Type D Associahedra are Unobstructed

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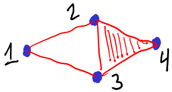
Stanley-Reisner Rings

Definition

A *simplicial complex* $\mathcal{K}$ with vertex set V is a collection of subsets of V, closed under taking subsets.



The *Stanley-Reisner ideal* of $\mathcal{K}$ is

$$I_{\mathcal{K}} = \left\langle \prod_{v \in W} x_v \mid W \notin \mathcal{K} \right\rangle \subseteq \mathbb{K}[x_v \mid v \in V].$$

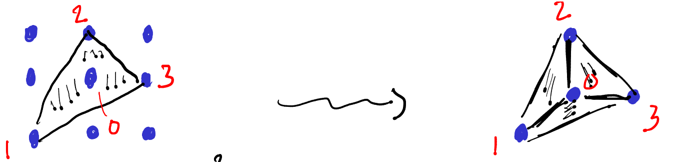
The *Stanley-Reisner ring* of $\mathcal{K}$ is $S_{\mathcal{K}} = \mathbb{K}[x_v \mid v \in V]/I_{\mathcal{K}}$

Ex: $I_K = \langle x_1 x_4, \ x_1 x_2 x_3 \rangle$

Stanley-Reisner Schemes

Geometry of $\mathbb{P}(\mathcal{K}) = \operatorname{Proj} S_{\mathcal{K}}$ is reflected in the geometry of $\mathcal{K}$:

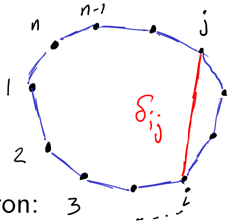
- Irreducible components $\leftrightarrow$ maximal faces;
- $\mathcal{K}$ a sphere $\implies$ $\mathbb{P}(\mathcal{K})$ Calabi-Yau;
- P lattice polytope with $\mathcal{K}$ as regular unimodular triangulation $\implies$ toric variety assoc. to P degenerates to $\mathbb{P}(\mathcal{K})$.



$x_1 x_2 x_3 - x_0^3 = 0$

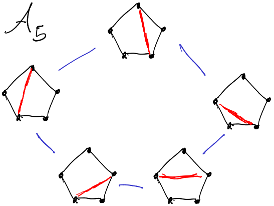
$x_1 x_2 x_3 = 0$

The Classical Associahedron



Boundary complex $\mathcal{A}_n$ of dual associahedron:

- Vertices are diagonals δ_{ij} of n-gon;
- Faces are sets of non-crossing diagonals.



The complex $\mathcal{A}_n$ is a sphere of dimension $n-4$.
The Grassmannian $G(2,n)$ degenerates to a cone over $\mathbb{P}(\mathcal{A}_n)$ (Sturmfels 1993).

Unobstructedness

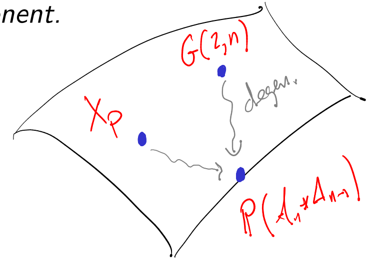
- S a $\mathbb{K}$-algebra $\rightsquigarrow$ T^2_S measures obstructions to deforming S.
- S graded and $(\dots)$: $(T^2_S)_0 = 0 \implies \operatorname{Proj} S$ is a smooth point of relevant Hilbert scheme.

Theorem (Christophersen, I- 2011)

The simplicial complex $\mathcal{A}_n$ is unobstructed, that is, $T^2_{S_{\mathcal{A}_n}} = 0$.

Corollary (Christophersen, I- 2011)

For P a lattice polytope with regular unimodular triangulation $\mathcal{A}_n * \Delta_{n-1}$, $G(2,n)$ and the toric variety corresponding to P lie on the same Hilbert scheme component.



Type D Associahedra

$\mathcal{D}_n$ is the *cluster complex* for type D_n cluster algebras, and the boundary complex of dual type D_n associahedra (Fomin and Zelevinsky 2001).

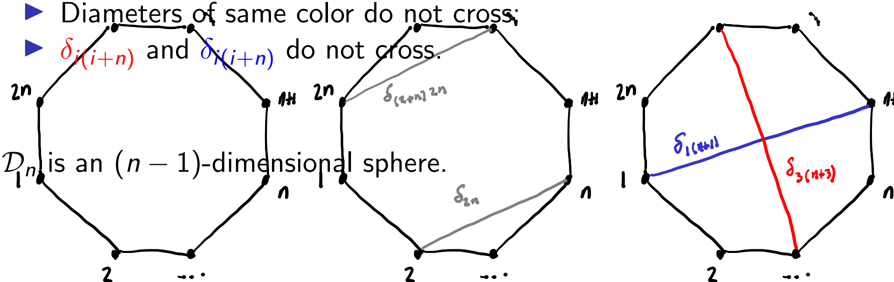
Vertices:

- Symmetric pairs $(\delta_{ij}, \delta_{(i+n)(j+n)})$ of non-diameter diagonals of $2n$-gon;
- Red or blue diameters $\delta_{i(i+n)}$ and $\delta_{i(i+n)}$.

Faces: sets of non-crossing diagonals.

- Diameters of same color do not cross;
- $\delta_{i(i+n)}$ and $\delta_{i(i+n)}$ do not cross.

$\mathcal{D}_n$ is an $(n-1)$-dimensional sphere.



Main Result

Theorem (I- 2020)

The simplicial complex $\mathcal{D}_n$ is unobstructed, that is, $T^2_{S_{\mathcal{D}_n}} = 0$.

Unobstructedness for Flag Complexes

In general, for $S = S_{\mathcal{K}}$:

- T^i_S is $\mathbb{Z}^{\#V}$-graded.
- $T^i_{S_{\mathcal{K}}}$ $(i = 1, 2)$ can be described via relative simplicial cohomology (Altmann and Christophersen, 2000)

$\mathcal{K}$ is a *flag complex* if minimal non-faces have at most two vertices.

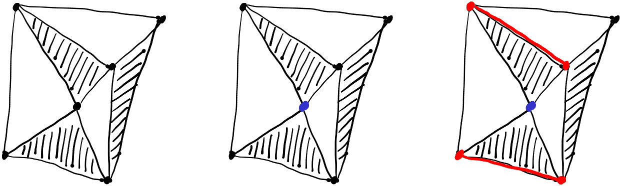
- $\mathcal{A}_n$ and $\mathcal{D}_n$ are flag complexes.

Lemma (Christophersen, I- 2011)

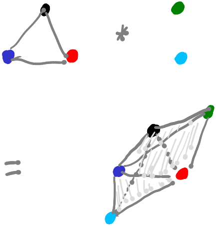
A flag complex $\mathcal{K}$ is unobstructed if

1. *$\mathcal{K}$ is a sphere;*
2. *$T^2_{S_{\mathcal{K}'}} = 0$ for all links $\mathcal{K}'$; and*
3. *L_b is contractible for all non-edge pairs of vertices $b \subset V$.*

Links and Joins

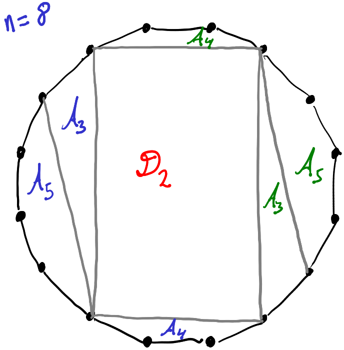


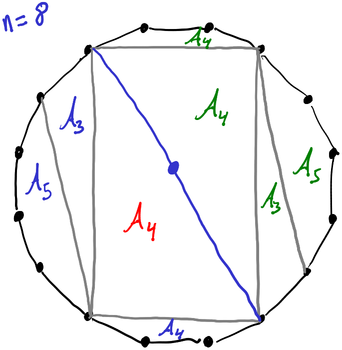
Joins: $\mathcal{K} * \mathcal{K}' = \{f \sqcup f' \mid f \in \mathcal{K},\ f' \in \mathcal{K}'\}$
Note $S_{\mathcal{K}*\mathcal{K}'} = S_{\mathcal{K}} \otimes S_{\mathcal{K}'}$!



Links are Unobstructed

$f \in \mathcal{D}_n$ induces subdivision of $2n$-gon:





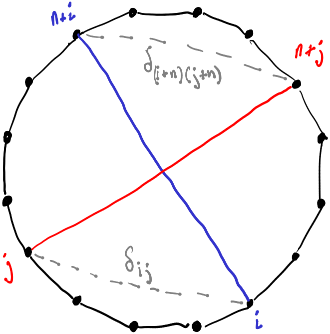
In general, $\operatorname{link}(f, \mathcal{D}_n) = \mathcal{D}_{n_0} * \mathcal{A}_{n_1} * \cdots * \mathcal{A}_{n_k}$.

Induction + Zariski-Jacobi sequence $\implies$ links are unobstructed!

L_b

For $b = \{v, w\}$, $L_b = \mathrm{link}(v, \mathcal{D}_n) \cap \mathrm{link}(w, \mathcal{D}_n)$.
Need L_b contractible for $b \notin \mathcal{D}_n$!

Case: $b = \{\delta_{i(n+i)}, \delta_{j(n+j)}\}$, $i \neq j$.



Anything crossing δ_{ij} crosses $\delta_{i(n+i)}$ or $\delta_{j(n+j)}$ $\implies$
$(\delta_{ij}, \delta_{(i+n)(j+n)})$ is in every face of L_b.

Applications

Smooth points of Hilbert schemes:

- $\mathbb{P}(\mathcal{D}_n)$ is a smooth point of its Hilbert scheme!

Toric Degenerations:

- $G(3,6)$ degenerates to a cone over $\mathbb{P}(\mathcal{D}_4)$ (Bossinger, Mohammadi, and Nájera Chávez, 2020).
- Other (skew) Schubert varieties degenerate to cones over $\mathbb{P}(\mathcal{D}_n)$ (Serhiyenko, Sherman-Bennett, and Williams, 2019, and ???)

Thanks for listening!