# On the cone conjecture for log Calabi-Yau mirrors of Fano 3-folds

Jennifer Li
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University of Nottingham

Morrison '93 (conj)

$X$: Calabi-Yau

$\mathrm{Nef}(X) \subset H^2(X, \mathbb{R})$

Then (i) $\mathrm{Aut}(X) \curvearrowright \mathrm{Nef}(X)$ with a rational polyhedral fundamental domain (R.P.F.D.); and

(ii) $\mathrm{PsAut}(X) \curvearrowright \mathrm{Mov}(X)$ with an R.P.F.D.

In part.,

$\mathrm{Aut}(X) \curvearrowright \{\text{faces of } \mathrm{Nef}(X)\}$ with finitely many orbits

$\mathrm{PsAut}(X) \curvearrowright \{\text{faces of } \mathrm{Mov}(X)\}$ " "

Notations  $Y$ : sm. proj. 3-fold

$\mathrm{Nef}^e(Y) = \mathrm{Eff}(Y) \cap \mathrm{Nef}(Y)$

$\mathrm{Mov}^e(Y) = \mathrm{Eff}(Y) \cap \mathrm{Mov}(Y)$

Important Idea :

If $Z$ is a flop of $Y$, then the extremal rays of $\mathrm{Curv}(Z)$ in $K_Z < 0$ region can be either

Type (1): blowup of a smooth $\Gamma$

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ & & \cup \\ & & \Gamma \end{array}$$

or Type (6) : conic bundle

## Mori's classification of extremal rays of $\overline{\mathrm{Curv}}(Y)$ (Kollár-Mori, Thm 1.32, p. 28)

$X$: nonsingular projective 3-fold over $\mathbb{C}$

$\mathrm{cont}_R : Y \to X$ contraction of a $K_Y$-negative extremal ray $R \subset \overline{\mathrm{Curv}}(Y)$

Then we have the following possibilities:

(1) $\mathrm{cont}_R$ is the (inverse of) the blowup of a smooth curve in the smooth 3-fold $Y$.

(2) $\mathrm{cont}_R$ is the (inverse of) the blowup of a smooth point of the smooth 3-fold $Y$.

(3) $\mathrm{cont}_R$ is the (inverse of) the blowup of a point of $Y$ that is locally analytically given by $x^2+y^2+z^2+w=0$.

(4) $\mathrm{cont}_R$ is the (inverse of) the blowup of a point of $Y$ that is locally analytically given by $x^2+y^2+z^2+w^3=0$.

(5) $\mathrm{cont}_R$ contracts a smooth $\mathbb{CP}^2$ with normal bundle $\mathcal{O}(-2)$ to a point of multiplicity 4 on $Y$, which is locally analytically the quotient of $\mathbb{C}^3$ by the involution

$$(x,y,z) \mapsto (-x,-y,-z).$$

(6) $\dim(Y) = 2$ and $\mathrm{cont}_R$ is a fibration whose fibers are plane conics (general fibers are smooth)

(7) $\dim(Y) = 1$ and the general fibers are del Pezzo surfaces.

(8) $\dim(Y) = 0$ and $-K_X$ is ample, so $X$ is a Fano variety.

## Main Theorem

$Y$: sm. proj. 3-fold admitting a K3 fibration $\begin{array}{c} Y \\ f \downarrow \\ \mathbb{P}^1 \end{array}$ s.t. $-K_Y = f^*\mathcal{O}(1)$.

Then the PsAut$(Y)$ acts on

(1) The Type (6) codim-one faces of $\mathrm{Mov}^e(Y)$ w/ finitely many orbits.

(2) The Type (1) codim-one faces of $\mathrm{Mov}^e(Y)$ w/ finitely many orbits if $H^3(Y, \mathbb{C}) = 0$.

<u>Remark</u> If $H^3(Y, \mathbb{C}) = 0$, then in case of Type (1) (b.u. of sm. curve $T$), the genus $g(T) = 0$.

<u>Conjecture</u> PsAut$(Y) \curvearrowright \{$Type (1) codim 1 faces of $\mathrm{Mov}^e(Y)\}$ w/ finitely many orbits for all $g \geq 0$.

**Thm** (Kawamata) $\mathrm{PsAut}(Y) \curvearrowright \{$codim 1 faces of $\mathrm{Mov}(Y)$ containing $-K_Y\}$ w/ finitely many orbits.

**Thm** Conj (assume true) + Kawamata's thm gives:

$\mathrm{PsAut}(Y) \curvearrowright \{$codim 1 faces of $\mathrm{Mov}^e(Y)\}$ with finitely many orbits.

**Remark** This statement is implied by the Kawamata-Morrison-Totaro (KMT) cone conj:

**KMT cone conj**

$(Y, \Delta)$: klt "Kawamata log terminal"

e.g. $Y$ smooth,

$\Delta = \sum a_i \Delta_i \subset Y$ n.c.d. with each $0 < a_i < 1$

$\rightsquigarrow$ $(Y, \Delta)$ klt

and $K_Y + \Delta = 0$ "klt log Calabi-Yau"

Then, (i) $\mathrm{Aut}(Y, \Delta) \curvearrowright \mathrm{Nef}^e(Y)$ with RPFD; and

(ii) $\mathrm{PsAut}(Y, \Delta) \curvearrowright \mathrm{Mov}^e(Y)$ with RPFD.

Pf (sketch)

Consider $Y$: CY3 with K3 fibration $f$ $\begin{array}{c} Y \\ \downarrow \\ \mathbb{P}^1 \end{array}$ s.t. $-K_Y = f^*\mathcal{O}(1)$.

Take $F$: a general fiber $f$

$F_1, F_2 \sim F$ smooth fibers

$\Delta = \frac{1}{2}F_1 + \frac{1}{2}F_2$.

Then $(Y, \Delta)$ : Klt log CY

By a result of Birkar-Cascini-Hacon-McKernan,

$$\mathrm{Mov}(Y) = \bigcup_{\substack{Y \dashrightarrow Z \\ \text{SQMs}}} \mathrm{Nef}(Z) \quad \text{is locally R.P. in } \mathrm{Int}(\mathrm{Mov}(Y))$$

Small $\mathbb{Q}$-factorial modification (SQM) $Y \dashrightarrow Z$

$Y, Z$: $\mathbb{Q}$-factorial

$Y \dashrightarrow Z$ is a birational map, an isom. away from 2 codim 2 subsets

Ex Flops are SQMs

In our setting, the <u>only</u> SQMs are flops / composition of flops.

A main idea:

$$S := \bigcup_{\substack{Y \dashrightarrow Z \\ \text{SQMs}}} \{\text{codim 1 faces of } \mathrm{Nef}(Z) \text{ in the boundary of } \mathrm{Mov}(Y)\} \twoheadrightarrow \{\text{codim 1 faces of } \mathrm{Mov}^e(Y)\}$$

$\updownarrow$ elements correspond

$$\bigcup_{\substack{Y \dashrightarrow Z \\ \text{SQMs}}} \{\text{extremal rays of } \overline{\mathrm{Curv}}(Z)\}$$

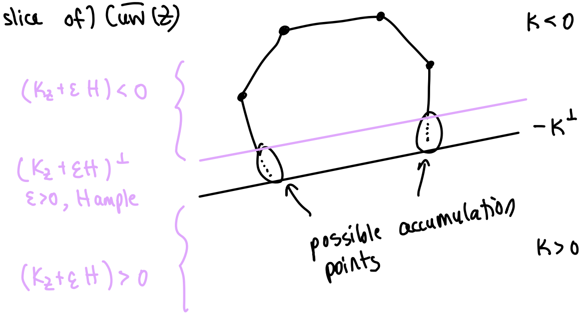
$$\overline{\mathrm{Curv}}(Z) = \mathrm{Nef}(Z)^*$$

(1) Mori's Cone Theorem

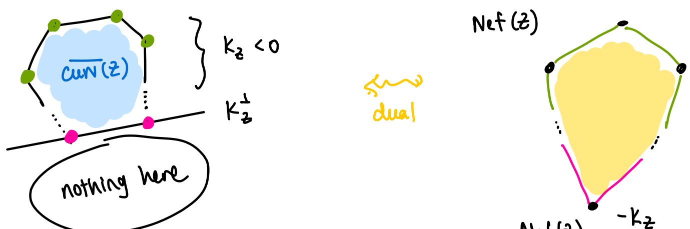
$Z$: sm. proj. variety

Tells us what $\overline{\mathrm{Curv}}(Z)$ looks like in $K_Z < 0$ region:

(a slice of) $\overline{\mathrm{Curv}}(Z)$




For us: $-K_Z$ (a fiber) is nef, so there are <u>no</u> curves in the $K>0$ region.





$\mathrm{Mov}(Y) = \bigcup\limits_{\substack{Y \dashrightarrow Z \\ \text{flop}}} \mathrm{Nef}(Z)$

(2) Mori's classification of $\underbrace{\text{extremal rays of } \overline{\mathrm{Curv}}(Z) \text{ in } K_Z < 0}$.

$\updownarrow$

codim 1 faces of Mov $(Y)$

We only have 2 possibilities:

Type (1): B.U. of a sm. curve

Type (6): conic bundle

① describe the green vertices in $\overline{\mathrm{Curv}}(Z)$

② pts on $-K^{\perp}$ are taken care of by result of Kawamata.

Type (6) Conic bundle corresponding to a ray of $\overline{\mathrm{Cuw}}(Z)$

$$\begin{tikzcd} Y \arrow[r, dashed, "\text{flops}"'] & Z \arrow[r, "g\,:\,\text{conic bundle}"] \arrow[d, "f\,:\,K3"] & \underbrace{S}_{\text{base of conic bundle}} \\ & \mathbb{P}^1 & \end{tikzcd}$$

Let $h := g|_F : \underbrace{F}_{\text{general fiber (K3)}} \longrightarrow S$ , finite , degree 2.

- There is an involution $i$ associated to $h$: $\begin{array}{c} F \circlearrowleft i \\ \downarrow h \\ S \end{array}$

- $S = F/\langle i \rangle$ , $F$ : K3

- (Mori) $S$ smooth

$\Rightarrow$ 2 possibilities : either $\begin{cases} (1)\ \mathrm{Fix}(i) = \text{union of smooth curves} & \Rightarrow S \text{ is rational surface} \\ (2)\ \mathrm{Fix}(i) = \emptyset & \Rightarrow S \text{ is an Enriques surface} \end{cases}$

Assume $S$ is rational surface. Run MMP on $S$

$$\pi : S \longrightarrow \underbrace{\bar{S}}_{\mathbb{P}^2 \text{ or } \mathbb{F}_n \ (0 \le n \le 4,\ n \ne 1)}$$

Now we have

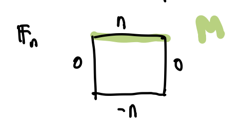
$$\begin{tikzcd} F \arrow[r, "h"] \arrow[rr, bend right, "\theta := \pi \circ h"'] & S \arrow[r, "\pi"] & \bar{S} \arrow[r, "\phi"] & \hat{S} \end{tikzcd}$$

and let $L = \theta^* M$, where:

- If $\bar{S} = \mathbb{P}^2$: take $\phi = \mathrm{id}$ and $M = H$ a hyperplane class

  $\Rightarrow L^2 = 2$

- If $\bar{S} = \mathbb{F}_n$ $(0 \le n \le 4,\ n \ne 1)$: take $\phi : \mathbb{F}_n \to \underbrace{\mathbb{P}(1,1,n)}_{\hat{S}}$ to be the contraction of the negative section of $\mathbb{F}_n$ and $M =$ the positive section of $\mathbb{F}_n$




  $$\Rightarrow L^2 = 2M^2 = 2n \le 8$$

<u>Thm</u> (Sterk, '85)

$F$: K3 surface
$L$: nef line bundle with $L^2 = 2k$ for some fixed $k \in \mathbb{N}$.
Then,

$$\mathrm{Aut}(F) \curvearrowright \{\text{all such } L\} \text{ with finitely many orbits.}$$

Moreover: $L$ gives the map $F \to \hat{S}$. Therefore this proves Main Thm (1).

<u>Type (1)</u> B.U. of a sm. curve, or contraction of a ruled surface $E$.

Have a $\mathbb{P}^1$-bundle $g: E \to T$ ( K3 fibr. $f: Y \to \mathbb{P}^1$)

Supp. $H^3(Y, \mathbb{C}) = 0$. Then $g(T) = 0$

$\Rightarrow T \cong \mathbb{P}^1$.

In our case, $g: E \to T$ is a trivial $\mathbb{P}^1$-bundle.

$\Rightarrow T \subset F$ for each fiber $F$ of $f$.

By A.F., $T \subset F$ is a $(-2)$-curve.

<u>Thm</u> (Sterk: Morrison's cone conj. for K3 surfaces)

$Y_\eta$: generic fiber (K3)

Then $\mathrm{Aut}(Y_\eta) \curvearrowright \{(-2)\text{-curves in } Y_\eta\}$ w/ finitely many orbits.

$\mathrm{Aut}(Y_\eta) \subset \mathrm{PsAut}(Y)$

$\Rightarrow \mathrm{PsAut}(Y) \curvearrowright \{(-2)\text{-curves in } Y_\eta\}$ w/ finitely many orbits

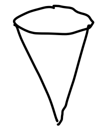
$\Rightarrow \mathrm{PsAut}(Y) \curvearrowright \{E \subset Y \mid E \text{ is an exceptional divisor of Type (1) on } Y\}$ w/ finitely many orbits

This proves Main Thm (2).

Thm (Kawamata)

$f: Y \to S$, K3 fibration, $\dim(Y) = 3$

Then $\mathrm{PsAut}(Y|S) \curvearrowright \mathrm{Mov}^e(Y|S)$ w/ finitely many orbits on faces

Kawamata's picture lives in $N^1(Y|S) = N^1(Y) \otimes \mathbb{R} \,/\, f^*N^1(S) \otimes \mathbb{R}$ $\supset$ 

$= N^1(Y) \otimes \mathbb{R} \,/\, \mathbb{R}\cdot[-K_Y]$

For us:
$S = \mathbb{P}^1$
so $N^1(S) = \mathbb{Z}\cdot[pt]$
$= \mathbb{Z}\cdot[f^*pt]$
$= \mathbb{Z}\cdot[-K_Y]$

Know: $\mathrm{PsAut}(Y/\mathbb{P}^1) \curvearrowright$ faces of a chamber decomp. containing $[D]$ w/ finitely many orbits.

$\cap$

$\mathrm{PsAut}(Y) \curvearrowright \mathrm{Mov}(Y) = \bigcup_{\substack{Y \dashrightarrow Z \\ \text{flops}}} \mathrm{Nef}(Z)$ w/ finitely many orbits.

Now, Main Thm + Kawamata + Conj. $\Rightarrow$ $\mathrm{PsAut}(Y) \curvearrowright \{\text{codim } 1 \text{ faces of } \mathrm{Mov}^e(Y)\}$ w/ finitely many orbits

<u>Remarks</u> There are many examples of $Y$ (sm. proj. 3fold) with a K3 fibration $f: Y \to \mathbb{P}^1$, with $-K_Y = f^*\mathcal{O}(1)$, and $|\mathrm{PsAut}(Y)| = \infty$.

[Coates-Corti-Galkin-Kasprzyk 2016, Cheltsov-Przyjalkowski 2018, Przyjalkowski 2018, Doran-Harder-Katzarkov-Ovcharenko-Przyjalkowski 2023]

$X$: Fano 3-fold, $E \subset X$: smooth divisor $K_X + E = 0$
Then there is sm. proj. 3-fold $Y$ and a K3 fibr. $f: Y \to \mathbb{P}^1$, $-K_Y = f^*\mathcal{O}(1)$ and $H^3(Y) = 0$ and $\exists$ fiber $\underbrace{D}_{\text{n.c.d. with a 0-stratum.}}$ of $f$.

(Total: 105 deform. types of Fano 3-folds) $- 7 - 6 = 92$ examples $|\mathrm{Aut}(Y_n)| = \infty$
$\Rightarrow |\mathrm{PsAut}(Y)| = \infty$.