Cascades of singular rational surfaces of Picard number one

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Outline

1. Motivation
2. toric case
3. Fano type

We only consider projective varieties defined over the field $\mathbb{C}$ of complex numbers.

Outline

Motivation: algebraic Montgomery-Yang problem

Conjecture (Algebraic Montgomery-Yang problem, Kollár 2008)

Let S be a normal projective surface with at worst quotient singularities such that $b_2(S) = 1$. If the smooth locus of S is simply-connected, then S has at most 3 singular points.

$\mathbb{Q}$-homology $\mathbb{P}^2$

Definition

A normal projective surface S with quotient singularities is called a $\mathbb{Q}$-homology projective plane ($\mathbb{Q}$-homology $\mathbb{P}^2$) if $b_2(S) = 1$.

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- It realizes the minimal possible Hodge diamond since $p_g = q = 0$.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Theorem (Prasad-Yeung(2007), Cartwright-Steger(2010))

Besides $\mathbb{P}^2$, there are exactly 100 smooth $\mathbb{Q}$-homology projective planes (fake projective planes) up to isomorphisms.

Trichotomy of K_S

Definition

Let S be a $\mathbb{Q}$-homology $\mathbb{P}^2$.

1. S is said to be of Fano type if $-K_S$ is ample.
2. S is said to be of Calabi-Yau type if K_S is numerically trivial.
3. S is said to be of general type if K_S is ample.

Let S be a $\mathbb{Q}$-homology $\mathbb{P}^2$ and S' be its minimal resolution.

- If S is of Fano type, then $\kappa(S') = -\infty$.
- If S is of Calabi-Yau type, then $\kappa(S') = -\infty, 0$.
- If S is of general type, then $\kappa(S') = -\infty, 0, 1, 2$.

Each case of $\kappa(S')$ can be realizable.

Known results

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Theorem (H & Keum 2011)

Let S be a $\mathbb{Q}$-homology $\mathbb{P}^2$. Then S has at most 5 singular points and S has exactly 5 singular points iff S is an Enriques surface with singularities of type $3A_1 + 2A_3$.

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Theorem (H & Keum 2011, 2013, 2014)

AMY holds true if either S has a non-cyclic singular points, S is non-rational or $-K_S$ is nef.

Remaining cases

AMY is open only for rational surfaces of Picard number one with cyclic singularities such that K_S is ample.

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- There exist infinite families of such surfaces with $|Sing(S)| \leq 3$. [Keel-McKernan(1999), Kollár(2008), H.-Keum(2012), Alexeev-Liu(2019)]

Remaining cases

AMY is open only for rational surfaces of Picard number one with cyclic singularities such that K_S is ample.

- There exist infinite families of such surfaces with $|Sing(S)| \leq 3$. [Keel-McKernan(1999), Kollár(2008), H.-Keum(2012), Alexeev-Liu(2019)]
- No such surface with 4 singular points is known, even without the simply-connectedness assumption on the smooth locus.

Definition

Definition (Cascades, general version)

Let S be a rational $\mathbb{Q}$-homology $\mathbb{P}^2$. We say S admits a cascade if there exists a diagram as follows:

$$\begin{array}{ccccccc}
S' = S'_t & \xrightarrow{\phi_t} & S'_{t-1} & \xrightarrow{\phi_{t-1}} & \dots & \xrightarrow{\phi_1} & S'_0 \\
\pi_t \downarrow & & \pi_{t-1} \downarrow & & & & \pi_0 \downarrow \\
S := S_t & & S_{t-1} & & \dots & & S_0
\end{array}$$

where for each k,

1. ϕ_k is a blowdown,
2. π_k is the contraction of all $(-n)$-curves with $(-n) \leq -2$,
3. S_k is a $\mathbb{Q}$-homology $\mathbb{P}^2$,
4. S_0 is a $\mathbb{Q}$-homology $\mathbb{P}^2$ of Fano type.

Cascade conjecture and AMY problem

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Every rational $\mathbb{Q}$-homology $\mathbb{P}^2$ of general type admits a cascade.

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Every rational $\mathbb{Q}$-homology $\mathbb{P}^2$ of general type admits a cascade.

Theorem (H)

Cascade conjecture implies the algebraic Montgomery-Yang problem.

- Detailed information obtained in the previous work
  + detailed analysis of $\mathbb{P}^1$-fibration.

Main question

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Unfortunately, I found an example of a rational $\mathbb{Q}$-homology $\mathbb{P}^2$ of Calabi-Yau type that does not admit a cascade.

Still, the existence is verified in the following cases:

- toric case
- Fano type (in preparation)

Outline

Toric case

Definition

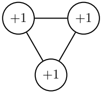
Let S be a toric log del Pezzo surface of Picard number one. We say that S admits a cascade if there exists a diagram as follows:

$$\begin{array}{ccccccc} S' = S'_t & \xrightarrow{\phi_t} & S'_{t-1} & \xrightarrow{\phi_{t-1}} & \dots & \xrightarrow{\phi_1} & S'_0 \\ \pi_t \big\downarrow & & \pi_{t-1} \big\downarrow & & & & \pi_0 \big\downarrow \\ S_t := S & & S_{t-1} & & \dots & & S_0 \end{array}$$

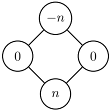
where for each k

1. ϕ_k is a toric blow-down,
2. π_k is the minimal resolution,
3. S_k is a toric log del Pezzo surface of Picard number one, and
4. S_0 is basic.

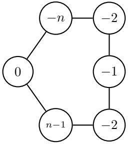
minimal surfaces and basic toric surfaces



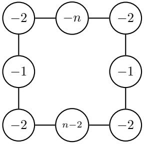
$G(\mathbb{P}^2)$



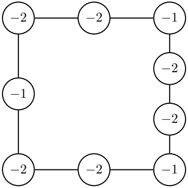
$G_n(0,0)$



$G_n(0,2)$



$G_n(2,2)$



$G_2(2,4)$

Results

Theorem (H.)

Let S be a toric $\mathbb{Q}$-homology $\mathbb{P}^2$. Then, S admits a cascade unless $S \cong \mathbb{P}(1,1,n)$ where each morphism in the cascade diagram is toric and there are 3 basic surfaces.

Corollary (H.)

Let S be a toric $\mathbb{Q}$-homology $\mathbb{P}^2$ of Fano type. Then, $|Sing(S)| \leq 3$ and

1. *If $|Sing(S)| = 0$, then $S \cong \mathbb{P}^2$.*
2. *If $|Sing(S)| = 1$, then $S \cong \mathbb{P}(1,1,n)$ where $n \geq 2$.*
3. *If $|Sing(S)| = 2$, then $S \cong \mathbb{P}(1,p,q)$ and it admits a cascade to $S_n(0,2)$.*
4. *If $|Sing(S)| = 3$, then S admits a cascade to either $S_n(2,2)$ or $S_2(2,4)$.*

Characterization for K-stability

Theorem (H., preprint)

Let S be a Kähler-Einstein toric log del Pezzo surface of Picard number one. Then

1. $K_S^2 = 3e_{orb}$.
2. *S is either isomorphic to $\mathbb{P}^2$ or S has exactly 3 singular points.*
3. *If S is not isomorphic to $\mathbb{P}^2$, it admits a cascade to $S_2(2,4)$, not to $S_n(2,2)$.*

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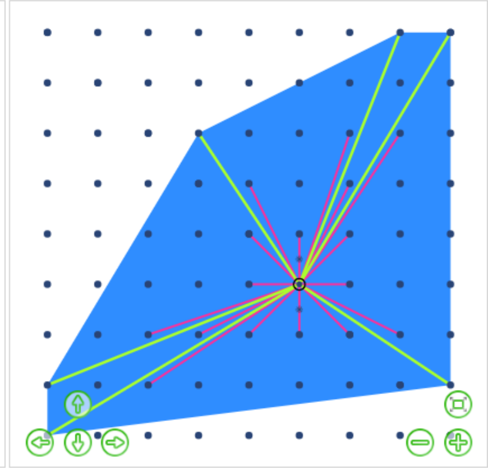
- In general, the Bogomolov–Miyaoka–Yau inequality

$$K_S^2 \leq 3e_{orb}$$

does not hold for (toric) log del Pezzo surfaces of Picard number one.

Generalization?

Unfortunately, this cannot be generalized to higher Picard rank case, even in the toric case.



Semicascades of toric del Pezzo surfaces

Definition

Let S be a toric log del Pezzo surface. We say that S admits a semicascade if there exists a diagram as follows:

$$\begin{array}{ccccccc} S' = S'_t & \xrightarrow{\phi_t} & S'_{t-1} & \xrightarrow{\phi_{t-1}} & \dots & \xrightarrow{\phi_1} & S'_0 \\ \pi_t \downarrow & & \pi_{t-1} \downarrow & & & & \pi_0 \downarrow \\ S_t := S & & S_{t-1} & & \dots & & S_0 \end{array}$$

where for each k

1. ϕ_k is a (toric) blow-down.
2. π_k is the minimal resolution,
3. Either $\rho(S_{k-1}) = \rho(S_k)$ or $\rho(S_{k-1}) = \rho(S_k) - 1$,
4. S_0 is either a surface of type (O) or $\mathbb{P}(1, 1, n)$.

Proposition

A toric graph of type (O) is one of the following graphs:

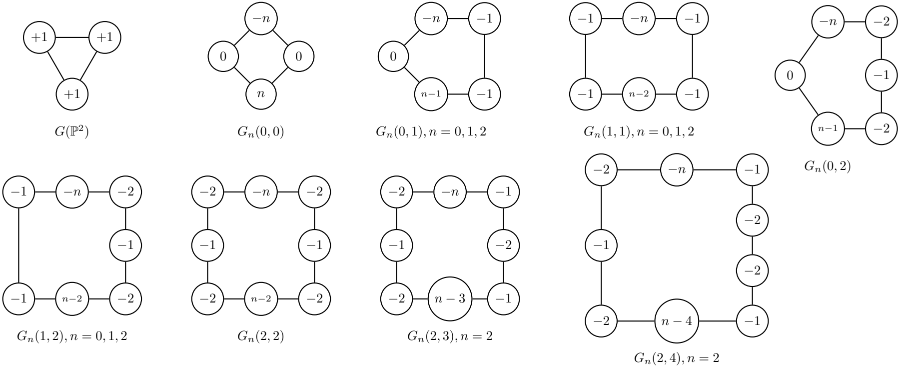


Figure: Toric graphs of type (O)

Results

Theorem

Every singular toric log del Pezzo surface admits a semicascade.

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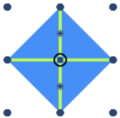
Every singular toric log del Pezzo surface admits a semicascade.

Theorem

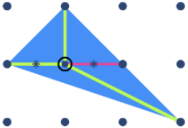
Let S be a singular toric log del Pezzo surface of Picard number ρ with t singular points. Then $\rho \leq t + 2$ and the equality holds if and only if S is the blow up of $\mathbb{P}(1, 1, n)$ at the two smooth torus-fixed points where $n \geq 2$.

- It generalizes the results of Dias and Suyama obtained for $t \leq 3$.

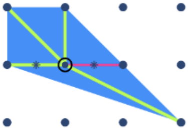
$P(\mathbb{P}^2)$



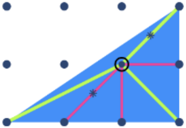
$P_0(0,0)$



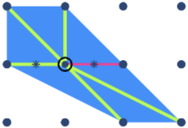
$P_2(0,0)$



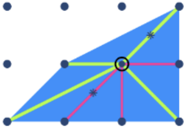
$P_2(0,1)$



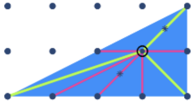
$P_2(0,2)$



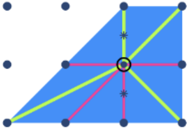
$P_2(1,1)$



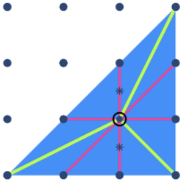
$P_2(1,2)$



$P_2(2,2)$



$P_2(2,3)$



$P_2(2,4)$

Application to K-stability

Recall that we always have $\rho \geq t - 2$.

Theorem

Let S be a singular Kähler–Einstein toric log del Pezzo surface of Picard number ρ with t singular points. Then we have $\rho = t - 2$. Moreover, S admits a semicascade to one of $S_1(2,2)$, $S_2(2,3)$, or $S_2(2,4)$.

Corollary

Let S be a Kähler–Einstein toric log del Pezzo surface. Then the maximal cones of the corresponding fan are either all smooth or all singular.

Outline

Cascades for Fano type

Definition

Let S be a rational $\mathbb{Q}$-homology $\mathbb{P}^2$ of Fano type. We say S admits a cascade if there exists a diagram as follows:

$$\begin{array}{ccccccc} S' = S'_t & \xrightarrow{\phi_t} & S'_{t-1} & \xrightarrow{\phi_{t-1}} & \dots & \xrightarrow{\phi_1} & S'_0 \\ \pi_t \downarrow & & \pi_{t-1} \downarrow & & & & \pi_0 \downarrow \\ S := S_t & & S_{t-1} & & \dots & & S_0 \end{array}$$

where for each k,

1. ϕ_k is a blowdown.
2. π_k is the contraction of all $(-n)$-curves with $(-n) \leq -2$.
3. S_k is a $\mathbb{Q}$-homology $\mathbb{P}^2$ of Fano type.
4. S_0 is basic, i.e. either Gorenstein or "KT"-type.

There are 12 types of basic $\mathbb{Q}$-homology $\mathbb{P}^2$'s of Fano type.

Smooth del Pezzo surfaces

Theorem (Pasquale del Pezzo(1885, 1887))

Every smooth del Pezzo surface is either a Hirzebruch surface or a blowup of $\mathbb{P}^2$ *at most* 8 *general points.*

Corollary

Let S *be a smooth del Pezzo surface. Then,* S *admits a morphism to* $\mathbb{P}^2$ *as follows:*

$$S := S_d \to S_{d+1} \to \ldots \to S_9 = \mathbb{P}^2$$

unless S *is a Hirzebruch surface.*

Classification of $\mathbb{Q}$-homology $\mathbb{P}^2$'s of Fano type

Theorem (H., in preparation)

Let S be a $\mathbb{Q}$-homology $\mathbb{P}^2$ of Fano type. Then, S admits a cascade unless $S \cong \mathbb{P}(1,1,n)$.

In principle, this gives a "classification" of such surfaces by inverting the cascade process.

Idea of main theorem

It is enough to show that following.

Theorem

Under the assumption in Cascade Conjecture, we further assume that the canonical divisor is ample and there exists a (-1)-curve E with $E.\mathcal{D} \leq 2$ where $\mathcal{D}$ is the reduced exceptional divisor of the minimal resolution of S. Then, S has at most three singular points.

Reference

1. D. Hwang, Algebraic Montgomery-Yang problem and cascade conjecture, arXiv:2012.13355.
2. D. Hwang, Cascades of toric log del Pezzo surfaces of Picard number one, arXiv:2012.13428.
3. D. Hwang, Semicascades of toric log del Pezzo surfaces, to appear in Bull. Korean Math. Soc. doi:10.4134/BKMS.b210211.

Thank you.