Using GIT to relate toric mirror constructions

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06.10.2022

Overview

Structures on a Calabi-Yau

Given a Calabi-Yau manifold X, we look at two kinds of structures on it.

- Symplectic structure (A-side): Gromov-Witten invariants, Lagrangian submanifolds, pseudo-holomorphic sphere counting,...
- Complex structure (B-side): Period integrals, complexes of coherent sheaves, deformations of complex structures,...

Mirror symmetry

Mirror symmetry is a duality that says claims that given a Calabi-Yau V, there should be a dual Calabi-Yau W such that the symplectic structure on V corresponds to the complex structure on W and vice versa.

Homological mirror symmetry

Given a Calabi-Yau X, denote by

- $\mathrm{D}^b(\mathrm{coh}\, X)$, the bounded derived category of coherent sheaves. Its objects are cochain complexes quasi-isomorphic to bounded derived complexes of coherent sheaves. This category encodes algebro-geometric information of X.
- $\mathrm{Fuk}(X)$, the Fukaya category. Its objects are Lagrangian submanifolds. This category encodes symplectic information.

Homological mirror symmetry

Homological mirror symmetry suggests that if V and W are two mirror Calabi-Yaus, then

$$\mathrm{D^b}(\mathrm{coh}\, V) \simeq \mathrm{Fuk}(W) \text{ and } \mathrm{D^b}(\mathrm{coh}\, W) \simeq \mathrm{Fuk}(V).$$

So if one finds two mirrors W, W' to a Calabi-Yau V, one should expect

$$\mathrm{D^b}(\mathrm{coh}\, W) \simeq \mathrm{D^b}(\mathrm{coh}\, W').$$

Homological mirror symmetry

Via the examples of a mirror construction by Libgober and Teitelbaum [**LT94**] and a construction by Batyrev and Borisov [**BB96**], I will present a way to use variations of geometric invariant theory (VGIT) to prove derived equivalences of toric mirror constructions. My research is based on [**Isi13**, **Shi12**, **Hirano**, **Seg11**, **FK17**, **FK19**, **HW12**].

The Libgober and Teitelbaum mirror construction

Libgober and Teitelbaum proposed the following mirror construction. Let

$$Q_{1,\lambda} = x_0^3 + x_1^3 + x_2^3 - 3\lambda x_3 x_4 x_5, \quad Q_{2,\lambda} = x_3^3 + x_4^3 + x_5^3 - 3\lambda x_0 x_1 x_2,$$

and consider the complete intersection $V_\lambda = Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq \mathbb{P}^5$.

The proposed mirror $W_{LT,\lambda}$ to V_λ is a (minimal) Calabi-Yau resolution of the variety $V_{LT,\lambda} = Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq \mathbb{P}^5 / G_{81}$, where $G_{81} \leq PGL(5, \mathbb{C})$ is a specified order 81 subgroup. They provide topological evidence of mirror symmetry by showing

$$\chi(V_\lambda) = -\chi(W_{LT,\lambda}).$$

The group G_{81}

The order 81 subgroup $G_{81} \leq PGL(5, \mathbb{C})$ is defined as the group generated by diagonal matrices of the form

$$g_{\alpha,\beta,\delta,\epsilon,\mu} := \text{diag}\left(\zeta_3^{\alpha}\zeta_9^{\mu},\ \zeta_3^{\beta}\zeta_9^{\mu},\ \zeta_9^{\mu},\ \zeta_3^{-\delta}\zeta_9^{-\mu},\ \zeta_3^{-\epsilon}\zeta_9^{-\mu},\ \zeta_9^{-\mu}\right)$$

Here, $\alpha, \beta, \delta, \epsilon \in \mathbb{Z} \pmod 3$ and $\mu \in \mathbb{Z} \pmod 9$ with $\zeta_9^{3\mu} = \zeta_3^{\alpha+\beta} = \zeta_3^{\delta+\epsilon}$.

How does $W_{LT,\lambda}$ relate to the Batyrev-Borisov mirror?

We note that Z_λ is different from $W_{LT,\lambda}$, so it is a fair question to ask how these two mirror candidates relate. An answer to this is given by my main result of today.

Main result

Theorem (M.'22)

Let $\lambda \in \mathbb{C}$ such that $\lambda^6 \neq 0, 1$. Consider the two polynomials

$$p_{1,\lambda} = x_0^3 x_6^3 + x_1^3 x_7^3 + x_2^3 x_8^3 - 3\lambda x_3 x_4 x_5 x_6 x_7 x_8,$$

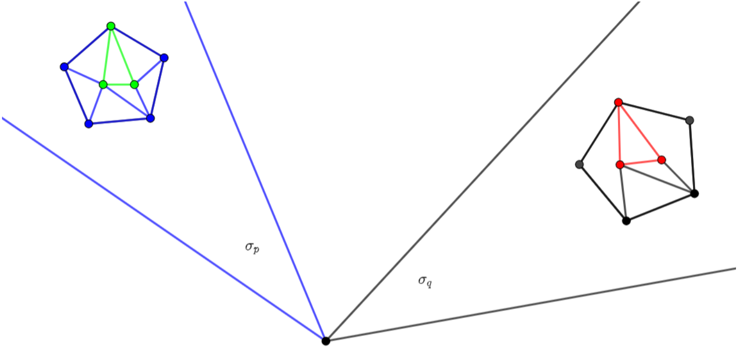
$$p_{2,\lambda} = x_3^3 x_9^3 + x_4^3 x_{10}^3 + x_5^3 x_{11}^3 - 3\lambda x_0 x_1 x_2 x_9 x_{10} x_{11}.$$

Let $\mathcal{Z}_\lambda = Z(p_{1,\lambda}, p_{2,\lambda}) \subseteq \mathcal{X}_\nabla$ and $\mathcal{V}_{LT,\lambda} = Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq [\mathbb{P}^5 / G_{81}]$. Then

$$\mathrm{D}^\mathrm{b}(\mathrm{coh}\, \mathcal{V}_{LT,\lambda}) \simeq \mathrm{D}^\mathrm{b}(\mathrm{coh}\, \mathcal{Z}_\lambda).$$

Here $\mathcal{X}_\nabla$ denotes the toric stack corresponding to the Batyrev-Borisov mirror construction.

Ansatz



$$D^b(\mathrm{coh}\, Z_p) \simeq D_{sg}([Y_p/S \times \mathbb{G}_m]) \simeq D_{sg}([Y_q/S \times \mathbb{G}_m]) \simeq D^b(\mathrm{coh}\, Z_q)$$

Figure: The idea of the proof

Setup

Consider a G-equivariant vector bundle $\mathcal{E}$ on X. Denote by Z the zero-locus of a G-invariant section $s \in H^0(X, \mathcal{E})$. Then $\langle -, s \rangle$ induces a global function on $\operatorname{tot} \mathcal{E}^\vee$. Let Y be the zero-section of this pairing and consider the fibrewise dilation action on the torus $\mathbb{G}_m$. We will be doing GIT on $\operatorname{tot} \mathcal{E}^\vee$.

GIT and the secondary fan

Recall the Cox construction for toric varieties, we can represent them as quotients

$$(\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))/G.$$

This is a good example of what GIT quotients are in general. We take an affine space X, remove a special set of points Z, called exceptional locus, and quotient by group S of the torus acting on the space X.

Choice of exceptional locus

So different choices of exceptional loci give different quotients. The different loci are obtained by looking at characters of the group S and assigning stability conditions based on them. Each character corresponds to a linearisation of the trivial line bundle.

These stability conditions define semi-stable loci, which then become the exceptional loci we remove before quotienting. It is a natural question to ask when distinct characters χ, ψ give the same exceptional loci and hence the same affine patches $U_\chi = U_\psi$ we take quotients of.

Secondary fan

Let us focus on a toric variety X with torus T. Denoting by $\tilde{S}$ the character group of G, we consider the vector space $\mathrm{Hom}(\tilde{S}, T) \otimes \mathbb{Q}$. We can think of this space as a parameter space for linearisation.

In [**GKZ**], Gelfand, Kapranov and Zelevinsky show that grouping characters together for corresponding to the same affine patch gives the parameter space a natural *GKZ*-fan structure. This is what we refer to as the secondary fan of the variety X, and every chamber of it corresponds to a GIT quotient.

The category of singularities

Let X be a variety and G an algebraic group acting on X.

Definition

An object of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}[X/G])$ is called perfect if it is locally quasi-isomorphic to a bounded complex of G-equivariant vector bundles on X. We denote the full subcategory of perfect objects by $\mathrm{Perf}([X/G])$. The Verdier quotient of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}[X/G])$ by $\mathrm{Perf}([X/G])$ is called the category of singularities and denoted

$$\mathrm{D}_{\mathrm{sg}}([X/G]) := \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[X/G]) / \mathrm{Perf}([X/G]).$$

This category can be viewed as studying the geometry of the singular locus by a result of Orlovs.

Proposition (Orlov, [O09])

Assume that $\mathrm{coh}[X/G]$ *has enough locally free sheaves. Let* $i : U \to X$ *be a* G*-equivariant open immersion such that the singular locus of* X *is contained in* $i(U)$*. Then the restriction,*

$$i^* : \mathrm{D}_{\mathrm{sg}}([X/G]) \to \mathrm{D}_{\mathrm{sg}}([U/G]),$$

is an equivalence of categories.

Theorem (Isik [Isi13], Shipman [Shi12], Hirano [Hirano])

Suppose the Koszul complex on s is exact. Then there is an equivalence of categories

$$\mathrm{D}_{\mathrm{sg}}([Y/(G \times \mathbb{G}_m)]) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[Z/G]).$$

Corollary (Corollary 3.4 in [**FK19**])

Let V be an algebraic variety with a $G \times \mathbb{G}_m$ action. Suppose there is an open subset $U \subseteq V$ such that U is $G \times \mathbb{G}_m$ equivariantly isomorphic to Y as above and that U contains the singular locus of X. Then

$$\mathrm{D}_{\mathrm{sg}}([V/(G \times \mathbb{G}_m)]) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[Z/G]).$$

Application to a GIT situation

Let $X := \mathbb{A}^{n+t}$ with coordinates x_i, u_j for $1 \leq i \leq n, 1 \leq j \leq t$. Let T be the standard torus $\mathbb{G}_m^{n+t}$, consider a subgroup $S \subseteq T$ with $\hat{S}$ the connected component containing the identity. We study the GIT quotients for actions of $\hat{S}$ on X, studying associated Landau-Ginzburg models.

The superpotential

The superpotential w, obtained from the pairing $\langle -, s \rangle$, can be written as

$$w = \sum_{j=1}^{t} u_j f_j.$$

Let $Z(w) \subseteq X$ be the zero-locus and $Y_p = Z(w) \cap U_p$, where U_p is the open affine patch associated to a chamber σ_p of the secondary fan.

An example

$X = \mathbb{P}^2$, $\mathcal{E} = \mathcal{O}(3)$. Then $\operatorname{tot}\mathcal{E}^\vee = \operatorname{tot}\mathcal{O}(-3)$.

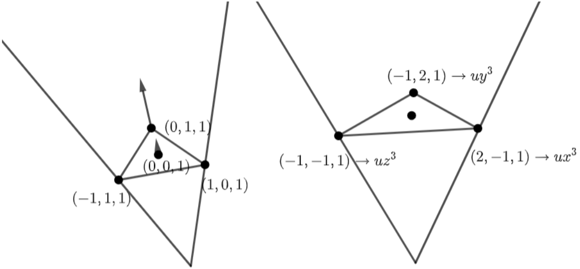


Figure: An example of how to get the superpotential

Given a section $s \in \Gamma(\mathbb{P}^2, \mathcal{O}(3))$, which is a point in the dual cone, we get the superpotential $\langle -, s \rangle = u \cdot s$.

Theorem (Theorem 3 in [HW12])

If S is quasi-Calabi-Yau, there is an equivalence of categories

$$\mathrm{D}_{\mathrm{sg}}([Y_p/S \times \mathbb{G}_m]) \simeq \mathrm{D}_{\mathrm{sg}}([Y_q/S \times \mathbb{G}_m])$$

for all $1 \leq p, q \leq k$, where k is the number of chambers in the GKZ fan.

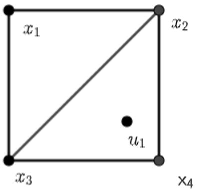
Two ideals

A chamber σ_p of the secondary fan corresponds to a regular triangulation $\mathcal{T}_p$ of a certain set of points $\nu_1(S), \ldots, \nu_{n+t}(S)$.

$$\mathcal{I}_p := \left\langle \prod_{i \notin I} x_i \prod_{j \notin J} u_j \;\middle|\; \bigcup_{i \in I} \nu_i(S) \cup \bigcup_{j \in J} \nu_{n+j}(S) \text{ gives a simplex in } \mathcal{T}_p \right\rangle .$$

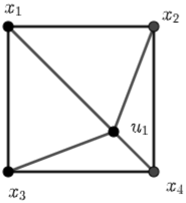
$$\mathcal{J}_p := \left\langle \prod_{i \notin I} x_i \;\middle|\; \bigcup_{i \in I} \nu_i(S) \cup \bigcup_{j=1}^{t} \nu_{n+j}(S) \text{ gives a simplex in } \mathcal{T}_p \right\rangle .$$

Example



$\mathcal{I} = \langle x_1u_1, x_4u_1 \rangle$

$\mathcal{J} = 0$



$\mathcal{I} = \langle x_1x_2, x_1x_3, x_3x_1x_4x_1 \rangle = \mathcal{J}$

Figure: An example of the ideals $\mathcal{I}, \mathcal{J}$.

We set $V_p := X \setminus Z(\mathcal{J}_p) \subseteq U_p$. Since $\mathcal{J}_p$ has no u_j in its generators, we can see it as ideal $\mathcal{J}_p^x$ in $\mathbb{C}[x_1, \ldots, x_n]$, giving an open subset of $\mathbb{A}^n$ by $V_p^x := \mathbb{A}^n \setminus Z(\mathcal{J}_p^x)$. This set gives us a toric stack $X_p := [V_p^x / S]$. In fact, we can view the function $\oplus f_j$ as a section of V_p which defines, for all p, a complete intersection $Z_p := Z(\oplus f_j) \subseteq X_p$.

Key result

Proposition (Proposition 4.7 in [**FK19**])

Suppose $\mathcal{J}_p$ is non-zero. If $\mathcal{I}_p \subseteq \sqrt{\partial w, \mathcal{J}_p}$, then

$$\mathrm{D_{sg}}([Y_p/S \times \mathbb{G}_m]) \simeq \mathrm{D^b}(\mathrm{coh}\, Z_p).$$

Corollary (†)

Assume S satisfies the quasi-Calabi-Yau condition and that $\mathcal{J}_p$ and $\mathcal{J}_q$ are non-zero. If $\mathcal{I}_p \subseteq \sqrt{\partial w, \mathcal{J}_p}$ and $\mathcal{I}_q \subseteq \sqrt{\partial w, \mathcal{J}_q}$ for some $1 \leq p, q \leq r$, then

$$\mathrm{D^b}(\mathrm{coh}\, Z_p) \simeq \mathrm{D^b}(\mathrm{coh}\, Z_q).$$

How to apply this machinery

Back to the problem motivating us, how do we use these results on the Libgober-Teitelbaum and Batyrev-Borisov constructions? While the Batyrev-Borisov is a toric construction, which makes is easy to set up a GIT problem, the Libgober-Teitelbaum construction is not. So the first thing to do is to find a toric description of the construction.

Expressing Libgober-Teitelbaum torically

First, we find a fan for $X_{LT} = \mathbb{P}^5 / G_{81}$. This is done by constructing $X_{tot} = \text{tot}\left(\mathcal{O}_{\mathbb{P}^5}(-3) \oplus \mathcal{O}_{\mathbb{P}^5}(-3)\right)$ as a toric variety with fan $\Sigma_{\mathbb{P}^5, T_a, T_b}$. We then construct its dual cone $|\Sigma_{\mathbb{P}^5, T_a, T_b}|^\vee$. Each point of this dual cone will correspond to a monomial on X_{tot}. Pick the eight points corresponding to $u_2 Q_{1,\lambda} + u_1 Q_{2,\lambda}$.

These eight points give eight rays $u_{\rho_0}, \ldots, u_{\rho_5}, u_{\tau_1}, u_{\tau_2}$ corresponding to a vector bundle $X_1 = \mathcal{O}_{X_{LT}}(-D_a) \oplus \mathcal{O}_{X_{LT}}(-D_b)$ with section $Q_{1,\lambda} \oplus Q_{2,\lambda}$. Quotienting by the bundle coordinates gives a fan that corresponds to X_{LT}.

Relating to Batyrev-Borisov

We compute the Batyrev-Borisov mirror and, similarly to above, build a vector bundle X_2 on it such that $p_{1,\lambda} \oplus p_{2,\lambda}$ is a section. We then obtain 14 rays $u_{\rho_0}, \ldots, u_{\rho_{11}}, u_{\tau_1}, u_{\tau_2}$. Next we find a chamber in the secondary fan of X_2 that partially compactifies X_1.

Proof of Theorem

By finding the triangulations associated to the chambers corresponding to X_2 and to the partial compactification of X_1, we can compute the associated ideals $\mathcal{I}, \mathcal{J}$ for either of them. A small computation allows to use Corollary (†), which completes the proof of the main Theorem.

Direct generalisation

This all motivates to try and do the same procedure in other dimensions. So let

$$Q_{1,n} = x_1^n + \cdots + x_n^n - x_{n+1} \ldots x_{2n}, \quad Q_{2,n} = x_{n+1}^n + \cdots + x_{2n}^n - x_1 \ldots x_n.$$

We want to try and find a Libgober-Teitelbaum style mirror to $Z(Q_{1,n}, Q_{2,n}) \subseteq \mathbb{P}^{2n-1}$.

Proposed mirror

Doing the analogous computations to the Libgober-Teitelbaum construction (which is the case $n = 3$) gives a mirror candidate $Z(Q_{1,n}, Q_{2,n}) \subseteq \mathbb{P}^{2n-1} / G_n$, where $G_n \cong (\mathbb{Z} / n\,\mathbb{Z})^{2(n-2)} \times (\mathbb{Z} / n^2\,\mathbb{Z})$.

$n = 2$

Theorem

Let $Q_1 = x_1^2 + x_2^2 - x_3x_4$, $Q_2 = x_3^2 + x_4^2 - x_1x_2$ *and let* $p_1 = x_1^2x_5^2 + x_2^2x_6^2 - x_3x_4x_5x_6$, $p_2 = x_3^2x_7^2 + x_4^2x_8^2 - x_1x_2x_7x_8$. *We define the group* $G_4 \subseteq PGL(3,\mathbb{C})$ *given by the four automorphisms*

$$\mathrm{diag}(1,1,1,1),\ \mathrm{diag}(\zeta_8, -\zeta_8, -\zeta_8^{-1}, \zeta_8^{-1}),$$

$$\mathrm{diag}(\zeta_4, \zeta_4, \zeta_4^{-1}, \zeta_4^{-1}),\ \mathrm{diag}(\zeta_8^3, -\zeta_8^3, -\zeta_8^{-3}, \zeta_8^{-3}),$$

where ζ_k *is a primitive* k^{th} *root of unity.*
The Batyrev-Borisov mirror to $Z(Q_1, Q_2) \subseteq \mathbb{P}^3$ *can be given as the zero locus* $Z_2 = Z(p_1, p_2) \subseteq \mathcal{X}_{BB}$, *where* $\mathcal{X}_{BB}$ *is the relevant toric stack. Consider* $\mathcal{V}_2 := Z(Q_1, Q_2) \subseteq [(\mathbb{C}^4 \setminus \{0\})/(\mathbb{C}^* \times G_4)]$. *Then*

$$\mathrm{D^b}(\mathrm{coh}\,\mathcal{V}_2) \simeq \mathrm{D^b}(\mathrm{coh}\,\mathcal{Z}_2).$$

$n \geq 4$

For $n \geq 4$, the variety $Z(Q_{1,n}, Q_{2,n}) \subseteq \mathbb{P}^{2n-1}$ is singular. This in particular poses a problem when relating the category of singularities to the derived category of coherent sheaves. So this seems like a dead end...

A way out!

Whenever singularities are the problem stopping us from advancing, a good instinct is to look for resolutions. And indeed, the notion of categorical resolution helps us in this case!

What is a categorical resolution?

Definition

Let $\tilde{D}$ be the homotopy category of a homologically smooth and proper pre-triangulated dg-category. A pair of exact functors

$$F : \tilde{D} \to D, G : D^{perf} \to \tilde{D}$$

is a categorical resolution of singularities if G is left adjoint to F and the natural morphism of functors $Id_{D^{perf}} \to FG$ is an isomorphism. We say such a resolution is crepant if G is also right adjoint to F.

A result

Theorem (M.,in preparation)

Let $n \geq 2$, $\lambda^{2n} \neq 0, n^{2n}$. Consider the 2 polynomials

$$Q_{1,n,\lambda} = x_1^n + \cdots + x_n^n - \lambda x_{n+1} \ldots x_{2n},$$

$$Q_{2,n,\lambda} = x_{n+1}^n + \cdots + x_{2n}^n - \lambda x_1 \ldots x_n.$$

Then there is a group $PGL(2n-1, \mathbb{C}) \supseteq G_n \cong (\mathbb{Z}/n\mathbb{Z})^{2(n-2)} \times (\mathbb{Z}/n^2\mathbb{Z})$ such that the hypersurfaces $\{Q_{i,n,\lambda} = 0\}$ are preserved under the action of G_n on $\mathbb{P}^{2n-1}$. Let $\mathcal{Z}_n = Z(Q_{1,n}, Q_{2,n}) \subseteq \left[(\mathbb{C}^{2n} \setminus \{0\})/(\mathbb{C}^* \times G_n)\right]$ and let $\mathcal{Y}_n$ be a Batyrev-Borisov mirror to $Z(Q_{1,n}, Q_{2,n}) \subseteq \left[(\mathbb{C}^{2n} \setminus \{0\})/\mathbb{C}^*\right]$. Then there is a categorical resolution

$$\mathrm{D^b}(\mathrm{coh}\,\mathcal{Y}_n) \to \mathrm{D^b}(\mathrm{coh}\,\mathcal{Z}_n)$$

Sketch of proof

The proof is based on results of [**FK18**].

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{abs}}([U_p, G, w]) & \xleftrightarrow{\simeq} & \mathrm{D}^{\mathrm{abs}}([U_q, G, w]) \\ \uparrow\downarrow & & \updownarrow\cong \\ \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Y_n) & & \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, Z_n) \end{array}$$

Steps in proof

1. Prove there exist chambers σ_p, σ_q in a common GKZ-fan partially compactifying $\mathcal{Z}_n, \mathcal{Y}_n$ respectively.
2. Prove $\mathrm{D}^{abs}([U_p, G, w]) \simeq \mathrm{D}^{abs}([U_q, G, w])$ using results of [**FK18**].
3. Prove $\mathrm{D}^{abs}([U_q, G, w]) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}_n)$ using results of [**FK18**].
4. Show $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathcal{Z}_n)$ is homologically smooth and dg-proper.

Thank you for your attention ☺