Unconditional Reflexive Polytopes

joint with McCabe Olsen & Raman Sanyal

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[KOS20]

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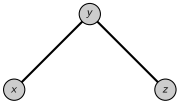
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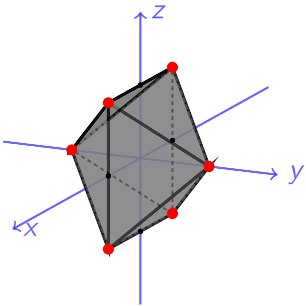
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Outline

- One of the most famous polytopes is the *Birkhoff polytope*, i.e.,
$\mathcal{B}(n) = \{M \in \mathbb{R}^{n\times n} \colon m_{ij} \geq 0, \text{row sums} = \text{column sums} = 1\}$.

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- Its dimension is $(n-1)^2 = n^2 - 2n + 1$.
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- ... but it is very complicated, and we do not even know its volume (for $n \geq 11$).

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- Moreover, it is also *reflexive* and has many more desirable properties.
- This led us to study unconditional reflexive polytopes in general.

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- Let $P \subset \mathbb{R}^d$ be a d-dimensional lattice polytope, i.e.,

$$P = \operatorname{conv}\left\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n \colon \ \boldsymbol{v}_i \in \mathbb{Z}^d\right\} := \left\{\sum_{i=1}^{d} \lambda_i \boldsymbol{v}_i \colon \lambda_i \geq 0, \sum_i \lambda_i = 1\right\}.$$

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- Equivalently, lattice polytopes can be defined by linear inequalities, i.e., $P = \{\boldsymbol{x} \colon A\boldsymbol{x} \leq \boldsymbol{b}\}$, where $A \in \mathbb{Z}^{m \times d}$, $\boldsymbol{b} \in \mathbb{Z}^m$.

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- For the rest of this talk, we assume that all polytopes are lattice.

Example

Let $\square = [-1,1]^2$. Then

$$\square = \mathrm{conv}\{(-1,-1),(-1,1),(1,-1),(1,1)\} = \{\boldsymbol{x} \colon \langle \pm \boldsymbol{e}_i, \boldsymbol{x}\rangle \leq 1\}.$$

Similarly, the 2-dimensional cross polytope $\diamondsuit$ can be written as

$$\diamondsuit = \mathrm{conv}\{\pm \boldsymbol{e}_i\} = \{\boldsymbol{x} \colon \langle \boldsymbol{v}, \boldsymbol{x}\rangle \leq 1,\ \boldsymbol{v} \in \{(\pm 1, \pm 1)\}\}.$$

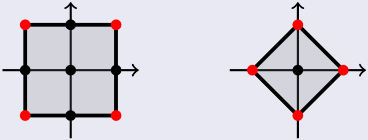

Figure: $\square$ and $\diamondsuit$ with their vertices marked red.

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$$\mathrm{ehr}_P(k) := \#(kP \cap \mathbb{Z}^d),$$

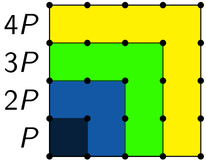
- or equivalently the *Ehrhart series of* P

$$\mathrm{Ehr}_P(z) := 1 + \sum_{k \geq 1} \mathrm{ehr}_P(k) z^k.$$

Example

Let $P = [0,1]^2$. Then

$$\mathrm{ehr}_P(k) = (k+1)^2.$$



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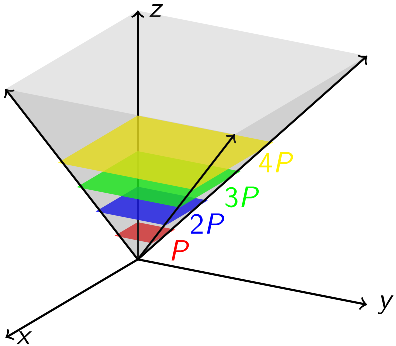
Let $P = [0,1]^2$. Then

$$\mathrm{Ehr}_P(k) = \frac{1+z}{(1-z)^3}.$$



Theorem [BR15, Thm. 3.12]

Let $P \subset \mathbb{R}^d$ be a d-dimensional lattice polytope. Then

$$\mathrm{Ehr}_P(z) = \frac{h_0^* + h_1^* z + \cdots + h_d^* z^d}{(1-z)^{d+1}},$$

and the coefficients h_i^* are non-negative integers.

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- The Ehrhart series of P is actually the Hilbert series of a graded algebra $\Bbbk[P]$.
- The *holy grail* of Ehrhart theory is to characterize the coefficients h_i^*.
- In particular, determining when h^* is *unimodal*, i.e., $h_0^* \leq h_1^* \leq \cdots \leq h_r^* \geq h_{r+1}^* \geq \cdots \geq h_s^*$, is of interest.

Definition

Let P be a d-lattice polytope with $0 \in P^\circ$. Then P is *reflexive* if $P = \{\boldsymbol{x} \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq 1\}$, where $\boldsymbol{a}_i \in \mathbb{Z}^d$.

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- P is *Gorenstein of degree c* if cP is reflexive, which is equivalent to h_P^* being palindromic, i.e., $h_i^* = h_{s-i}^*$ where $s = \deg h_P^*$.

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Let $P = \{\boldsymbol{x} \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq \boldsymbol{b}_i\}$ with $\gcd(\boldsymbol{a}_i) = 1$. Then P is *compressed* if for all i

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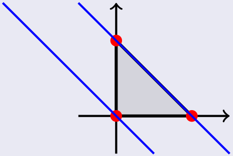
The standard triangle $\Delta = \{\boldsymbol{x} \colon \langle \boldsymbol{v}, \boldsymbol{x} \rangle$ where $\boldsymbol{v} \in \{(1,1), -\boldsymbol{e}_i\}\}$ is compressed.



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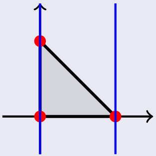

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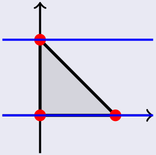

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- Compressed polytopes have (regular) unimodular triangulations, i.e., triangulations into simplices $\operatorname{conv}\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ so that $\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_d - \boldsymbol{v}_0$ is a lattice basis of $\mathbb{Z}^d$.

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- In fact, being compressed is equivalent to all pulling triangulations being unimodular.

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A polytope P is *unconditional* if $\boldsymbol{p} \in P$ implies $(\pm p_1, \ldots, \pm p_d) \in P$.

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- P is convex, so we have

$$P_{\geq 0} \; = \; \{\boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq 1 \text{ for } i = 1, \ldots, m\} \tag{1}$$

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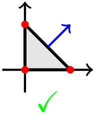

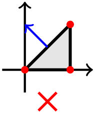

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  for some $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^d_{\geq 0}$.
- A polytope satisfying (1) is called an *anti-blocking polytope*.

- Given an anti-blocking polytope $Q \subset \mathbb{R}^d_{\geq 0}$, the polytope $\mathrm{U}Q := \{\boldsymbol{p} \in \mathbb{R}^d : \overline{\boldsymbol{p}} \in Q\}$ is an unconditional convex body, where $\overline{\boldsymbol{p}} := (|p_1|, |p_2|, \ldots, |p_d|)$.

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- This establishes a bijection

anti-blocking polytopes $\longleftrightarrow$ unconditional polytopes

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- P is reflexive $\Longrightarrow \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq 1$.
- $P_{\geq 0}$ has inequalities of the form $x_i \geq 0$ and $\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq 1$ where $\boldsymbol{a}_i \in \mathbb{Z}^d_{\geq 0}$.

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- $P_{\geq 0}$ has inequalities of the form $x_i \geq 0$ and $\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq 1$ where $\boldsymbol{a}_i \in \mathbb{Z}^d_{\geq 0}$.
- This implies $\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle = 0,$ or 1 and $x_i = 0,$ or 1. Hence $P_{\geq 0}$ is compressed.
- The other direction follows similarly.

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Theorem

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- Unconditional polytopes are reflexive if and only if they arise from compressed anti-blocking polytopes. This leads to the following question:

- Compressed polytopes have regular unimodular triangulations (RUT) $\implies P_{\geq 0}$ has RUT.
- Reflecting this triangulation across all hyperplanes yields:

Theorem

Let P be unconditional and reflexive. Then P has a RUT. In particular, h^*_P is symmetric and unimodal.

- Unconditional polytopes are reflexive if and only if they arise from compressed anti-blocking polytopes. This leads to the following question:

Question

When are anti-blocking polytopes compressed?

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Theorem ([CFS17, Prop. 3.10])

Let $P_{\geq 0}$ be anti-blocking. Then $P_{\geq 0}$ is compressed if and only if it is the stable set polytope of a perfect graph.

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Let $P_{\geq 0}$ be anti-blocking. Then $P_{\geq 0}$ is compressed if and only if it is the stable set polytope of a perfect graph.

Enter perfect graphs!

Outline

- Let $G = ([d], E)$ be a graph on d vertices.

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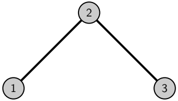

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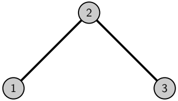

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- The size of the biggest clique of G is denoted $\omega(G)$.
- A proper k-coloring of a graph $G = ([d], E)$ is a function $c\colon [d] \to [k]$ such that $c(u) \neq c(v)$ whenever $uv \in E$.
- The smallest k such that there is a proper coloring of G is denoted $\chi(G)$ and it is called the *chromatic number*.

Definition

A graph is *perfect* if for all induced $H \subset G$, $\chi(H) = \omega(H)$.

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Example and Non-example

Let K_3 be the triangle and C_5 be the 5-cycle. Then K_3 is perfect, but C_5 is not, as $\chi(C_5) = 3$, but $\omega(C_5) = 2$.

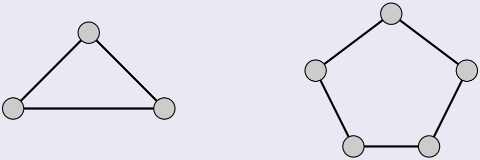

Figure: K_3 and C_5.

Recall:

Theorem ([CFS17, Prop. 3.10])

Let P be anti-blocking. Then P is compressed if and only if it is the stable set polytope of a perfect graph.

Definition

The stable set polytope of a graph $G = ([d], E)$ is
$P_G := \operatorname{conv}\{\mathbf{1}_S : S \subset [d] \text{ stable}\}$.

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Example

Let C_2 be the cycle on 2 vertices. Then there are three stable sets, namely $\emptyset$, $\{1\}$, and $\{2\}$. Therefore, P_{C_2} has vertices $(0,0)$, $(1,0)$, and $(0,1)$.

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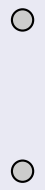
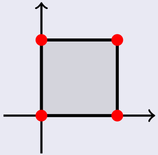
Example

Let G be the graph on 2 vertices without edges. Then there are four stable sets, namely $\emptyset$, $\{1\}$, $\{2\}$, and $\{1, 2\}$. Therefore, P_G has vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$.

G





P_G

Definition

The stable set polytope of a graph $G = ([d], E)$ is $P_G := \operatorname{conv}\{\mathbf{1}_S \colon S \subset [d] \text{ stable}\}$.

- Stable set polytopes are always anti-blocking.
- The dimension of a stable set polytope P_G equals the number of vertices of G.

Perfect graphs can be characterized purely geometrically:

Theorem [Lov72]

A graph $G = ([d], E)$ is perfect if and only if

$$P_G \;=\; \{\boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : \sum_{i \in C} x_i \leq 1 \text{ for all max. cliques } C \subseteq [d]\}.$$

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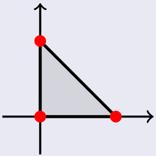
Example

C_2

$x + y \leq 1$



P_{C_2}

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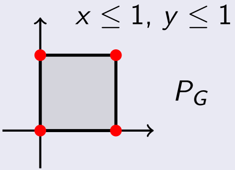

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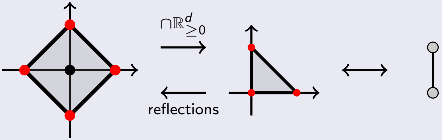

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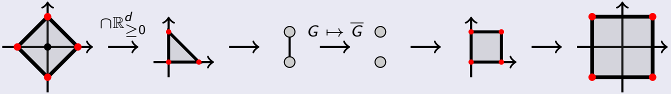
- Equivalently, P is unconditional and reflexive if and only if $P_{\geq 0} = P_G$ for some perfect G.
- Dualizing the polytope corresponds to taking the complement!

Example

Let's dualize $\diamondsuit = \mathrm{conv}\{\pm \boldsymbol{e}_i\}$!



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Theorem

Two unconditional reflexive polytopes are unimodularly equivalent if and only if they arise from isomorphic graphs.

- Therefore, we have a bijection between perfect graphs on d vertices and unconditional reflexive d-polytopes.
- We can now count unconditional reflexive polytopes!

n	3	4	5	6	7	8	9	10	11	12
p(n)	4	11	33	148	906	8887	136756	3269264	115811998	5855499195

Table: Number $p(n)$ of unlabeled perfect graphs; OEIS sequence A052431.

THANK YOU!

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