Toric and tropical Bertini theorems

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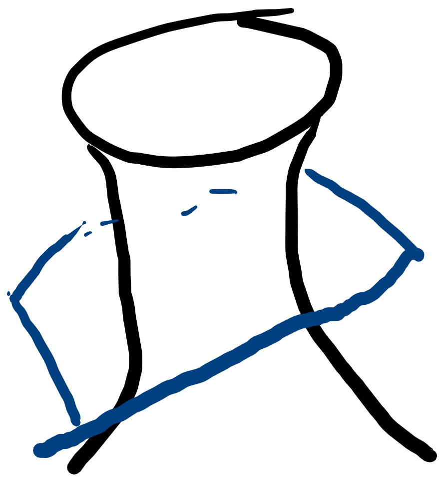
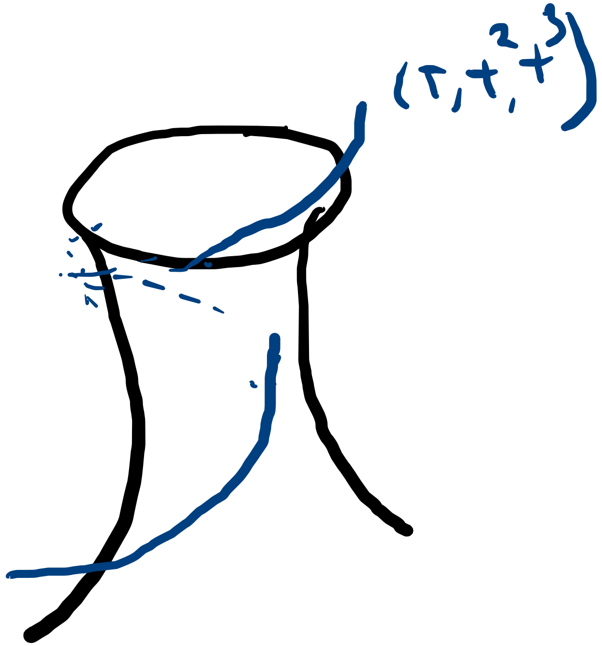
w/ Gandini, Hering, Mohammadi, Rajchgot, Wheeler + Yu.

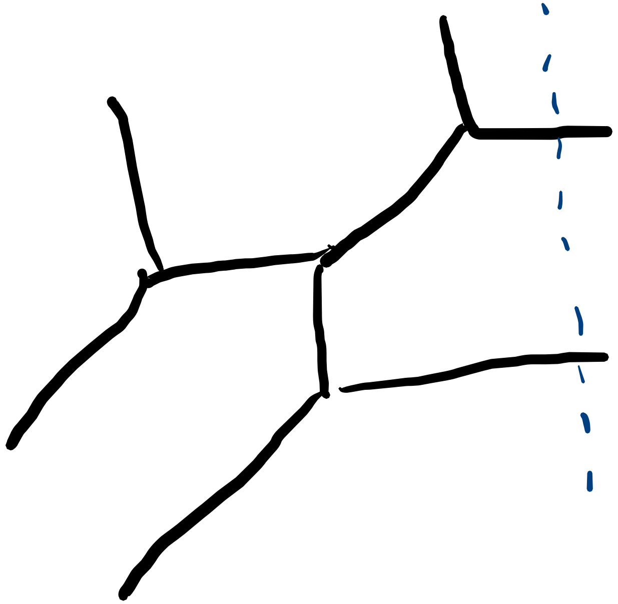
Classical



toric



tropical



Classical Bertini:

Recall: The classical Bertini theorem states that if $X \subseteq \mathbb{P}^n$ is irreducible over $K = \overline{K}$, & $\dim(X) \geq 2$, then the set of hyperplanes $H \subseteq \mathbb{P}^n$ with $X \cap H$ irreducible is dense in $\mathbb{P}^{n\vee}$

Key for induction proofs!

This talk: Two variants

a) $H \to$ torus
b) $X \to \operatorname{trop}(X)$.

Toric version:

For $X \subseteq T^n \simeq (K^\circ)^n$, replace H by a subtorus $T' \leq T^n$.

Problem: Can't expect only a few "bad T'"

eg $X = V(x^2 - yz^2) \subseteq (K^\circ)^3$

$T' = (t_1, t_2^2, t_2)$

$X \cap T' = V(t_1^2 - t_2^2 t_2^2) = V(t_1 - t_2^2) \cup V(t_1 + t_2^2)$

$T'_{abc} = (t_1^a, t_2^{2b}, t_2^c)$ $\quad V(t_1^{2a} - t_2^{2b+2c})$

$= V(t_1^a - t_2^{b+c}) \cup V(t_1^a + t_2^{b+c})$.

Solution (Fuchs, Mantova, Zannier JAMS 2017)

Rule this out:

Let $\pi : X \to T^d \simeq (K^*)^d$ be a dominant finite map.

We say π satisfies PB if for every isogeny (surjective with finite kernel) $\mu : T^d \to T^d$ the pullback

is irreducible

$$\begin{tikzcd} X \times_\mu T^d \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ T^d \arrow[r, "\mu"'] & T^d \end{tikzcd}$$

eg $V(x^2 - yz^2)$ does not satisfy PB since for $\mu : T^2 \to T^2$, $(x,y) \mapsto (x, y^2)$, $X \times_\mu T^2 = V(x^2 - y^2 z^2)$ ← reducible

Theorem [FMZ] If $\pi : X \to T^d$ is a dominant finite map satisfying PB then there is a finite union Σ of subtori such that if $T' \subseteq T^d$ is a subtorus not in Σ then $\pi^{-1}(\theta \cdot T')$ is irreducible.

[FMZ] is written for $K = \mathbb{C}$, and the proof requires $\mathrm{char}(K) = 0$.

(it is a sideproduct of a proof of a bound on the number of terms in a root $y = g(x)$ of $f(x, y) = 0$ when f is monic of degree d in y — There is no bound in $\mathrm{char}(K) = p > 0$)

My motivation:

A tropical Bertini theorem

Let $X \subseteq T^n \cong (K^*)^n$, where K is a valued field

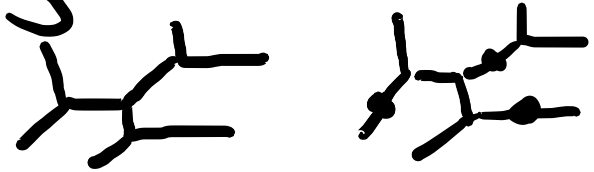
The tropicalization of X is

$$\operatorname{trop}(X) = \operatorname{cl}(\operatorname{val}(X)) \subseteq \mathbb{R}^n$$
$$= \operatorname{cl}\{(\operatorname{val}(x_1), \ldots, \operatorname{val}(x_n)) : x = (x_1, \ldots, x_n) \in X\}$$

Euclidean topology

Theorem: [Bieri-Groves, BJSSST, Cartwright-Payne]
When X is irreducible, $\operatorname{trop}(X)$ is the support of a polyhedral complex that is pure of dim d.

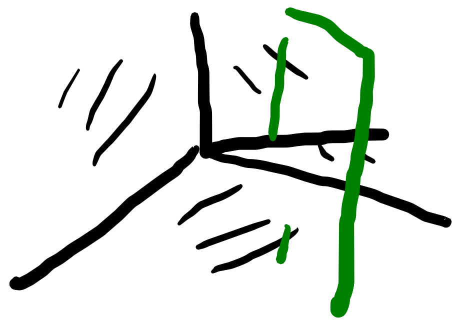
connected



all max polyhedra have dim d.

Theorem [M–Yu in char 0 + Gandini–Hering–Mohammadi–Rajchgot–Wheeler in arbitrary characteristic]

Let $\Sigma \subseteq \mathbb{R}^n$ be the tropicalization of $X \subseteq T^n / K=\bar{K}$ where X is irreducible of dimension $d \geq 2$. The set of rational affine hyperplanes $H \subseteq \mathbb{R}^n$ with $\Sigma \cap H$ the tropicalization of an irreducible variety is dense in the Euclidean topology on $\mathbb{P}^n_{\mathbb{Q}}$



Answers question of Cartwright–Payne

Q: What do these topics have to do with each other?

A: An affine rational hyperplane is the tropicalization of a coset of a subtorus of T^n:

eg $H = \{(x,y,z) : 4x + 5y - 3z = 1\}$

$$= \operatorname{trop}\big(\underbrace{V(x^4y^5 - \alpha z^3)}_{= (\alpha^{1/4},1,1)\cdot\{(t_1^3,\, t_2^3,\, t_1^4 t_2^5) \,:\, (t_1,t_2)\in (K^*)^2\}}\big) \quad \text{where } \operatorname{val}(\alpha) = 1$$

For sufficiently general α,

$$\mathrm{trop}(X \cap \alpha T') = \mathrm{trop}(X) \cap H.$$

$X \cap \alpha T'$ might be reducible, but components have the same tropicalization.

$\rightsquigarrow$ reduce to a PB situation and deduce from the toric Bertini theorem.

Thm [GHMMRWY]

A toric Bertini theorem holds in arbitrary characteristic.

Theorem [GHMMRWY]

Fix $K = \overline{K}$. If $\pi : X \to T^d$ is finite, dominant, and satisfies PB, then the set of $T' \subseteq T^d$ with $\pi^{-1}(\theta \cdot T')$ irreducible for any θ is dense in $Gr(\dim(T'), d)$.

eg $X = V(y^3 + 1 + t_1 + t_2 + t_3) \in \overline{\mathbb{F}}_2[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, y^{\pm 1}]$

$$\begin{array}{c} X \\ \downarrow \pi \\ T^3_{t_1 t_2 t_3} \end{array}$$

$T' = (t^a, t^b, t^c)$

$\pi^{-1}(T')$ is irreducible unless $a + b = c$

Ideas of proof:

- Reduce to the case that $X = V(f) \subseteq T^{d+1}$
  $f \in K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, y]$
- Reduce to considering $\dim(T') = 1$.

$\rightsquigarrow$ Question becomes: For which $(n_1, \ldots, n_d) \in \mathbb{Z}^d$ is $f(x^{n_1}, \ldots, x^{n_d}, y) \in K[x^{\pm 1}, y^{\pm 1}]$ irreducible?

- View f as a polynomial in y with coefficients in $K(t_1, \ldots, t_d)$.

Write $f = \prod_{i=1} (y - \alpha_i)$ with $\alpha_i \in \overline{K(t_1, \ldots, t_d)}$

Bulk of pf - understand a field containing this.

Generalisations of Puiseux series

Recall:

If $\mathrm{char}(k)=0$, $f \in K[x,y]$ is monic in y

then $f = \prod(y-\alpha_i)$ with $\alpha_i \in K\{\{x\}\}$

Puiseux series

$\bigcup_{n\geq 1} K((x^{\frac{1}{n}}))$

(ie $\overline{K(x)} \subseteq K\{\{x\}\}$.)

↯ There is a convention choice:

Could work with $\bigcup_{n\geq 1} K((x^{-\frac{1}{n}}))$

$\frac{1}{1-x} = 1 + x + x^2 + \cdots$

$= -x^{-1} - x^{-2} - x^{-3} - \cdots$

When $\mathrm{char}(K) = p > 0$

$K\{\{x\}\}$ is not algebraically closed

Chevally $f : y^p - y + x^{-1} \in K(x)[y]$

has no root in $K\{\{x\}\}$

Abhyankar $f = \prod_{i=0}^{p-1} \left(y + i - \sum_{i=0}^{\infty} x^{\frac{1}{p^i}} \right)$

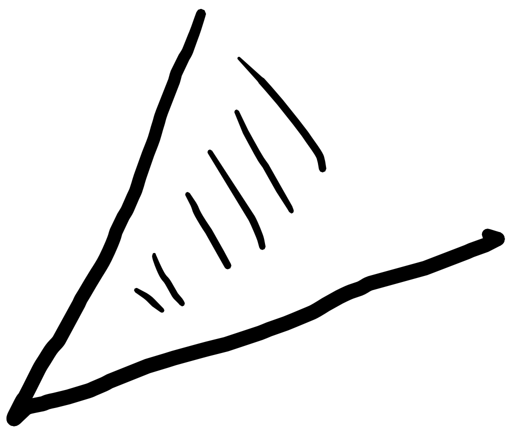
In general, allow series in x with exponents of the form $\frac{m}{np^i}$ – for a fixed series, n is fixed, but i can be unbounded.

$p=2$ $\quad x^{\frac{1}{3}} + x^{1+\frac{1}{6}} + x^{2+\frac{1}{12}} + x^{3+\frac{1}{24}} + \cdots$

Multivariate analogues of Puiseux series

[MacDonald, Saavedra]

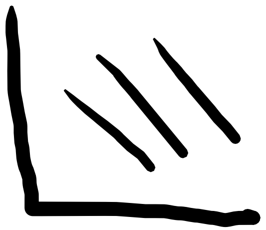
Look at series supported in a cone



Allowed denominators have the form np^i

Fixed for a given series

eg



$$1 + x + y + x^{\frac{1}{3}}y^{\frac{1}{2}} + x^{\frac{1}{3}}y^{\frac{1}{4}} + x^{3}y^{\frac{1}{8}} + \dots$$

$p=2,\ n=3$

Our contribution:

We describe an explicit algebraicaly closed subfield K^w containing $K(t_1 , t_d)$ that has more constraints on where exponents can lie.

This lets us define a specialization map on a subring $R \subseteq K^w$

$$R \longrightarrow K\{\{x\}\}$$

$$t_i \longmapsto x^{n_i}$$

$R \hookrightarrow K^w$

as $K[[x]] \hookrightarrow K((x))$

or $\mathbb{Z} \hookrightarrow \mathbb{Q}$

Back to the proof

$$\begin{array}{ccc} V(f) \subseteq T^{d+1} & & (t_1, \ldots, t_d, y) \\ \downarrow \pi & & \downarrow \\ T^d & & (t_1, \ldots, t_d) \end{array}$$

$f = \prod (y - \alpha_i)$

$\alpha_i \in K^\omega$

can assume in $R \subseteq K^\omega$.

$T' = (t^{n_1}, \ldots, t^{n_d})$

$\pi^{-1}(T')$ is $V(\underbrace{f(x^{n_1}, \ldots, x^{n_d},}_{\text{Specialization of } f} y))$

Specialization

$R \to K\{\{x\}\}$

$t_i \mapsto x^{n_i}$

. $f(x^{n_1}, \ldots, x^{n_d}, y)$ factors if the specialization of $\prod_{\text{subset}} y - \alpha_i$ is a polynomial in $K[x^{\pm}, y^{\pm}]$