Symmetries of Fano varieties

University of Nottingham Online AG Seminar
(joint work with Louis Esser and Joaquín Moraga)

Work over $\mathbb{C}$

§1. Introduction

Question: How "big" can the aut. group of a Fano variety be?

infinite

dim $n=1$: $g=0$ $(K_C<0)$: $\mathbb{P}^1$ $\mathrm{Aut}(\mathbb{P}^1)=PGL_2$

$g=1$ $(K_C\equiv 0)$: $\mathrm{Aut}(C)=C(\mathbb{C})\rtimes \mathrm{Aut}(C,0)$

$g\geq 2$ $(K_C>0)$: $|\mathrm{Aut}(C)|\leq 84(g-1)$

Question': How "non-abelian" can Aut(Fano) be?

Defn: a group is Jordan $\overset{\text{def}}{\iff}$ $\exists$ constant $J=J(G)$ such that any finite subgroup of G has a normal abelian subgroup of index $\leq J$

a family $\mathcal{G}$ of groups is uniformly Jordan $\overset{\text{def}}{\iff}$ every $G\in\mathcal{G}$ is Jordan, and the same constant J works for all $G\in\mathcal{G}$

Thm (Jordan 1878) $GL_n(\mathbb{C})$ is Jordan

Thm (Collins 2007) For $n\geq 71$, the Jordan constant $J(GL_n(\mathbb{C}))=(n+1)!$
from the standard rep. of S_{n+1}

$S_{n+1} \circlearrowright \mathbb{C}^{n+1}$ by permuting the $n+1$ coordinates
Have a 1-dim'l invariant subspace span$\{e_1+\cdots+e_{n+1}\}$
The complement is the standard repn $S_{n+1}\hookrightarrow GL_n(\mathbb{C})$

Conditionally on boundedness of terminal Fanos

proved BAB

Thm (Prokhorov–Shramov–Birkar) (Fano means klt)

For $n\geq 1$, the family of groups

$$\{\mathrm{Bir}\,X \mid X \text{ n-dimensional rationally connected variety}\} \text{ is uniformly Jordan}$$
$$\cup\!\!\mid$$
$$\{\mathrm{Aut}\,X \mid X \text{ n-dimensional (klt) Fano variety}\}$$

Consequence: In any fixed dim n, get a (non-explicit) bound on the size of a semi-simple subgroup of $\mathrm{Aut}\,X$ for any n-dim'l Fano X
has no nontrivial abelian subgroups

symmetric group on m elements

In particular, $\exists$ $M(n)=$ maximal m such that $S_m\hookrightarrow \mathrm{Aut}$(n-dim'l Fano variety) symmetric group on m elements

In particular, $\exists$ $M(n)$ = maximal m such that $S_m \hookrightarrow \mathrm{Aut}(n\text{-diml Fano variety})$

<u>Examples:</u> 1) $S_{n+1} \circlearrowright \mathbb{P}^n$ by permuting coordinates $[x_0 : \cdots : x_n]$

$S_{n+2} \circlearrowright \mathbb{P}^n$ by standard repn

2) Among rational varieties, can do 1 better

$$X := \left(\sum_{i=0}^{n+2} x_i = \sum_{i=0}^{n+2} x_i^2 = 0 \right) \subseteq \mathbb{P}^{n+2}$$

S_{n+3}-action on $\mathbb{P}^{n+2}$ descends to X

$X \cong$ smooth quadric $\Rightarrow$ X is rational

For $n = 1, 2$, S_{n+3} is the largest action:

n	$M(n)$	optimal examples
1	4	$\mathbb{P}^1$
2	5	$\mathbb{P}^1 \times \mathbb{P}^1$, Clebsch cubic, dP5 (Dolgachev–Iskovskikh 2009) $\Sigma x_i = \Sigma x_i^2 = 0$, $\Sigma x_i = \Sigma x_i^3 = 0$, $X \cong \overline{\mathcal{M}}_{0,5} \cong (\mathbb{P}^1)^5 /\!/ SL_2$
3	$7 \neq 3+3$	only only exmp up to conj. (Prokhorov 2022) $(\Sigma x_i = \Sigma x_i^2 = \Sigma x_i^3 = 0) \subseteq \mathbb{P}^6$ $\leftarrow$ is irrational (Beauville 2012)
≥ 4	$\geq 8 \neq 4+3$	???? no classification

<u>Defn:</u> a Fano variety X is maximally symmetric if it admits a faithful $S_{M(n)}$-action

<u>Question 1:</u> In dim $n \geq 3$, are the n-dimensional maximally symmetric Fano varieties <u>bounded</u>?

<u>Question 2:</u> In dim $n \geq 3$, are the n-dimensional maximally symmetric Fano varieties <u>irrational</u>?

<u>Rem:</u> Q1: This behavior is very different from <u>abelian</u> actions

<u>Exmp:</u> $M(4) \geq 8$:

$$\left.\begin{aligned} X_{123} &= (\Sigma x_i = \Sigma x_i^2 = \Sigma x_i^3 = 0) \subseteq \mathbb{P}^7 \\ X_{124} &= (\Sigma x_i = \Sigma x_i^2 = \Sigma x_i^4 = 0) \subseteq \mathbb{P}^7 \end{aligned}\right\} \text{smooth Fano 4-folds with faithful } S_8\text{-actions}$$

<u>**Theorem 1**</u> (Esser–J.–Moraga) The maximally symmetric Fano 4-folds form a bounded family.

(Show S_8-equivariant Fano 4-folds are bounded)

<u>Rem:</u> S_7-equivariant Fano 4-folds are <u>unbounded</u>

<u>Reason:</u> $S_7 \circlearrowright$ Fano 3-fold Y

can make some family (proj bundles $/Y$) where mlds form an unbounded sequence

§2. Bounds on symmetric actions

Theorem 2 (EJM) For $n \geq 1$, let $p_n :=$ smallest prime number $> n+1$.

Then $M(n) < p_{n+1}(n+1)$.

$\rightsquigarrow M(n) < (1+\varepsilon)(n+1)^2$

Use results of J. Xu on p-groups acting on RC varieties

$\left[M(n) = \text{maximal } m \text{ such that } S_m \hookrightarrow \mathrm{Aut}(n\text{-diml Fano variety}) \right]$

For certain classes of Fano varieties, get sharp bounds

Exmp: Let $X = \left(\sum x_i = \sum x_i^2 = \cdots = \sum x_i^m = 0 \right) \subseteq \mathbb{P}^{n+m}$

Choose largest m such that X is Fano (i.e. want $-n-m-1+(1+2+\cdots m) < 0$)

X is a smooth n-diml Fano variety with a faithful S_{n+m+1}-action

Get $n+m+1 = n + \left\lceil \frac{1+\sqrt{8n+9}}{2} \right\rceil =: M_{WCI}(n)$

Theorem 3 (EJM) Let $X \subseteq \mathbb{P}(a_0, \ldots, a_N)$ be a quasismooth weighted complete intersection, with a faithful S_k-action. Then $k \leq M_{WCI}(n)$, and this bound is sharp.

Moreover: 1) If $S_{M_{WCI}(n)} \curvearrowright X$, then there is a finite cover $X \to Y \subseteq \mathbb{P}'$

defining ideal of Y is gen. by symmetric polynomials

2) If $S_{M_{WCI}(n)} \curvearrowright X$ and if X has maximal Fano index, then X is equivariantly isom to a complete intersection in proj. space defined by Fermat polynomials

n	1	2	3	4	5	6	7	8
$M_{WCI}(n)$	4	5	7	8	9	11	12	13
$n+3$ $(S_{n+3} \hookrightarrow Cr_n(\mathbb{C}))$	4	5	6	7	8	9	10	11

for $n \geq 3$, $M_{WCI}(n) > n+3$

(Recall: max symm Fano 3-fold is irrational)

Some ingredients of pf of Thm 3:

Lift S_k action to $\mathbb{P}(a_0, \ldots, a_N) =: \mathbb{P}$

$1 \to \mathbb{C}^* \to \mathrm{Aut}\, R \to \mathrm{Aut}\, \mathbb{P} \to 1$

$R = \mathbb{C}[x_0, \ldots, x_N]$, $\mathrm{wt}(x_i) = a_i$

$\mathrm{Aut}\, R \supseteq \prod_\ell GL_{N_\ell}(\mathbb{C}) \rightsquigarrow$ get $\tilde{S}_k \hookrightarrow GL_{N_\ell}(\mathbb{C})$ (Schur cover)

Use proj rep theory of S_k to bound k

Fano assumption on X is used in numerics

Theorem 4 (EJM) Let $S_k \curvearrowright$ n-diml simplicial toric variety

n	maximal k	optimal examples
1	4	$\mathbb{P}^1$
2	5	$\mathbb{P}^1 \times \mathbb{P}^1$
3	6	$\mathbb{P}^3$
4	6	$\mathbb{P}^4$, $\mathbb{P}^2 \times \mathbb{P}^2$
≥ 5	$n+2$	$\mathbb{P}^n$

* smaller than $S_{n+3} \curvearrowright$ quadric

* smaller then $M_{WCI}(n)$

Idea: Use structure of Aut (toric variety) (Cox), use (proj) repn theory of S_k and A_k

Question 3: For $n \geq 1$, is $M_{WCI}(n) = M(n)$? $\leftarrow$ optimal among Fanos

$\uparrow$ optimal among quasism WCI Fanos

Recall $S_{n+3} \hookrightarrow Cr_n(\mathbb{C})$ by $\curvearrowright$ Fermat CI quadric

Question 2': Is S_{n+3} the largest symmetric subgroup of $Cr_n(\mathbb{C})$?

§3. Pf of Theorem 1 (boundedness of S_8-Fano-4-folds)

Idea: $S_8 \curvearrowright X$ = Fano 4-fold, $\pi: X \to Y = X/S_8$ quotient

$$\pi^*(K_Y + \underset{\text{define}}{B_Y}) = K_X$$

(Y, B_Y) log Fano pair, get 3 cases depending on $\operatorname{coreg}(Y, B_Y)$

Defn: (Moraga). the coregularity of a log CY pair (Y,Γ) is

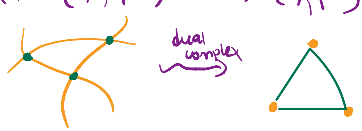
$$\operatorname{coreg}(Y,\Gamma) := \dim Y - \dim \mathcal{D}(Y,\Gamma) - 1$$

• the coregularity of a log Fano pair (Y,B) is

$$\operatorname{coreg}(Y,B) := \min\{\operatorname{coreg}(Y,\Gamma) \mid \Gamma \geq B \text{ and } (Y,\Gamma) \text{ is log CY}\}$$

$\mathcal{D}(Y,\Gamma)$ = dual complex: $(Y',\Gamma') \xrightarrow{\text{dlt modif}} (Y,\Gamma)$

E_i



CW complex: vertices $\leftrightarrow E_i$

fill in k-cells based on intersections of E_i's

Case 1: coreg = 4 — directly show boundedness (using Birkar's BAB)

Case 2: coreg = 3

Case 2: coreg ≤ 2 — $(*)$

<u>Case 3:</u> coreg ≤ 2.

$\rightsquigarrow$ get pair with dual complex $\underset{\text{PL-homeo}}{\sim}$ to $\mathbb{S}^k$ or $\mathbb{D}^k$ with $k \leq 3$ (Kollár-Xu)

use results of Pardon and classification of actions on spheres

to get a contradiction

"S_8" acts on this

$\Rightarrow$ this case doesn't happen

(need many results in topology).

Ingredients of pf of Thm 1:

- S_8 doesn't act on Fanos of dim $\leq n-1=3$
- S_8 doesn't act on spheres of dim ≤ 3
- dual complex of log CY pair of dim ≤ 4 is a quotient of sphere or disk of dim ≤ 3
- boundedness of Fano 4-folds with log discrep bounded away from 0

in dim $n \geq 5$

? $M(n-1)$ is unknown for $n \geq 5$

?

? (but expected to be true)

✓ (Birkar)

<u>Question 4:</u> Is $M(n)$ strictly increasing?