The tropical geometry of monotone Hurwitz numbers

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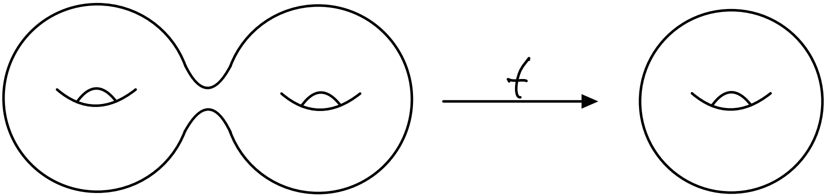
Nottingham University
Online Algebraic Geometry Seminar, 20.01.22

Introduction and context

Hurwitz numbers: Important enumerative invariants in algebraic geometry



Introduction and context

Hurwitz numbers: Important enumerative invariants in algebraic geometry

Introduced by Hurwitz in the 1890s.

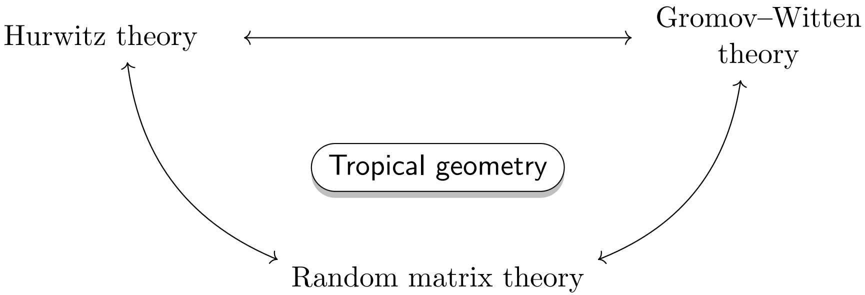
Rapid developments since 1990s due to deep connections with **Gromov–Witten theory** and **string theory**.

Connections to other fields

Introduction and context

Today: Focus on *monotone Hurwitz numbers* from random matrix theory.

Progress report: programme towards connecting monotone Hurwitz numbers to Gromov–Witten theory via combinatorial methods of **tropical geometry**.



Overview

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Riemann surfaces

Recall: **Riemann surfaces** are one-dimensional complex manifolds.

Hurwitz theory is concerned with the enumeration von holomorphic maps between *compact Riemann surfaces*, i.e. finite regular morphisms between complex smooth algebraic curves.

Maps between Riemann surface

Let S_1, S_2 compact Riemann surfaces, $f\colon S_2 \to S_1$ a non-constant holomorphic map.
For all $y \in S_2$, we have that f locally at y is given by

$$z \mapsto z^{n_y},$$

where $n_y \in \mathbb{N}_{\geq 1}$.

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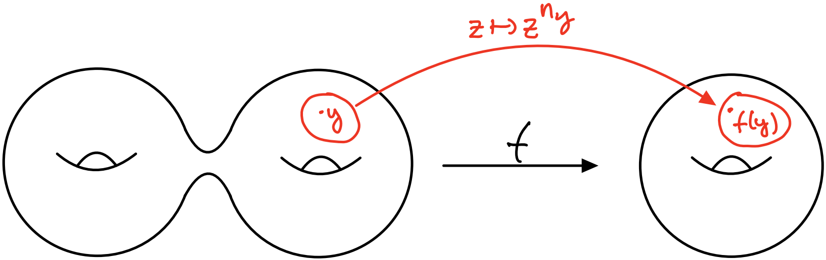

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We call n_y the **ramification index of** y.

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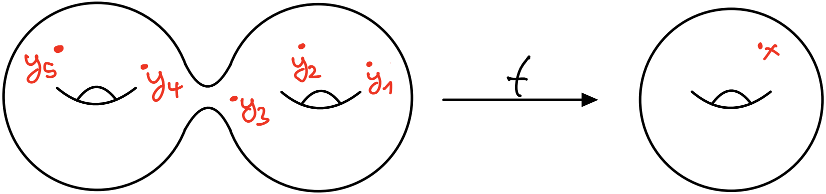
For all but finitely many y, we have $n_y = 1$.

Maps between Riemann surface

For $f : S_2 \to S_1$, let $x \in S_1$ and

$$f^{-1}(x) = \{y_1, \ldots, y_s\},$$

then, we call $\mu_x = (n_{y_1}, \ldots, n_{y_s})$ the **ramification profile** of x.



$\mu_x = (n_{y_1}, n_{y_2}, n_{y_3}, n_{y_4}, n_{y_5})$

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Fact

All ramification profiles μ_x are partitions of the same number d, i.e. $d = n_{y_1} + \cdots + n_{y_s}$.
We call d the **degree** $\deg(f)$ **of** f.

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If $\mu_x = (2, 1 \dots, 1)$, we call x **simply ramified**.

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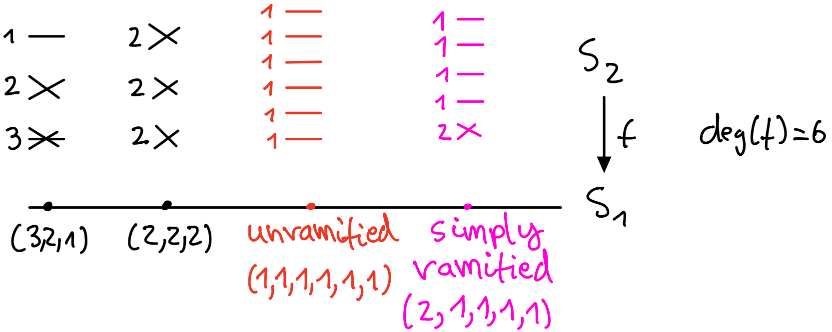

Double Hurwitz numbers

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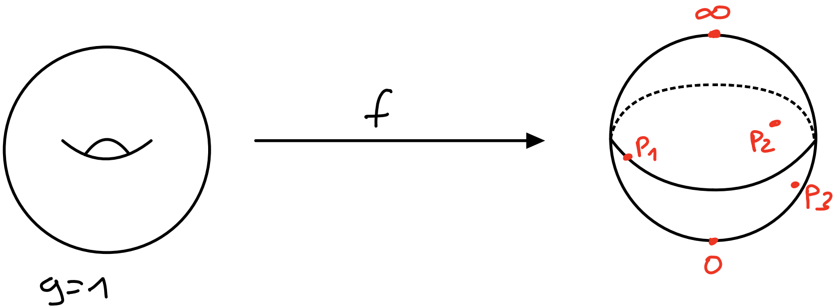
- the ramification profile of the **north pole** ∞ is ν;
- p_i is simply ramified with $\mu_{p_i} = (2, 1, \dots, 1)$;
- all other points are unramified.

Double Hurwitz numbers



Computation via the symmetric group

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E.g. $C((1\,2\,3)(4\,5)(6)) = (3, 2, 1)$.

Computation via the symmetric group

Theorem (Hurwitz '1892)

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Then, we have:

$$H_r(\mu,\nu) = \frac{1}{d!} \cdot \left| \left\{ \begin{array}{l} (\sigma_1, \tau_1, \ldots, \tau_r, \sigma_2) : \\ \bullet\ \sigma_1, \sigma_2, \tau_i \in S_d \\ \bullet\ C(\sigma_1) = \mu,\ C(\sigma_2) = \nu \\ \bullet\ \tau_i \text{ transposition} \\ \bullet\ \sigma_1 \tau_1 \cdots \tau_r = \sigma_2 \end{array} \right\} \right|$$

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Proceeding through all tuples, we obtain $H_r(\mu, \nu) = 14$.

Overview

Gromov–Witten theory

Gromov–Witten theory originates in mathematical physics

- Study of the intersection theory of parameter spaces of curves, more precisely their moduli spaces.
- Central objects are so-called **Gromov–Witten invariants**
- Revolutionised enumerative algebraic geometry

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Connections between Gromov–Witten and Hurwitz theory via the celebrated ELSV formula.

ELSV formula

ELSV formula (Ekedahl, Lando, Shapiro, Vainshtein '99/'01)

Let $d > 0$, $r \geq 0$ and $\mu, \nu \vdash d$ with $\nu = (1, \ldots, 1)$. Then, we have

$$H_r(\mu, \nu) =$$
$$(\text{Comb. factor}) \cdot (\text{Gromov–Witten invariants of } \overline{M}_{g,n}) .$$

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In other words: **Hurwitz numbers enumerate certain Gromov–Witten invariants!**

Applications of the ELSV formula

The ELSV formula implies various important theorems on the intersection theory of $\overline{M}_{g,n}$.

- Witten's conjecture/Kontsevich's theorem
- Virasoro constraints
- Faber's λ_g conjecture

Polynomiality of Hurwitz numbers

It also allows applications in Hurwitz theory

E.g. it implies the Goulden–Jackson conjecture '99.

Conjecture (Goulden, Jackson '99)

Let $d > 0$, $r \geq 0$ and $\mu, \nu \vdash d$ with $\nu = (1, \ldots, 1)$. Then, we have

$$H_r(\mu, \nu) = (\text{Comb. factor}) \cdot P_g(\mu),$$

where $P_g(\mu)$ is a polynomial in the entries of μ.

Generalisations

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What about other ν?

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2005: Goulden, Jackson and Vakil propose a programme towards an ELSV type formula for all double Hurwitz numbers.

Idea: Derive polynomial behaviour of $H_r(\mu, \nu)$ for any ν and reconstruct an ELSV type formula.

Theorem of Goulden, Jackson and Vakil

Let $m, n \in \mathbb{N}_{\geq 1}$ and consider

$$\mathcal{H}_{m,n} := \{(\mu, \nu) \in \mathbb{N}^m \times \mathbb{N}^n \mid \sum \mu_i = \sum \nu_i\}.$$

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Theorem (Goulden, Jackson, Vakil '05)

Let $r \geq 0$. The map

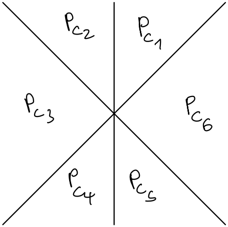
$$\begin{aligned} \mathcal{H}_{m,n} &\to \mathbb{Q} \\ (\mu, \nu) &\mapsto H_r(\mu, \nu). \end{aligned}$$

is **piecewise polynomial**.

Theorem of Goulden, Jackson and Vakil

Hyperplane arrangement in $\mathcal{H}_{m,n}$.

- Polynomial in every maximal cell.



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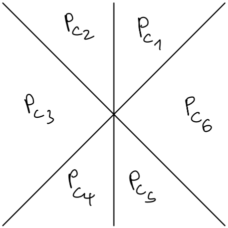

Conjecture (Goulden, Jackson, Vakil '05; Bayer, Cavalieri, Johnson, Markwig '12)

Concrete proposal for an ELSV type formula for double Hurwitz numbers

Polynomiality of double Hurwitz numbers

Question (Shadrin, Shapiro, Vainshstein '06)

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The difference $P_{C_1}(\mu,\nu) - P_{C_2}(\mu,\nu)$ may be expressed recursively in terms of Hurwitz numbers with smaller input data. (**Recursive wall-crossing formula**)

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Since 2010: Strong progress, but Goulden–Jackson–Vakil/ Bayer–Cavalieri–Johnson–Markwig conjecture remains open.

Overview

Harish-Chandra–Itzykson–Zuber integral

Harish-Chandra–Itzykson–Zuber integral: Central object in random matrix theory.

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(**Motivation:** Conjecture about the convergence of the HCIZ integral – proved in 2020 by Novak.)

Monotone double Hurwitz numbers

Definition (Goulden, Guay-Paquet, Novak '11)

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Then, we define **monotone double Hurwitz numbers**:

$$\vec{H}_r(\mu,\nu) = \frac{1}{d!} \cdot \left| \left\{ \begin{array}{l} (\sigma_1, \tau_1, \dots, \tau_r, \sigma_2): \\ \bullet\ \sigma_1, \sigma_2, \tau_i \in S_d \\ \bullet\ C(\sigma_1) = \mu,\ C(\sigma_2) = \nu \\ \bullet\ \tau_i \text{ Transposition} \\ \bullet\ \sigma_1 \tau_1 \cdots \tau_r = \sigma_2 \\ \bullet\ \tau_i = (r_i\, s_i) \text{ mit } r_i < s_i, \text{ then: } s_i \leq s_{i+1} \end{array} \right\} \right|$$

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Proceedings through all tuples, we obtain $\vec{H}_r(\mu,\nu) = \frac{25}{4}$.

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1. Our ansatz is via **tropical geometry**.
2. Long-term goal: **ELSV-Typ formula** for monotone double Hurwitz numbers.

Overview

Tropical geometry

Tropical geometry: Conceptualisation of a combinatorial approach to algebraic geometry

- Inception in 2000s through work of Mikhalkin and Sturmfels following a suggestion of Kontsevich
- **Tropicalisation:** Transformation of algebro–geometric to combinatorial objects

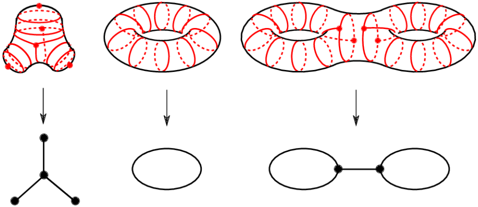

Figure: Colla, Marelli - "Pair of pants decomposition of 4-manifolds"

Tropical Hurwitz numbers

Cavalieri, Johnson, Markwig '08: Graph-theoretic interpretation of classical double Hurwitz numbers.

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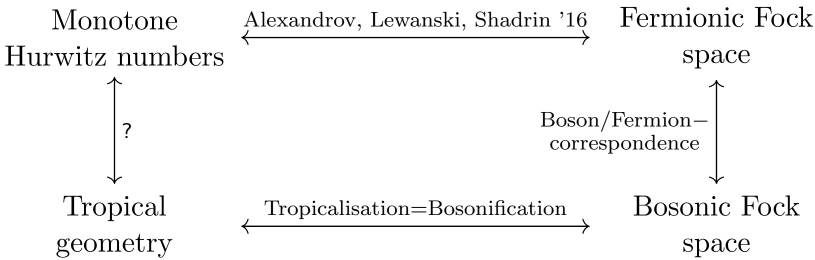
Alternative approach: Fermionic/bosonic Fock space



Tropical monotone Hurwitz numbers

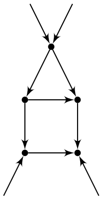
Theorem (H. '17; H., Lewanski '18)

Let $d > 0$, $r \geq 0$, $\mu, \nu \vdash d$. Then, we have

$$\vec{H}_r(\mu, \nu) = \sum_{\Gamma}$$

where

- Γ finitely many oriented graphs



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$$\vec{H}_r(\mu, \nu) = \sum_{\Gamma} \mathrm{GW}(\Gamma),$$

where

- Γ finitely many oriented graphs
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Tropical monotone Hurwitz numbers

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In other words: **Relative Gromov–Witten invariants compute monotone double Hurwitz numbers!**

Overview

ELSV-Typ formula for $\vec{H}_r(\mu,\nu)$

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- Purely an existence theorem;
- Wall-crossing formulae not approachable with the involved methods $\longrightarrow$ **Open problem**

Wall-Crossing formulae for $\vec{H}_r(\mu, \nu)$

Using the tropical interpretation for monotone double Hurwitz numbers, we could prove the following.

Theorem (H. '17; H., Kramer, Lewanksi '17; H., Lewanski '19)

- Algorithms to compute monotone double Hurwitz numbers
- Monotone double Hurwitz numbers admit recursive wall-crossing formulae.

Wall-Crossing formulae for $\vec{H}_r(\mu, \nu)$

Sketch of proof

- Use the tropical interpretation $\vec{H}_r(\mu, \nu) = \sum_\Gamma \mathrm{GW}(\Gamma)$
- Observe: $\mathrm{GW}(\Gamma)$ is a discrete integral over a polytope $\longrightarrow$ Polynomiality via Ehrhart theory
- Combinatorial analysis of the structure of the polytopes in different maximal cells of the hyperplane arrangement yields wall-crossing formulae.

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Our results also hold for **strictly monotone Hurwitz numbers** ($s_i < s_{i+1}$), that are equivalent to an enumeration of Grothendieck dessins d'enfants.

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Elliptic Hurwitz numbers

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- S is a compact Riemann surface;
- p_i is simply ramified with $\mu_{p_i} = (2, 1, \ldots, 1)$;
- all other points are unramified.

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Dijkgraaf '95: Confirms two predictions of mirror symmetry regarding the structure of elliptic Hurwitz numbers.

Quasimodularity

One of the predictions is the following:

Theorem (Dijkgraaf '95)

Let E_2, E_4, E_6 be the Ehrhart series given by

$$E_k(q) = -\frac{B_{2k}}{2k} + \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} m^{k-1} q^{mr}.$$

Then, for fixed $r \geq 2$, we have

$$\sum_{d=1}^{\infty} N_{d,r} q^d \in \mathbb{Q}[E_2, E_4, E_6],$$

i.e. it is a **quasimodular form**.

Quasimodularity

Eskin–Okounkov '99: This quasimodularity property allows to study the asymptotics of elliptic Hurwitz numbers as d approaches infinity.

Quasimodularity

Elliptic Hurwitz numbers equivalently enumerate factorisations $(\tau_1, \ldots, \tau_r, \alpha, \beta)$, with τ_i transpositions, α, β any permutations and

$$\tau_r \cdots \tau_1 = \alpha\beta\alpha^{-1}\beta^{-1}.$$

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From this interpretation, it follows that for fixed $r \geq 2$, the series

$$\sum_{d=1}^{\infty} \vec{N}_{d,r} q^d$$

is a quasimodular form as well.