Nikulin orbifolds

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25th March 2021
Nottigham-online

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IHS manifolds

Definition

An irreducible holomorphic symplectic (IHS) manifold is a simply connected complex compact Kähler manifold X such that $H^0(X, \Omega_X^2) = \mathbb{C}\omega$, where ω is a holomorphic 2-form on X which is nowhere degenerate.

- These manifolds have even dimension and trivial canonical class.
- They play an important role, as building blocks, in the classification of Kähler manifolds with trivial first Chern class.
- IHS manifolds of dimension 2 are K3 surfaces.

Known types

In higher dimensions IHS manifolds are very hard to construct. The known deformation types are:

1. deformations of Hilbert schemes of n-points on K3 surfaces. $K3^{[n]}$-type
2. deformations of generalised Kummer manifolds
3. O'Grady type manifolds in dimensions 10 and 6

Note that a general element in these deformation families is not projective.

Known projective models

In fact, there is only a few known complete families of projective IHS manifolds.

- Beauville, Donagi: Hilbert scheme of lines on cubics $X_3 \subset \mathbb{P}^5$
- Iliev, Ranestad: varieties of sum of powers $VSP(X_3, 10)$
- O'Grady: double EPW sextics associated to Lagrangian spaces in $\wedge^3 V_6$
- Debarre Voisin: zero loci of sections of the vector bundle $\wedge^3 \mathcal{U}^*$ on $G(6, 10)$
- Lehn, Lehn, Sorger, van Straten: from Hilbert scheme of twisted cubics on X_3
- Iliev, Ranestad, G.Kapustka, MK: double EPW cubes associated to Lagrangian spaces in $\wedge^3 V_6$

All of them are $K3^{[n]}$-type manifolds and this type of IHS manifolds will also be the main focus of this talk.

Lattice structure

The theory of IHS manifolds of higher dimension has some similarities to that of K3 surfaces thanks to the existence of an intersection form called the Beauville-Bogomolov form on $H^2(X, \mathbb{Z})$ and associated to the symplectic form. For $K3^{[2]}$- type manifolds the resulting intersection lattice is

$$L := L_{K3} \oplus \langle -2 \rangle = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle$$

There is also a related Torelli theorem (proved by Verbitsky) which permits to study IHS manifolds in terms of the related integral Hodge structure which is a structure on the lattice.

Singular symplectic manifolds

All known types of IHS manifolds arise as moduli spaces of sheaves on K3 or abelian surfaces. These by results of Mukai admit symplectic forms on the smooth locus but are singular in general. It makes it natural to study singular symplectic manifolds.

There are various definitions of symplectic varieties of different generality. The most general is the following.

Definition

A variety X is called to have symplectic singularities if it admits a non-degenerate symplectic form on the smooth locus which extends to some resolution of singularities.

Usually we consider manifolds satisfying some stronger conditions.

IHS orbifolds

Another natural way to obtain a singular symplectic manifold is to take a quotient of an IHS manifold by a group action preserving the symplectic form. This leads us to the study of IHS orbifolds.

Definition

An orbifold X is said to be an IHS orbifold if $X \setminus \operatorname{Sing} X$ is simply connected and admits a unique, up to a scalar multiple, nondegenerate holomorphic 2-form.

For any varieties X with symplectic singularities (fixing a symplectic form) and in particular for IHS orbifolds we also have a Beauville-Bogomolov type intersection form on $H^2(X, \mathbb{Z})$. Furthermore, by results of Menet the global Torelli theorem works for IHS orbifolds at almost the same extent.

Automorphisms of IHS manifolds

The study of IHS orbifolds is related to the study of automorphisms on IHS manifolds and has been the subject of recent work of many authors. Automorphisms on IHS manifolds are divided into two classes:

- those that preserve the symplectic form - symplectic automorphisms
- those that do not preserve the symplectic form - non-symplectic automorphisms

Non-symplectic automorphisms of IHS manifolds of $K3^{[2]}$ type have been more or less completely classified (Mongardi-Wandel, Boissiére-Camere-Sarti, Camere-Mongardi-G.Kapustka- MK).

Nikulin involutions and Nikulin surfaces

The study of symplectic automorphisms of K3 surfaces is a classical subject started by Nikulin. A special case are symplectic involutions (automorphisms of order 2) nowadays called Nikulin involutions.

- A Nikulin involution admits 8 fixed points
- The resolution of the quotient of a K3 surface by a symplectic involution is another K3 surface called a Nikulin surface.
- Van Geemen and Sarti classified very general K3 surfaces admitting a symplectic involution.
- Associated Nikulin surfaces have been furthermore characterised by Garbagnati and Sarti.

Existence of symplectic involutions on $K3^{[2]}$ type manifolds

Similarly to the case of K3 surfaces, we study symplectic involutions on IHS manifolds of $K3^{[2]}$ type in terms of lattices.

Proposition (Mongardi)

An IHS fourfold of $K3^{[2]}$ type X admits a symplectic involution if and only if $E_8(-2)$ is primitively embedded in $\mathrm{NS}(X)$.

To get a projective IHS manifold we need an additional ample class.

Proposition

If X is a projective IHS fourfold of $K3^{[2]}$ type with symplectic involution then there exists $d \in \mathbb{N}$ such that $\Lambda_{2d} := < 2d > \oplus E_8(-2) \subset \mathrm{NS}(X)$.

The Neron-Severi lattice

We then consider the Neron-Severi lattice of a very general element X in a component of the family of $\langle 2d \rangle$-polarized $K3^{[2]}$-type fourfolds admitting a symplectic involution. The Neron-Severi lattice of such a very general X is an overlattice of Λ_{2d} containing both $< 2d >$ and $E_8(-2)$ as primitive sublattices. It follows by a result of van Geemen-Sarti that we have the following possibilities:

1. if d odd then $\mathrm{NS}(X) = \Lambda_{2d}$;
2. if d even then either $\mathrm{NS}(X) = \Lambda_{2d}$ or $\mathrm{NS}(X) = \widetilde{\Lambda}_{2d}$.

Here $\widetilde{\Lambda}_{2d}$ is a unique nontrivial overlattice of Λ_{2d} described explicitly by choosing an additional generator: $\frac{h+b_1}{2}$ if $d \equiv 2 \mod 4$ or $\frac{h+b_1+b_3}{2}$ if $d \equiv 0 \mod 4$.

The transcendental lattice

There still might be different families of IHS manifolds with involution having the same Neron-Severi lattice. These are distinguished by the embedding of $\mathrm{NS}(X) \subset L$ or by the transcendental lattice $T_X = NS(X)^{\perp}$.

Condition on d	NS(X)	T_X
$\forall d \in \mathbb{N}$	Λ_{2d}	$T_{2d,1} := U^{\oplus 2} \oplus E_8(-2) \oplus \langle -2d \rangle \oplus \langle -2 \rangle$
$d \equiv 1 \mod 2$	Λ_{2d}	$T_{2d,2} := U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d \rangle \oplus \langle -2 \rangle^5$
$d \equiv 3 \mod 4$	Λ_{2d}	$T_{2d,3} := U^{\oplus 2} \oplus E_8(-2) \oplus K_d$
$d \equiv 0 \mod 2$	$\widetilde{\Lambda}_{2d}$	$\widetilde{T}_{2d} \simeq U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d \rangle \oplus \langle -2 \rangle^5$

Here K_d is the rank 2 lattice defined by $\begin{bmatrix} -\frac{d+1}{2} & 1 \\ 1 & -2 \end{bmatrix}$.

Description as moduli spaces

From the lattice we can recover X as a moduli space of twisted sheaves on a K3 surface.

d	$K3$	model	v
$d \equiv 1 \mod 2$	S_d	$M_v(S_d, \beta)$	$(0, H', 2)$
$d \equiv 0 \mod 2$	S_d	$M_v(S_d, \beta)$	$(4, \sum_{i=1}^7 n_i, 2)$
$d \equiv 1 \mod 2$	S_d	$S_d^{[2]}$	$-$
$d \equiv 3 \mod 4$	Z_d	$M_v(Z_d, \beta)$	$(0, H', 2)$
$d \equiv 0 \mod 2$	S_d	$M_v(S_d)$	$(2, \sum_{i=1}^7 n_i, 4)$

Here S_d, Z_d are very general K3 surface with Neron-Severi lattice $\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7}$ or its extension $\left(\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7}\right)'$ by the vector $\frac{t+\sum n_i}{2}$ and β is an explicitly described in each case Brauer class.

A special case

Note that S_d and Z_d admit no symplectic involution. It is a nontrivial task to construct the involution on the related moduli space. The question is particularly interesting for $S_d^{[2]}$.
We can describe it for $d = 1$.
In this case S_1 has two related descriptions:

- a double cover of a del Pezzo surface of degree 2
- a symmetric quartic surface

We hence have two non-symplectic involutions on $S_1^{[2]}$

- The involution induced by the covering involution
- The Beauville involution on the quartic

Their composition is the symplectic involution we are looking for.

Nikulin orbifolds

By the works of Camere and Mongardi a symplectic involution on a $K3^{[2]}$- type IHS fourfold admits 28 fixed points and a fixed K3 surface.
The quotient of the IHS manifold by the action of a symplectic involution admits no crepant resolution but admits a partial crepant resolution which is an IHS orbifold that we call a **Nikulin orbifold**.

Theorem (Menet)

The Beauville Bogomolov intersection form for a Nikulin orbifold Y endows $H^2(Y,\mathbb{Z})$ with a lattice structure isometric to $U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2 \rangle$.

Using the results on projective IHS manifolds with involution we characterise very general projective Nikulin orbifolds.

d	NS(Y)	T_Y
$\forall d$	$\langle 4d\rangle \oplus \langle -4\rangle$	$U(2)^{\oplus 2} \oplus E_8(-1) \oplus \langle -4d\rangle \oplus \langle -4\rangle$
$d \equiv 1 \mod 2$	$\begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}$	$U(2)^{\oplus 2} \oplus E_7(-1) \oplus K_d(2) \oplus \langle -2\rangle$
$d \equiv 3 \mod 4$	$\begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}$	$U(2)^{\oplus 2} \oplus K_d(2) \oplus E_8(-1)$
$d \equiv 0 \mod 2$	$\langle d\rangle \oplus \langle -4\rangle$	$U^{\oplus 2} \oplus \langle -d\rangle \oplus N \oplus \langle -4\rangle$

where N is the Nikulin lattice of rank 8 extending $\langle -2\rangle^{\oplus 8}$ by $\frac{\sum n_i}{2}$.

Fixed locus K3

It is an interesting question to compare the Hodge structure on the transcendental lattice T_F of the K3 surface F contained in the fixed locus of a symplectic involution on a $K3^{[2]}$- type manifold with that on the transcendental lattice T_Y of the corresponding Nikulin orbifold Y.
Note that the K3 surfaces F can be seen as generalisations of Nikulin surfaces.

Conjecture

The integral Hodge structures on T_F and T_Y are isometric.

Proposition

The conjecture is true with rational coefficients and also for all families that we are able to construct.

Orbifolds of Nikulin type

Corollary

Let X be a very general fourfold of $K3^{[2]}$-type with a symplectic involution such that $NS(X) \simeq E_8(-2)$; then the corresponding Nikulin orbifold Y has $NS(Y) \simeq \langle -4 \rangle$.

In particular, Nikulin orbifolds are not general in their deformation family. This is is also valid for projective Nikulin orbifolds.

Definition

Deforming a Nikulin orbifold we obtain orbifolds which aren't Nikulin orbifolds anymore. We call these orbifolds of Nikulin type.

Riemann-Roch formula

We would like to construct projective models for some complete families of orbifolds of Nikulin type. We need some Riemann-Roch type formulas.

Proposition

Let Y be a four-dimensional orbifold of Nikulin type. If $D \in \mathrm{Pic}(Y)$ then $\chi(Y, D) = \frac{1}{4}(q(D)^2 + 6q(D) + 12)$.

We also prove a RR formulas for Weil divisors on Nikulin orbifolds. It depends on the number of points on which the divisor fails to be Cartier and the restriction to the exceptional divisor over the resolved K3 surface.

Projective models of Nikulin orbifolds

Special Nikulin orbifolds in each degree can be constructed by considering Nikulin surfaces and their Hilbert schemes or moduli spaces of bundles on them and taking the quotient.
General Nikulin orbifolds in some cases can be constructed by taking symmetric starting data in the known constructions of families of IHS manifolds:

1. Symmetric cubics and their Fano varieties of lines – $T_{6,3}$
2. Symmetric Lagrangians and associated double EPW sextics– $T_{2,1}$
3. Symmetric Verra fourfolds and associated scheme of (1,1)-conics – $\widetilde{T}_4$

We also constructed the family for the case $T_{2,2}$ using its Hilbert scheme description.

Projective models of orbifolds of Nikulin type

It is much more difficult to describe a complete family of orbifolds of Nikulin type. In fact, no complete family of IHS orbifolds has been constructed up to now.

Let Y be an orbifold of Nikulin type of dimension four such that there exists an ample Cartier divisor H on Y with degree $q(H) = 2$ and divisibility 1.

Theorem

The map $\varphi_{|H|} : Y \to \mathbb{P}^6$ is $2:1$ map and its image is a special fourfold of codimension 2 in $\mathbb{P}^6$ being the intersection of a cubic and a quartic. The map is branched along a surface of degree 48.

Idea of proof

We construct elements corresponding to some boundary component in this family giving Nikulin orbifolds. For that, one can consider IHS manifolds with symplectic involution defined a double cover of a so-called EPW quartic section in a cone over $\mathbb{P}^2 \times \mathbb{P}^2$. The involution is a lift of the involution defined by exchanging the two $\mathbb{P}^2$ factors.

Proposition

In this case $\varphi_{|H|} : Y \to \mathbb{P}^6$ is a 2:1 map to a complete intersection $Z_3 \cap T_4 \subset \mathbb{P}^6$ of two hypersurfaces Z_3 and T_4 of degrees 3 and 4 respectively and Z_3 is a cone over a symmetric determinantal cubic.

Using RR and this special case by deformation arguments we are able to prove that $\varphi_{|H|} : Y \to \mathbb{P}^6$ is 2:1 in general. Then the image must be of degree 12 and finally, by investigating its surface sections, also a complete intersection (3,4).

It is still not clear for us which $(3,4)$ complete intersections in $\mathbb{P}^6$ should correspond to orbifolds of Nikulin type. An attempt to solve this problem is related to the Beilinson spectral sequence and leads to the choice of a family of 28 dimensional linear subspaces in

$$3\bigwedge^5 V_7 \oplus \bigwedge^3 V_7 \oplus 3V_7,$$

isotropic with respect to the map

$$b : (3V_7 \oplus \bigwedge^3 V_7 \oplus 3\bigwedge^5 V_7)^2 \to \bigwedge^6 V_7$$

given by the formula

$$b((l_1, l_2, l_3, \alpha, w_1, w_2, w_3), (L_1, L_2, L_3, \beta, W_1, W_2, W_3)) =$$

$$= L_1 \wedge w_1 + L_2 \wedge w_2 + L_3 \wedge w_3 + \alpha \wedge \beta + l_1 \wedge W_1 + l_2 \wedge W_2 + l_3 \wedge W_3.$$

Thank you!