Counting bitangents of plane quartics — tropical, real and arithmetic

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Tropicalized plane curves

Field: $K = k\{\{t\}\}$, i.e., Puiseux series over a field k with characteristic not 2. The tropicalization map

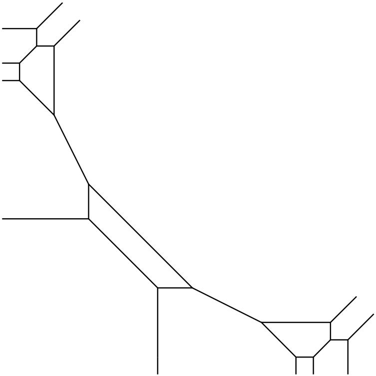
$$(x, y) \mapsto (-\operatorname{val}(x), -\operatorname{val}(y)).$$

The plane quartic $V(f)$ for

$$\begin{aligned} f(x,y) = \quad & t^{36}x^4 + t^{18}x^3y + t^2x^2y^2 + t^{18}xy^3 + t^{36}y^4 + t^{23}x^3 \\ & + t^6x^2y + t^6xy^2 + t^{23}y^3 + t^{12}x^2 + xy + t^{12}y^2 + t^2x \\ & + t^2y + 1. \end{aligned}$$

Tropicalization of a plane quartic

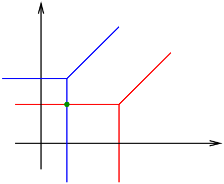
The tropicalization of $V(f)$:

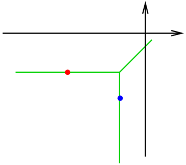


The tropical dual $\mathbb{R}^2$

$$(\mathbb{R}^2)^\vee \to \mathbb{R}^2 :$$
$$\text{line with center at } (x,y) \mapsto (-x,-y)$$

respects incidence relations: $(L_1 \cap L_2)^\vee = L_1^\vee \cup L_2^\vee$.



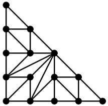


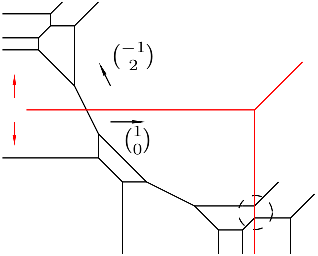
However, for better drawing of bitangents, we use $(\mathbb{R}^2)^\vee \to \mathbb{R}^2$: line centered at $(x,y) \mapsto (x,y)$.

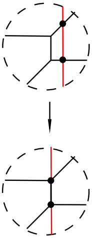
Bitangents to quartics

- A plane quartic has 28 bitangents (Plücker, 1834).
- A tropical plane quartic may have infinitely many bitangents.
- We identify: $L_1 \sim L_2$ if we can continously move L_1 to L_2 while maintaining bitangency.

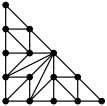
Example

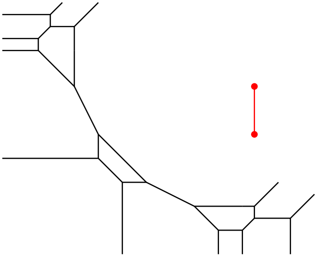




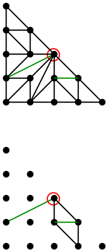


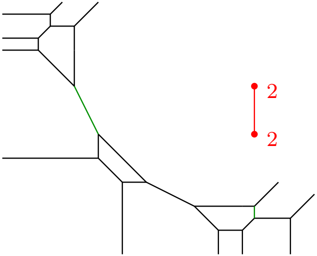
Example





Example





For $q \in \mathbb{C}\{\{t\}\}[x, y]$ a (generic) quartic polynomial with $\mathrm{Trop}(V(q)) = C$, exactly 2 of the 28 bitangent lines to $V(q)$ tropicalize to the tropical line with vertex the upper red point, exactly 2 to the one with vertex the lower red point, and none to a point in the interior of the red segment.

Bitangents to quartics

- A plane quartic has 28 bitangents (Plücker, 1834).
- A tropical plane quartic may have infinitely many bitangents.
- We identify: $L_1 \sim L_2$ if we can continously move L_1 to L_2 while maintaining bitangency.
- Then: A tropical quartic in $\mathbb{R}^2$ has 7 bitangent classes (Baker, Len, Morrison, Pflueger, Ren, 2014).
- If the skeleton of the tropical quartic is a K_4, then each bitangent class has 4 lifts (Chan, Jiradilok, 2015).
- For any generic *smooth* tropical quartic in $\mathbb{R}^2$, each bitangent class has 4 lifts (Len, M, 2017).

Real bitangents

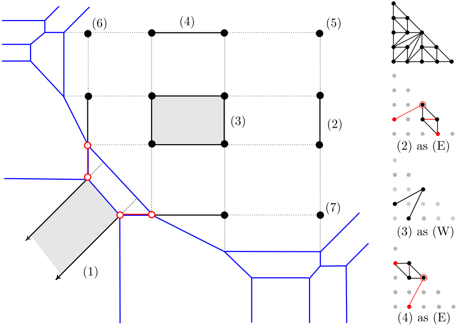
- A real plane quartic can have 4, 8, 16 or 28 real bitangents (depending on the ovals).

Theorem (Cueto-M, 2020)

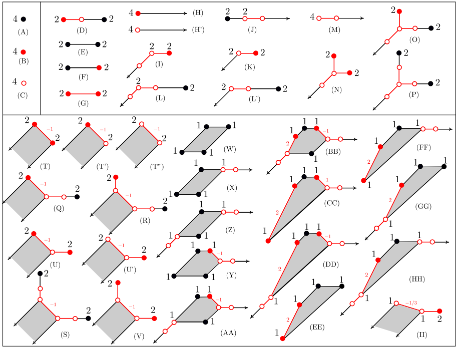
A tropical bitangent class of a generic smooth tropical quartic in $\mathbb{R}^2$ has either 0 or 4 real lifts.

Techniques of proof: Combinatorial classification and local lifting computations.

Combinatorics: Example



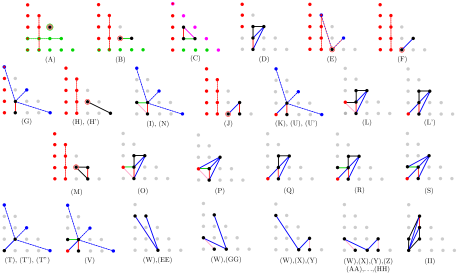
Combinatorial Classification

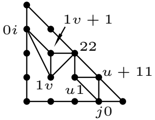


40 **shapes** for bitangent classes, up to symmetry.

The black cells of each bitangent class miss the curve, whereas the red ones lie on it. The unfilled vertices indicate points that must be vertices.

Relevant parts of the dual subdivision



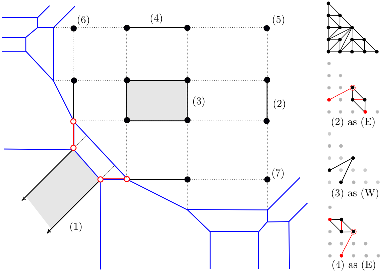


Lifting conditions

Shape	Lifting conditions
(A)	$(-1)^i (s_{1v} s_{1,v+1})^i s_{0i} s_{22} > 0$ and $(-1)^j (s_{u1} s_{u+1,1})^j s_{j0} s_{22} > 0$
(B)	$(-1)^{i+1} (s_{1v} s_{1,v+1})^{i+1} s_{0i} s_{21} > 0$ and $(-1)^{j+1} {s_{21}}^{j+1} {s_{31}}^j s_{1v} s_{1,v+1} s_{j0} > 0$
(C)	If $j = 2$: $(-1)^i {s_{11}}^i s_{0i} s_{20} > 0$ and $(-1)^k {s_{21}}^k s_{k,4-k} s_{20} > 0$ If $j = 1, 3$: $(-1)^{i+1} {s_{11}}^{i+1} s_{21} s_{0i} s_{j0} > 0$ and $(-1)^k {s_{21}}^{k+1} s_{11} s_{k,4-k} s_{j0} > 0$
(D),(L)	$(-1)^i (s_{10} s_{11})^i s_{0i} s_{22} > 0$
(E),(F),(J)	$(-1)^i (s_{1v} s_{1,v+1})^i s_{0i} s_{20} > 0$
(G)	$(-1)^i (s_{10} s_{11})^i s_{0i} s_{p,4-p} > 0$
(H)	$(-1)^{i+1} (s_{1v} s_{1,v+1})^{i+1} s_{0i} s_{21} > 0$ and $-s_{1v} s_{1,v+1} s_{21} s_{40} > 0$
(I),(K)	$(-1)^i (s_{10} s_{11})^i s_{0i} s_{p,4-p} > 0$
(M)	$(-1)^{i+1} (s_{1v} s_{1,v+1})^{i+1} s_{0i} s_{21} > 0$ and $s_{1v} s_{1,v+1} s_{30} s_{31} > 0$
(N)	$-s_{01} s_{10} s_{11} s_{p,4-p} > 0$
(O),(P)	$-s_{01} s_{10} s_{11} s_{22} > 0$
(Q),(R),(S)	$s_{00} s_{22} > 0$
(T),(U),(V)	$s_{00} s_{p,4-p} > 0$
rest	no conditions

Bitangent classes and their real-lifting sign conditions.

Example for Lifting



Negative signs	Lifting tropical bitangents	Total # of real lifts	Topology
—	1, 3	8	2 non-nested ovals
s_{31}	1, 2, 3, 7	16	3 ovals
s_{13}, s_{31}, s_{22}	3	4	1 oval
s_{13}, s_{31}	1, . . . , 7	28	4 ovals

Corollaries

Corollary

A tropical bitangent class is a tropical convex set.

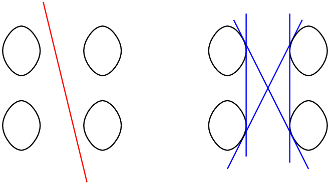
Corollary

Any real lift of a tropical bitangent to a generic smooth quartic is **totally real**, *i.e. the points of tangency are also real.*

Questions

- What are the tropicalizations of real quartics which have real, but not totally real, bitangents?
- How can we show that altogether, there are 4, 8, 16 or 28 real lifts? (Geiger-Panizzut)
- What about bitangents of tropical quartics which are not in $\mathbb{R}^2$, but in a different model of the tropical plane?

Avoidance loci



Theorem (Kummer, Vinnikov,...)

Every connected component of the avoidance locus of a smooth real quartic contains precisely 4 bitangents in its closure.

Theorem (Payne-Shaw-M (in progress))

A tropical bitangent class which is liftable to the reals is (roughly) the tropicalization of a connected component of the avoidance locus.

Further perspective: arithmetic counts

Definition

Let k be a field. The Grothendieck-Witt ring $\mathrm{GW}(k)$ contains all formal sums of isomorphism classes of quadratic forms $V \times V \to k$ over k.

Example

For $k = \mathbb{C}$,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

since

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

but not for $k = \mathbb{R}$.

Arithmetic counts

Associates an element in $\mathrm{GW}(k)$ to a geometric object to be counted.

There exist arithmetic counts of

- ... lines in cubic surfaces (Kass-Wickelgren),
- ... plane curves satisfying point conditions (Levine),
- ... bitangents of a quartic (Larson-Vogt).

Insert

- $k = \mathbb{C} \rightsquigarrow \dim \equiv$ "number"
- $k = \mathbb{R} \rightsquigarrow$ other meaningful real invariants (e.g. Welschinger invariants)

Tropical geometry plays intermediary role, e.g. quantum counts of plane curves.

Bitangents

Tropical curves
The dual plane
Bitangents
Real bitangents
Arithmetic

Bitangent to quartics

Theorem (Payne-Shaw-M (in progress))

For any field of characteristic $\neq 2$, a tropical bitangent class to a smooth tropicalized quartic has either 0 or 4 lifts. We give all lifting conditions.

Conjecture (Payne-Shaw-M (in progress))

The element in $GW(k)$ that belongs to the 4 bitangents in an equivalence class can be determined with tropical methods and is a Laurent monomial in the coefficients of the quartic.

bitangent	
(II)a	$\langle -2\rangle, \langle 2\rangle, \langle 2 s_{12} s_{00} s_{13} a_{13} s_{11} a_{11}\rangle, \langle -''\rangle$
(II)b	$\langle 1\rangle, \langle -1\rangle, \langle 2\rangle, \langle -2\rangle$
(II)c	$\langle -1\rangle, \langle 1\rangle, \langle 1\rangle, \langle 1\rangle$
(A)a	$\langle a_{1k}^{k+1} a_{l1}^{l+1} a_{1k+1}^{k} a_{l+11}^{l} s_1 s_2\rangle, \langle -''\rangle, \langle ''\rangle, \langle -''\rangle$
(A)b	$\langle 2 a_{1k}^{k} a_{1k+1}^{k+1} a_{m3-m}^{3-m} a_{m+13-m-1}^{3-m-1} s_1 s_2^2\rangle, \langle -''\rangle, \langle ''\rangle, \langle -''\rangle$
(D)a	$\langle s_{12} 2 \vert a_{10} \vert a_{22} s_1\rangle, \langle -''\rangle, \langle a_{22} a_{10} s_1\rangle, \langle -''\rangle$
(D)b	$\langle 2 s_1^2\rangle, \langle -''\rangle, \langle 2 s_1^2\rangle, \langle -''\rangle$
(D)c	$\langle a_{31} a_{12} s_1\rangle, \langle -''\rangle, \langle 2 a_{02} a_{21} \vert a_{31} a_{11} \vert s_1\rangle, \langle -''\rangle$
(E)a	$\langle (-1)^k a_{21} a_{1k}^{k} a_{1k+1}^{k+1} s_1\rangle, \langle -''\rangle, \langle (-1)^k a_{20} a_{31} a_{30} a_{1k}^{k} a_{1k+1}^{k+1} s_1\rangle, \langle -''\rangle$
(E)b	$\langle (-1)^k a_{22} a_{1k}^{k+1} a_{1k+1}^{k} s_1\rangle, \langle -''\rangle, \langle ''\rangle, \langle -''\rangle$
(E)c	$\langle (-1)^{2-k} 2 s_1^2 a_{10} a_{k3-k}^{2-k} a_{k+13-k-1}^{3-k}\rangle, \langle -''\rangle, \langle ''\rangle, \langle -''\rangle$
(F)a	$\langle (-1)^k a_{21} a_{1k}^{k} a_{1k+1}^{k+1} s_1\rangle, \langle -''\rangle, \langle (-1)^k a_{20} a_{1k}^{k+1} a_{1k+1}^{k} s_1\rangle, \langle -''\rangle$
(F)b	$\langle (-1)^k a_{22} a_{1k}^{k+1} a_{1k+1}^{k} s_1\rangle, \langle -''\rangle, \langle 2''\rangle, \langle -2''\rangle$
(F)c	$\langle (-1)^{2-k} 2 s_1^2 a_{10} a_{k3-k}^{2-k} a_{k+13-k-1}^{3-k}\rangle, \langle -''\rangle, \langle 2''\rangle, \langle -2''\rangle$
(G)a I	$\langle 2 s_{12} a_{22} \vert a_{11} \vert s_1\rangle, \langle -''\rangle, \langle 2 s_{21} a_{11} \vert a_{22} \vert s_1\rangle, \langle -''\rangle$
(G)a II	$\langle 2 s_{11} s_{03} s_1\rangle, \langle -''\rangle, \langle 2 s_{04} s_{12} s_1\rangle, \langle -''\rangle$
(G)a III	$\langle 2 s_{11} s_{21} s_1\rangle, \langle -''\rangle, \langle 2 s_{40} s_{30} s_1\rangle, \langle -''\rangle$
(G)b I	$\langle 2 s_{02} s_{21} s_1\rangle, \langle -''\rangle, \langle 2 a_{02} s_{21} \vert a_{11} \vert s_1\rangle, \langle -''\rangle$
(G)b II	$\langle 2 s_{04} a_{21} \vert a_{13} \vert s_1\rangle, \langle -''\rangle, \langle 2 a_{04} s_{21} \vert a_{12} \vert s_1\rangle, \langle -''\rangle$
(G)b III	$\langle 2 s_{00} a_{21} \vert a_{11} \vert s_1\rangle, \langle -''\rangle, \langle 2 a_{00} s_{21} \vert a_{10} \vert s_1\rangle, \langle -''\rangle$
(G)c	$\langle s_1^2\rangle, \langle s_1^2\rangle, \langle -s_1^2\rangle, \langle -s_1^2\rangle$
(H)a	$\langle a_{1k}^{k} a_{1k+1}^{k+1} a_{21} s_1 s_2^2\rangle, \langle -''\rangle, \langle ''\rangle, \langle -''\rangle$
(H)b	$\langle 2 a_{k3-k}^{3-k-1} a_{k+13-k-1}^{3-k} a_{11} s_2 s_1^2\rangle, \langle -''\rangle, \langle ''\rangle, \langle -''\rangle$
(N)a I	$\langle 2 s_{12} a_{11} s_1\rangle, \langle -''\rangle, \langle 2 s_{21} a_{11} s_1\rangle, \langle -''\rangle$
(N)a II	$\langle 2 s_{03} a_{04} s_1\rangle, \langle -''\rangle, \langle 2 s_{12} a_{04} s_1\rangle, \langle -''\rangle$
(N)a III	$\langle 2 s_{21} a_{40} s_1\rangle, \langle -''\rangle, \langle 2 s_{30} a_{40} s_1\rangle, \langle -''\rangle$
(N)b I	$\langle 2 s_1^2\rangle, \langle 2 s_1^2\rangle, \langle 2 s_{12} a_{02} s_1\rangle, \langle -''\rangle$
(N)b II	$\langle 2 s_1^2\rangle, \langle 2 s_1^2\rangle, \langle 2 a_{21} a_{04} \vert a_{31} \vert s_1\rangle, \langle -''\rangle$
...	...