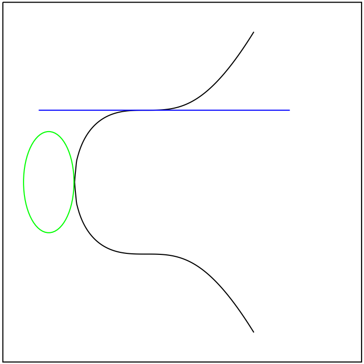
**Definition (vdim $= 0$)**

$$N_d := \int_{[\![\mathscr{M}(X,\beta)]\!]} 1 \in \mathbb{Q}$$

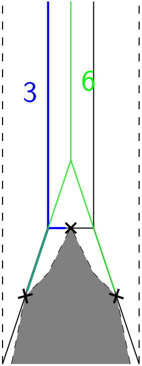
# Tropical correspondence for $(\mathbb{P}^2, E)$

$N_d = \#$ rational degree $d$ curves in $\mathbb{P}^2$ meeting $E$ in a single point

| | | | | |
|---|---|---|---|---|
| $N_1 = 9$ | $N_2 = \frac{135}{4} = 27 + 9 \cdot \frac{3}{4}$ | $N_3 = 244$ | $N_4 = \frac{36999}{16}$ | $N_5 = \frac{635634}{25}$ |



$\longleftrightarrow$



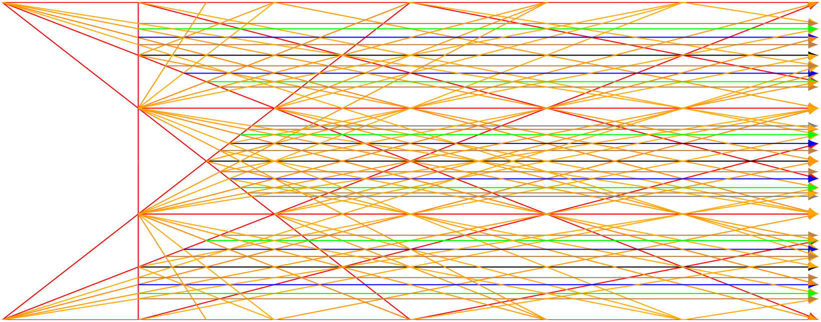

## Theorem

$$N_d = N_d^{\text{trop}}$$

# Scattering

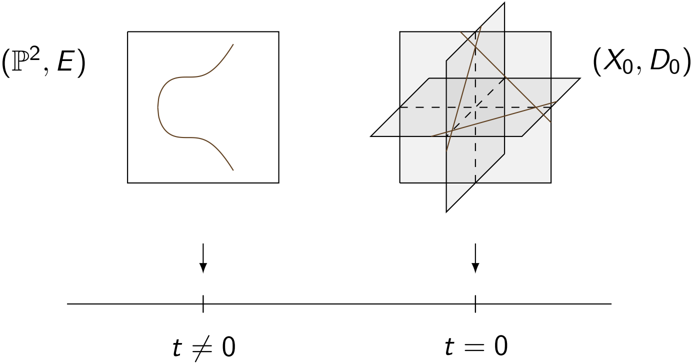
## Theorem

$$\log\, f_{\text{out}} = \sum_{d=1}^{\infty} 3d \cdot N_d \cdot x^{3d} \quad \text{for} \quad f_{\text{out}} := \prod_{\mathfrak{p} \text{ outgoing wall}} f_{\mathfrak{p}}$$



# Toric degeneration

Idea: deform the complicated object $(\mathbb{P}^2, E)$ into something simpler such that $N_d$ can still be calculated




$$\mathfrak{X} = \{XYZ = t^3(W + f_3(X,Y,Z))\} \subset \mathbb{P}(1,1,1,3) \times \mathbb{A}^1 \to \mathbb{A}^1$$

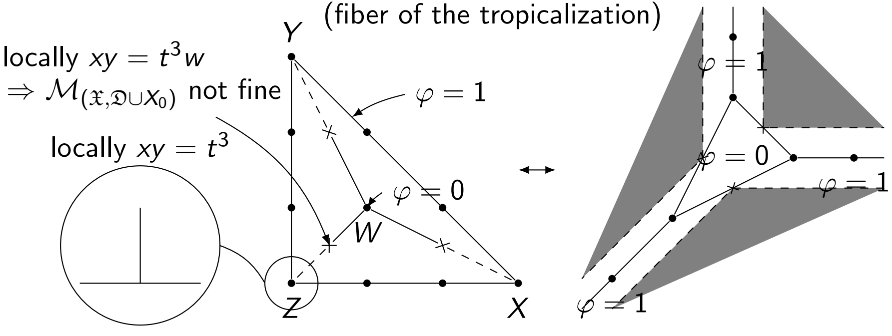
$$\mathfrak{D} = \{W = 0\} \qquad X_0 = \bigcup_3 \mathbb{P}(1,1,3) \text{ intersecting along } \mathbb{P}^1\text{'s}$$

# Toric degeneration

$$\mathfrak{X} = \{XYZ = t^3(W + f_3(X,Y,Z))\} \subset \mathbb{P}(1,1,1,3) \times \mathbb{A}^1 \to \mathbb{A}^1$$
$$\mathfrak{D} = \{W = 0\} \qquad X_0 = \bigcup_3 \mathbb{P}(1,1,3) \text{ intersecting along } \mathbb{P}^1\text{'s}$$

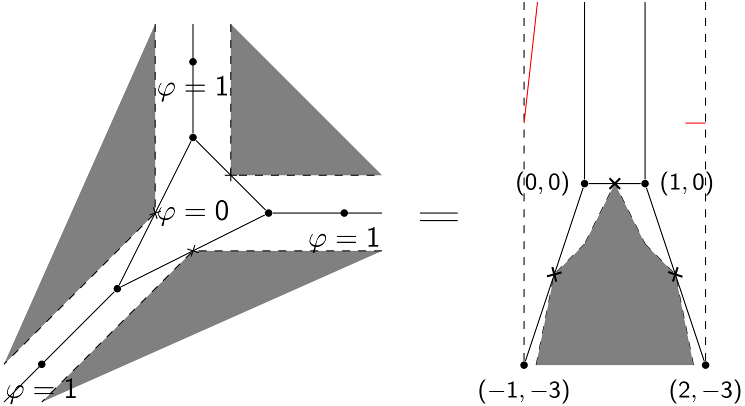
intersection complex: glue polytopes, $\varphi$ describes family locally
($\mathfrak{X}$ defined by upper convex hull of $\varphi$)

dual intersection complex: glue fans, $\varphi$ gives divisor class (polarization)
(fiber of the tropicalization)



# Dual intersection complex $(B, \mathscr{P}, \varphi)$

chart at unbounded cell: all unbounded rays are parallel
monodromy transformation: $\Lambda_B \to \Lambda_B, m \mapsto \left(\begin{smallmatrix} 1 & 9 \\ 0 & 1 \end{smallmatrix}\right) \cdot m$



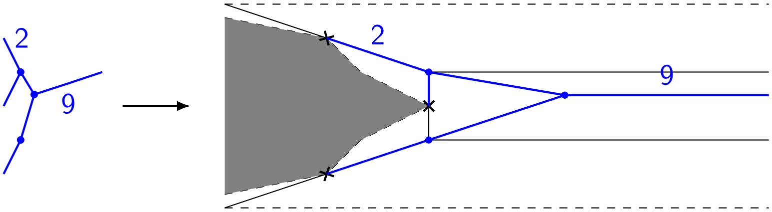
# Tropical curves

## Definition (Tropical curve)

$h : \Gamma \to B$, $\Gamma$ weighted graph, $h$ continuous, integral affine linear

- balancing condition $\forall V$: $\sum_{E \ni V} u_{(V,E)} = 0$
- legs can end at affine singularities with prescribed direction

## Example $(\mathbb{P}^2, E)$




$$\mathfrak{H}_d := \{h : \Gamma \to B \mid \text{one unbounded leg with weight } w_{\text{out}} = 3d\}$$

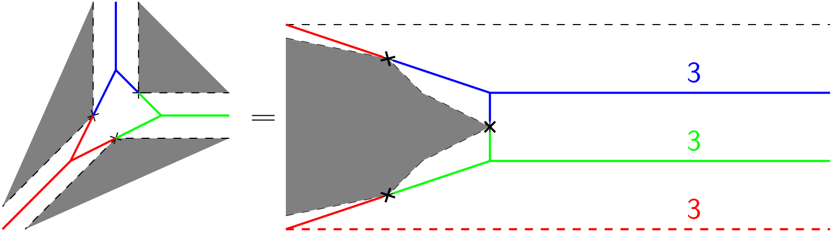
# Tropical correspondence

## Theorem (Tropical correspondence)

$$N_d = N_d^{\mathrm{trop}} := \sum_{h \in \mathfrak{H}_d} \mathrm{Mult}(h)$$

## Example

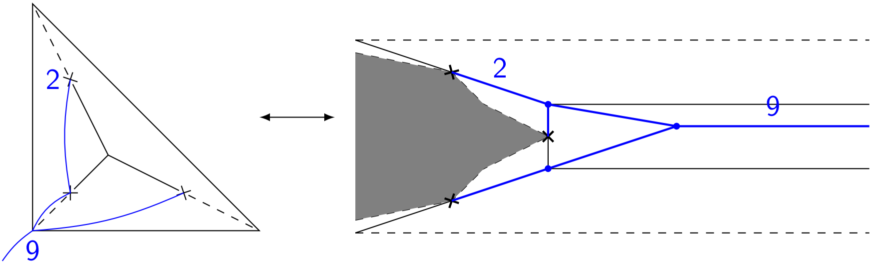
$N_1^{\mathrm{trop}} = 9$



# Tropical correspondence

## Idea of Proof

Tropical curves describe combinatorics of curves on $X_0$




## Problem

Logarithmic Gromov-Witten invariants are constant in log smooth families.
The toric degeneration $\mathfrak{X} \to \mathbb{A}^1$ is not log smooth (wrt. $\mathcal{M}_{(\mathfrak{X},\mathfrak{D}\cup X_0)}$)
$\Rightarrow$ cannot calculate $N_d$ on $X_0$

# Step 1: Resolution of log singularities

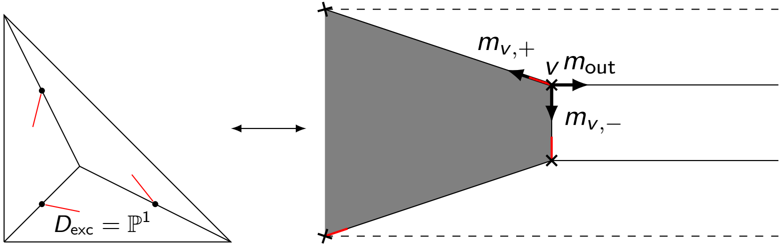
Locally $\{xy = t^3 w\} \subset \mathbb{A}^4$

Blow up two irreducible components of $X_0$ or blow up the interior edges and contract one component of $D_{\text{exc}} = \mathbb{P}^1 \times \mathbb{P}^1$ in a symmetric way.

$\rightsquigarrow$ log smooth degeneration $\tilde{\mathfrak{X}} \to \mathbb{A}^1$ (not a toric degeneration!)

$\Rightarrow$

$$N_d := \int_{[\![\mathscr{M}(\tilde{\mathfrak{X}},\beta)]\!]} 1 = \int_{[\![\mathscr{M}(\tilde{X}_t,\beta)]\!]} 1 \quad \text{for all } t \in \mathbb{A}^1$$



# Tropicalization and stable log maps

**Definition (Tropicalization)**

$$\mathrm{Trop}(X) := \left( \coprod_{x \in X} \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0}) \right) / \sim$$

union over scheme-theoretic points, equiv. relation by generization maps.

Consider a basic stable log map in $\mathscr{M}(\tilde{X}_0/\mathrm{pt}_{\mathbb{N}}, \beta)$

$$\begin{array}{ccc} C & \xrightarrow{f} & \tilde{X}_0 \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \tilde{\pi}_0} \\ \mathrm{pt}_{\mathbb{N}} & \xrightarrow{g} & \mathrm{pt}_{\mathbb{N}} \end{array} \quad \xrightarrow{\mathrm{Trop}} \quad \begin{array}{ccc} \mathrm{Trop}(C) & \xrightarrow{\mathrm{Trop}(f)} & \mathrm{Trop}(\tilde{X}_0) \\ \downarrow{\scriptstyle \mathrm{Trop}(\gamma)} & & \downarrow{\scriptstyle \mathrm{Trop}(\tilde{\pi}_0)} \\ \mathbb{R}_{\geq 0} & \xrightarrow{\mathrm{Trop}(g)} & \mathbb{R}_{\geq 0} \end{array}$$

We have $\mathrm{Trop}(\gamma)^{-1}(1) \simeq \Gamma_C$ and $\mathrm{Trop}(\tilde{\pi}_0)^{-1}(1) \simeq \tilde{B}$.

This gives a tropical curve $\tilde{h} : \Gamma_C \to \tilde{B}$ with modified balancing condition.

# Tropical curves

## Proposition

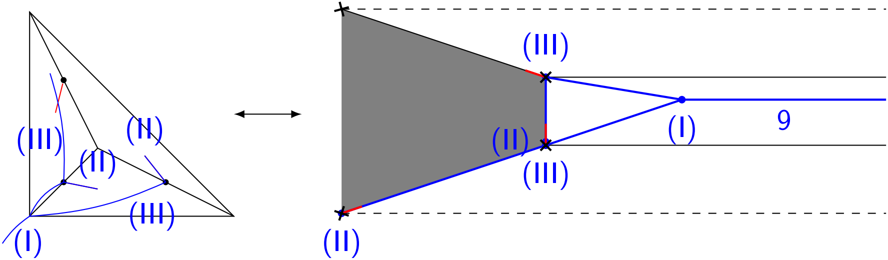
$$\tilde{\mathfrak{H}}_d := \{\tilde{h} : \tilde{\Gamma} \to \tilde{B} \text{ tropicalization of } f \in \mathscr{M}(\tilde{\mathfrak{X}}/\mathbb{A}^1, \beta)\}$$

modified balancing condition: 3 types of vertices

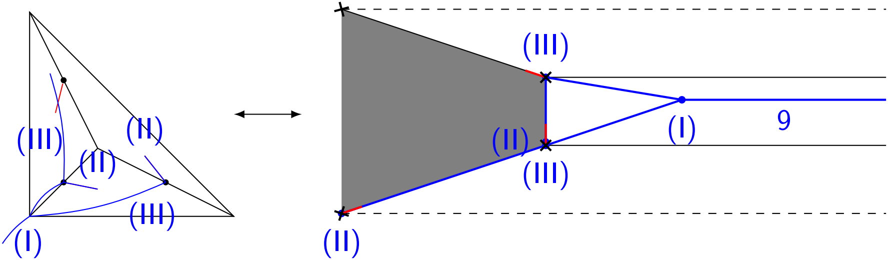
(I) $\sum_{E \ni V} u_{(V,E)} = 0$

(II) $u_{(V,E)} = km_{v,+}$

(III) $\sum_{E \ni V} u_{(V,E)} + km_{v,+} = 0$



# Step 2: Refinement and logarithmic modification




Refine $\mathscr{P}$ by tropical curves in $\tilde{\mathfrak{H}}_d$ (base change $t \mapsto t^e \rightsquigarrow$ integral vert.)
$\rightsquigarrow$ logarithmic modification via subdivision of Artin fans
$\rightsquigarrow$ log smooth degeneration $\tilde{\mathfrak{X}}_d \to \mathbb{A}^1$ such that stable log maps to the central fiber $Y := \tilde{X}_{d,0}$ are torically transverse

Log GW invariant under log modifications (Abramovich, Wise '18)

# Step 3: Degeneration formula

Calculate $N_d$ on $Y := \tilde{X}_{d,0}$

Theorem (Decomposition formula, ACGS '17)

$$\mathscr{M}_d := \mathscr{M}(Y,\beta) = \coprod_{\tilde{h}\in\tilde{\mathfrak{H}}_d} \mathscr{M}_{\tilde{h}}$$

$$[\![\mathscr{M}_d]\!] = \sum_{\tilde{h}\in\tilde{\mathfrak{H}}_d} \frac{\ell_{\tilde{\Gamma}}}{|\mathrm{Aut}(\tilde{h})|} F_\star [\![\mathscr{M}_{\tilde{h}}]\!]$$

for $\ell_{\tilde{\Gamma}} = \mathrm{lcm}\{w_E \mid E \in E(\tilde{\Gamma})\}$.

Proof.

The affine length of the image of an edge $E \in E(\tilde{\Gamma})$ is $\ell_E w_E \in \mathbb{Z}$, so the scaling necessary to obtain integral edge lengths $\ell_E$ is $\ell_{\tilde{\Gamma}}$. □

# Gluing

$\mathop{\times}\limits_V \mathscr{M}_V^\circ$ contains stable log maps in $\prod\limits_V \mathscr{M}_V^\circ$ matching along divisors

$$\begin{array}{ccc} \mathop{\times}\limits_V \mathscr{M}_V^\circ & \longrightarrow & \prod\limits_V \mathscr{M}_V^\circ \\ \downarrow & & \downarrow \text{ev} \\ \prod\limits_{E\in E(\tilde{\Gamma})} D_E^\circ & \xrightarrow{\delta} & \prod\limits_V \prod\limits_{\substack{E\in E(\tilde{\Gamma}) \\ V\in E}} D_E^\circ \end{array}$$

## Proposition (KLR '18)

There is a morphism $\mathrm{cut} : \mathscr{M}_{\tilde{h}} \to \times_V \mathscr{M}_V^\circ$, étale of degree

$$\deg(\mathrm{cut}) = \frac{\prod_{E\in E(\tilde{\Gamma})} w_E}{\ell_{\tilde{\Gamma}}}$$

## Proof.

For each node there is a choice of $w_E$-th root of unity in the log structure of $C$. Isomorphic log structures correspond to $\ell_{\tilde{\Gamma}}$-th roots of unity. □

# Gluing

## Proposition (Gluing formula, KLR '18)

$$[\![\mathscr{M}_{\tilde{h}}]\!] = \mathrm{cut}^\star \delta^! \prod_{V \in V(\tilde{\Gamma})} [\![\mathscr{M}_V^\circ]\!]$$

## Corollary

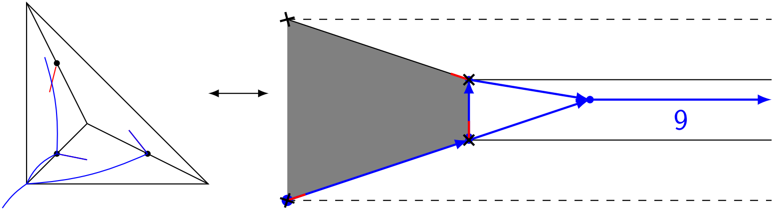
$$\int_{[\![\mathscr{M}_{\tilde{h}}]\!]} 1 = \frac{1}{\ell_{\tilde{\Gamma}}} \prod_{E \in E(\tilde{\Gamma})} w_E \cdot \int_{\prod_V [\![\mathscr{M}_V]\!]} \mathrm{ev}^\star[\delta]$$

for the class of the diagonal in $\prod_V \prod_{\substack{E \in E(\tilde{\Gamma}) \\ V \in E}} D_E$ (note $D_E = \mathbb{P}^1$)

$$[\delta] = \prod_{E \in E(\tilde{\Gamma})} (\mathrm{pt}_E \times 1 + 1 \times \mathrm{pt}_E)$$

# Gluing

$\tilde{\Gamma}$ is a rooted tree, thus has a natural orientation:




Notation: $E$ points from $V_{E,-}$ to $V_{E,+}$

### Proposition (Identifying the pieces)

The only term of $\mathrm{ev}^\star[\delta] = \prod_{E\in E(\tilde{\Gamma})} \left((\mathrm{ev}_{V_{E,-}})^\star[\mathrm{pt}_E] + (\mathrm{ev}_{V_{E,+}})^\star[\mathrm{pt}_E]\right)$ giving a nonzero contribution after integration is $\prod(\mathrm{ev}_{V_{E,+}})^\star[\mathrm{pt}_E]$.

### Proof.

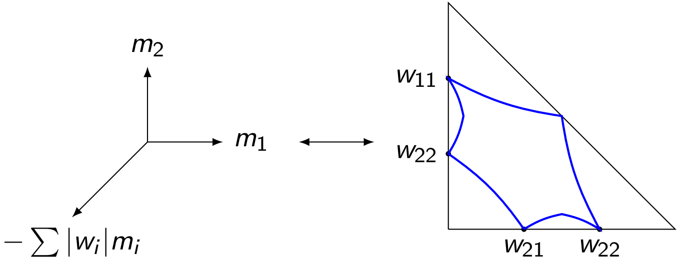
All other gluings give negative virtual dimension. □

# Contributions of the vertices

## Definition (Toric invariants)

$\mathbf{m} = (m_1, \ldots, m_n) \in (\mathbb{Z}^2)^n$, $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$, $\mathbf{w}_i = (w_{i1}, \ldots, w_{il_i}) \in \mathbb{Z}^{l_i}_{>0}$

$N_{\mathbf{m}}(\mathbf{w}) :=$



# Contributions of the vertices

## Definition

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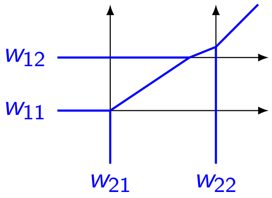
$N^{\text{trop}}_{\mathbf{m}}(\mathbf{w}) :=$




## Proposition (GPS '09)

$$N^{\text{trop}}_{\mathbf{m}}(\mathbf{w}) = N_{\mathbf{m}}(\mathbf{w}) \cdot \prod_{i,j} w_{ij}$$

# Contributions of the vertices

With $\mathbf{m}$, $\mathbf{w}$ defined by edges of $\tilde{h}$ pointing towards $V$:

$$\begin{aligned}
\text{(I)}: N_V &= N_{\mathbf{m}}(\mathbf{w}) \\
\text{(II)}: N_V &= \frac{(-1)^{w_E-1}}{w_E^2}(w_E\text{-fold multiple covers of } D_{\text{exc}} = \mathbb{P}^1) \\
\text{(III)}: N_V &= \sum_{\mathbf{w}_{V,+}} \frac{N_{\mathbf{m}}(\mathbf{w})}{|\text{Aut}(\mathbf{w}_{V,+})|} \prod_{i=1}^{l} \frac{(-1)^{w_{V,i}-1}}{w_{V,i}}
\end{aligned}$$

sum over $\mathbf{w}_{V,+} = (w_{V,1}, \ldots, w_{V,l_V})$ such that $|\mathbf{w}_{V,+}| := \sum_{i=1}^{l_V} w_{V,i} = k$ for $\sum_{E \ni V} u_{(V,E)} + k m_{v,+} = 0$

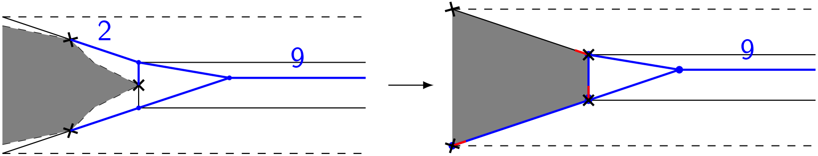
## Proposition (Degeneration formula)

$$N_d = \sum_{\tilde{h} \in \tilde{\mathfrak{H}}_d} \frac{1}{|\text{Aut}(\tilde{h})|} \cdot \prod_{E \in E(\tilde{\Gamma})} w_E \cdot \prod_{V \in V(\tilde{\Gamma})} N_V.$$

# Balanced tropical curves

**Lemma**

$$\mathfrak{H}_d = \{\text{balanced } h : \Gamma \to B\} \twoheadrightarrow \tilde{\mathfrak{H}}_d = \{\tilde{h} : \tilde{\Gamma} \to \tilde{B}\}$$




**Theorem**

$$N_d = \sum_{h \in \mathfrak{H}_d} \left( \frac{1}{|\mathrm{Aut}(h)|} \cdot \underbrace{\prod_{E \text{ compact}} w_E}_{\text{(gluing)}} \cdot \underbrace{\prod_{E \text{ bd leg}} \frac{(-1)^{w_E - 1}}{w_E}}_{\text{(multicover)}} \cdot \underbrace{\prod_V N_{\mathbf{m}}(\mathbf{w})}_{\text{(components)}} \right)$$

# Tropical correspondence

## Theorem (Tropical correspondence)

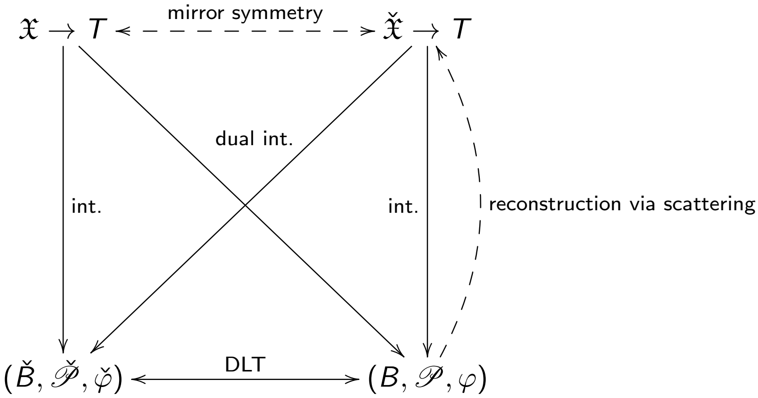
$$\begin{aligned} N_d &= \sum_{h\in\mathfrak{H}_d} \left( \frac{1}{|\mathrm{Aut}(h)|} \cdot \underbrace{\prod_{E \text{ compact}} w_E}_{\text{(gluing)}} \cdot \underbrace{\prod_{E \text{ bd leg}} \frac{(-1)^{w_E-1}}{w_E}}_{\text{(multicover)}} \cdot \underbrace{\prod_V N_{\mathbf{m}}(\mathbf{w})}_{\text{(components)}} \right) \\ &= \sum_{h\in\mathfrak{H}_d} \left( \frac{1}{|\mathrm{Aut}(h)|} \cdot \prod_{E \text{ bd leg}} \frac{(-1)^{w_E-1}}{w_E^2} \cdot \prod_V N_{\mathbf{m}}^{\text{trop}}(\mathbf{w}) \right) = N_d^{\text{trop}} \end{aligned}$$

## Remark

- Should also apply to other problems, inserting points conditions, etc.
- New: bounded legs have multiplicity $\frac{(-1)^{w_E-1}}{w_E^2}$ (multiple cover contribution of $D_{\text{exc}} = \mathbb{P}^1$)

# Gross-Siebert program

1. Find a toric degeneration $\mathfrak{X} \to T = \text{Spec } \mathbb{C}[\![t]\!]$ of $X$ or $(X, D)$.
2. Form the dual intersection complex $(B, \mathscr{P}, \varphi)$ of $\mathfrak{X}$.
3. Construct another toric degeneration $\check{\mathfrak{X}} \to T = \text{Spec } \mathbb{C}[\![t]\!]$.
4. $\check{X}_{t \neq 0}$ is the mirror CY/LG to $X$ (superpotential via broken lines).

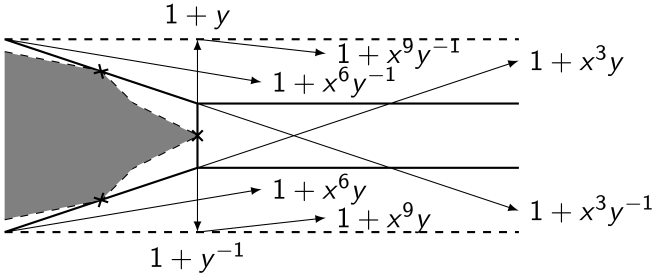


# Initial wall structure

Wall structure: collection of codim. 1 polyhedral subsets (walls) $\mathfrak{p}$ with attached functions $f_{\mathfrak{p}} \in R_\varphi$ satisfying some conditions.

$$P_\varphi = \{p = (\overline{p}, h) \in M \oplus \mathbb{Z} \mid h \geq \varphi(\overline{p})\}$$
$$R_\varphi = \varprojlim \mathbb{C}[P_\varphi]/(t^k)$$

$x := z^{((-1,0),0)}, y := z^{((0,-1),0)}, t := z^{((0,0),1)}$

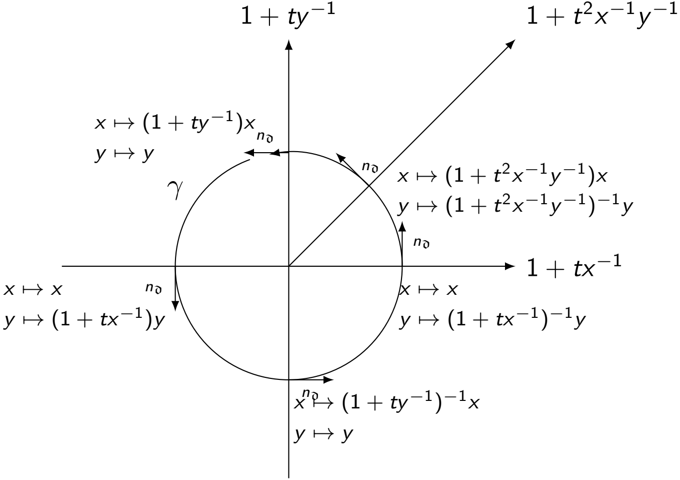



Note $z^{(\overline{p},0)} = \left(z^{(-\overline{p},\varphi(-\overline{p}))}\right)^{-1} t^{\varphi(-\overline{p})}$.

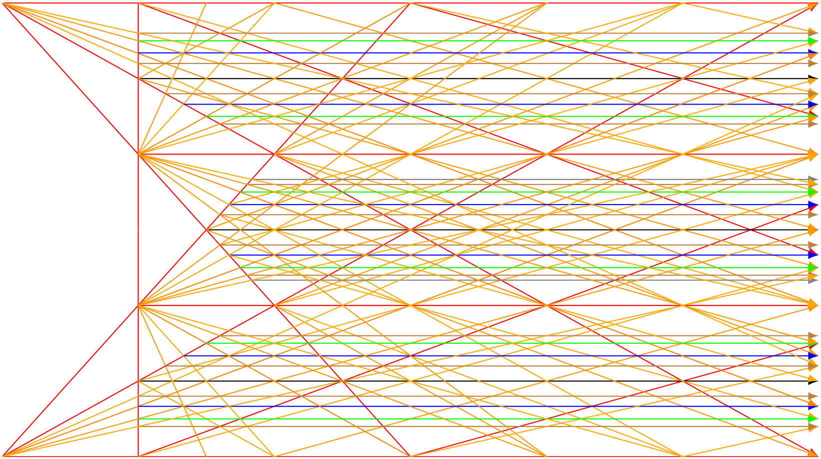
# Scattering

$\theta_{\mathfrak{d}}(z^p) = f_i^{-\langle n_{\mathfrak{d}}, \overline{p} \rangle} z^p$
$\theta_{\gamma,\mathfrak{D}} = \theta_{\mathfrak{d}_s} \circ \ldots \circ \theta_{\mathfrak{d}_1}$

$\mathfrak{D}$ consistent to order $k$ if
$\theta_{\gamma,\mathfrak{D}} \equiv 1 \bmod (t^k)$ for all $\gamma$.



# Scattering



# The tropical vertex

## Proposition (GPS '09)

If $\mathfrak{D}$ consists of lines in direction $m_i$ with

$$\log f_i = \sum_{w=1}^{\infty} a_{iw} z^{(-wm_i,0)}, \quad a_{iw} \in \mathbb{C},$$

then

$$\log f_{\mathfrak{d}} = \sum_{w=1}^{\infty} \sum_{\mathbf{w}} w \frac{N_{\mathbf{m}}(\mathbf{w})}{|\mathrm{Aut}(\mathbf{w})|} \left( \prod_{\substack{1\leq i\leq n \\ 1\leq j\leq l_i}} a_{iw_{ij}} \right) z^{(-wm_{\mathfrak{d}},0)},$$

where the sum is over all $n$-tuples of weight vectors $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ satisfying

$$\sum_{i=1}^{n} |\mathbf{w}_i| m_i = wm_{\mathfrak{d}}.$$

# Main theorem

**Theorem**

$$\log f_{\mathrm{out}} = \sum_{\underline{\beta} \in H_2^+(X,\mathbb{Z})} (D \cdot \underline{\beta}) \cdot N_d \cdot x^{D \cdot \underline{\beta}}.$$

**Proof.**

- Applying [GPS] inductively gives a formula similar to the degeneration formula.
- Note that

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} x^k.$$

□

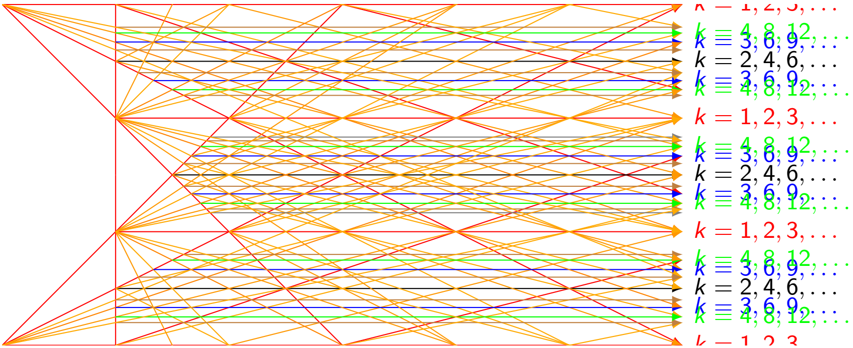
# Torsion points

Group law on $E$ with identity a flex point

A curve contributing to $N_d$ intersects $E$ in a point of order $3k$, $k \leq d$

$N_{d,k} := \#$ curves meeting $E$ in a fixed point of order $3k$.

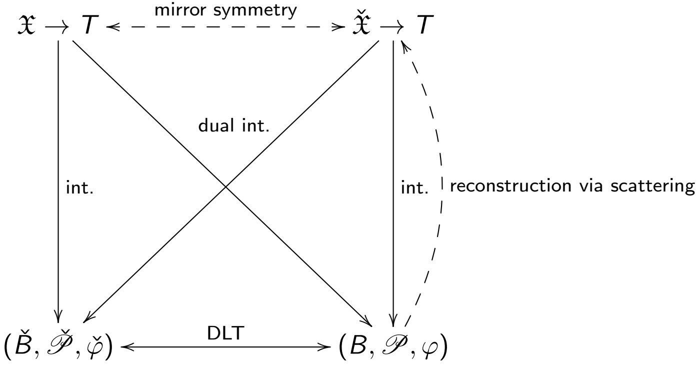
$n_{d,k} :=$ log BPS numbers (subtract multiple cover contributions)




| | | | | | |
|---|---|---|---|---|---|
| $n_{4,1} = 14$ | $n_{4,2} = 14$ | $n_{4,4} = 16$ | $n_{6,1} = 927$ | $n_{6,2} = 938$ | $n_{6,3} = 936$ |

# Reconstruction

- Scattering gives a refinement of $(B, \mathscr{P}, \varphi)$.
- Mirror constructed by gluing pieces corresponding to maximal cells.
- This is well-defined by the scattering construction (Gross-Siebert '10).
- Superpotential via broken lines $\rightsquigarrow$ theta functions, intrinsic MS.
- Main theorem: curve counts $\overset{\text{MS}}{\longleftrightarrow}$ deformations of complex structure.



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