Combinatorial mutations and deformations of dimer models

(joint work with A. Higashitani)

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Mirror symmetry for Fano varieties (cf. Coates - Corti - Galkin - Golyshev - Kasprzyk)

- X: n-dimensional Fano variety
- X is expected to correspond to a certain Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

The regularised quantum period of X = The classical period π_f of f

↖ mirror partner of X

- There are so many mirror partners of X.

⇝ want to understand the relationship between mirror partners.

Mutations of f [Akhtar - Coates - Galkin - Kasprzyk]

$$f \overset{\text{mut.}}{\sim} g \implies \pi_f = \pi_g$$

Thus, mutating a mirror partner f of X, we can obtain a lot of mirror partners of X.

$$
\begin{tikzcd}
\text{Laurent polynomials :} & f \arrow[rr, leftrightarrow, "\text{mutation}"] \arrow[d, rightsquigarrow] & & g \arrow[d, rightsquigarrow] \\
\text{Newton polytopes :} & P := \mathrm{Newt}(f) \arrow[rr, leftrightarrow, "\text{combinatorial mutation}"] \arrow[d, rightsquigarrow] & & \mathrm{Newt}(g) =: Q \arrow[d, rightsquigarrow] \\
\text{toric varieties :} & X_P \arrow[rr, leftrightarrow, "\text{``deformation equiv.'' in the following sense}"'] & & X_Q
\end{tikzcd}
$$

Today (especially $d = 2$)

<u>Thm</u> [Ilten]

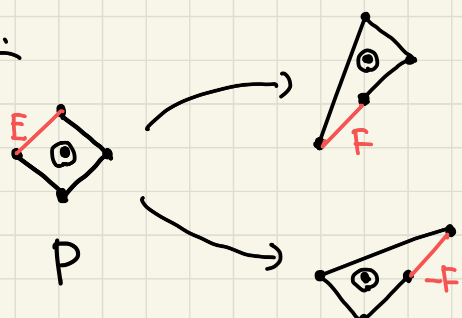
There exists a flat family $\mathcal{X} \to \mathbb{P}^1$ s.t. $\mathcal{X}_0 \cong X_P$ and $\mathcal{X}_\infty \cong X_Q$.

Combinatorial mutations (dim 2)

- P : lattice polygon , $0 \in P$
- E : edge of P
- $w \in \mathbb{Z}^2$: primitive inner normal vector for the edge E.
- $h_{max} := \max\{\langle w, v\rangle \mid v \in P\}$, $h_{min} = \min\{\langle w, v\rangle \mid v \in P\}$
- $v_E \in \mathbb{Z}^2$: primitive lattice vector with $\langle w, v_E\rangle = 0$, $F := \mathrm{Conv}\{0, v_E\}$

Roughly ,
the combinatorial mutation of P w.r.t. w and F, denoted by $\mathrm{mut}_w(P, F)$,
is given by removing $-h_{min}$ primitive segments from E and
adding $h_{max} F$ to the "opposite side".

e.g.



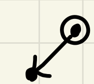
$\mathrm{mut}_w(P, F)$

unimodularly equiv.

$\mathrm{mut}_w(P, -F)$

$w = (1, -1)$, $v_E = (-1, -1)$

$h_{max} = 1$, $h_{min} = -1$

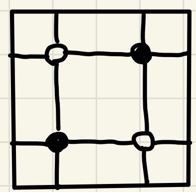
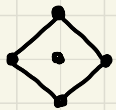
$F = \mathrm{Conv}\{(0,0), \underset{v_E}{\underset{\shortparallel}{(-1,-1)}}\} =$ 

What is a dimer model? (cf. Hanany et. al., Gulotta, Ishii-Ueda, etc.)

- A dimer model Γ is a finite bipartite graph described on the real 2-torus T.

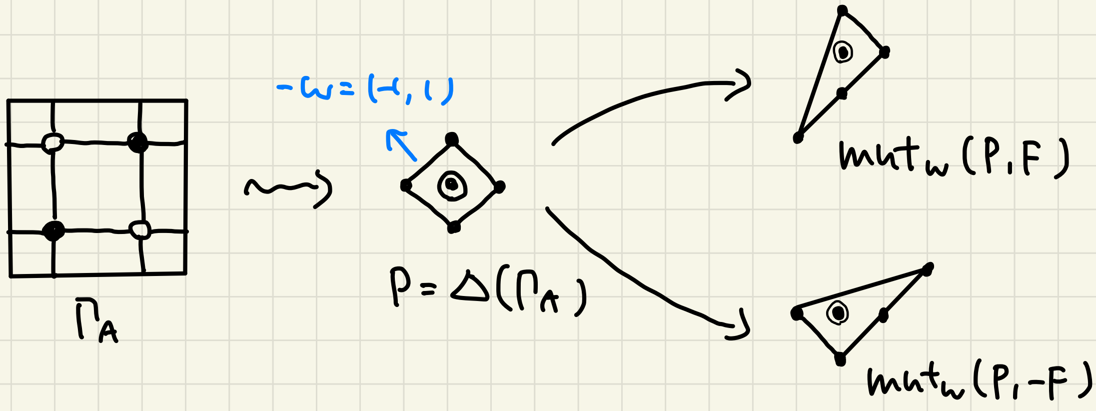
- We can obtain the lattice polygon $\Delta(\Gamma)$ from a dimer model Γ.

e.g.

Γ_A :  $\rightsquigarrow$  : $\Delta(\Gamma_A)$

- For any lattice polygon P, there is a dimer model Γ s.t. $P = \Delta(\Gamma)$

Remark Such a dimer model is not unique.



$\exists\, \Gamma_B$ s.t.

$\Delta(\Gamma_B) = \mathrm{mut}_w(P, F)$

$\exists\, \Gamma'_B$ s.t.

$\Delta(\Gamma'_B) = \mathrm{mut}_w(P, -F)$

where $w = (1,-1)$, $F = \mathrm{conv}\{(0,0), (-1,-1)\}$

<u>Question</u>: Can we construct Γ_B (or Γ'_B) from Γ_A ?
(Even if we do not know the shape of $\mathrm{mut}_w(P, \pm F)$)

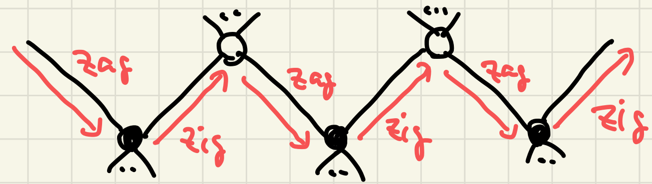
<u>Today</u>: I will introduce the <u>deformation of dimer models</u>
to answer this question. ← different from the mutation of DM
(this does not change $\Delta(\Gamma)$)

How to construct $\Delta(\Gamma)$?

Can lift on the univ. cover $\mathbb{R}^2 \to T$

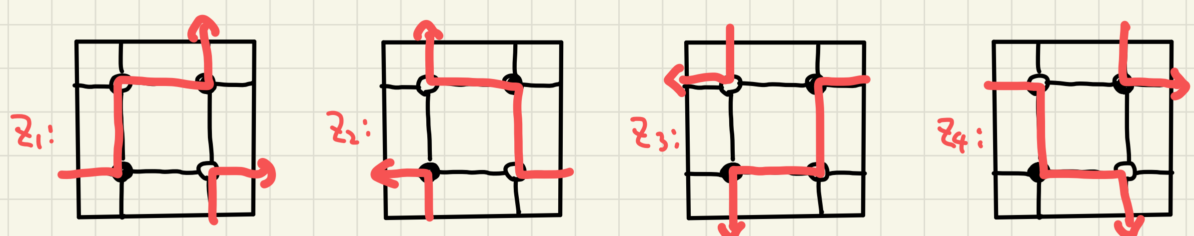
Def A path z on a dimer model is called a zigzag path if it makes a maximum turn to the right on ○

—— " —— to the left on ●



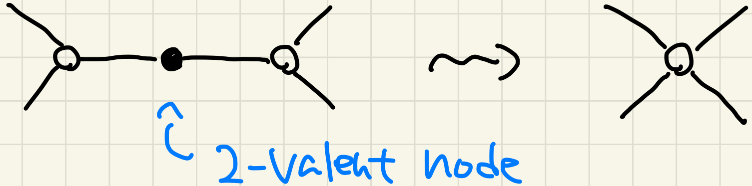
the slope of z

• z is a 1-cycle on T $\rightsquigarrow$ determines the element $[z] \in H_1(T) \cong \mathbb{Z}^2$



$[z_1] = (1,1)$ $[z_2] = (-1,1)$ $[z_3] = (-1,-1)$ $[z_4] = (1,-1)$

Remark



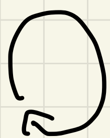
This does not change the slope.

- In the rest, we assume that $\nexists$ 2-valent nodes in a dimer model.

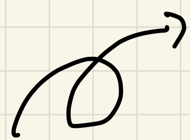
  $\Rightarrow$ <u>length</u> of z : $\ell(z) = \#(\text{zigs}) + \#(\text{zags})$

- We also assume that a dimer model is <u>consistent</u>.

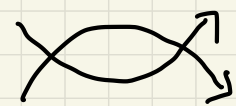
  <u>i.e.</u> A dimer model does not have zigzag paths with



homologically trivial



self-intersection on the universal cover



intersect each other on the universal cover in the same direction more than once.

For a consistent dimer model Γ

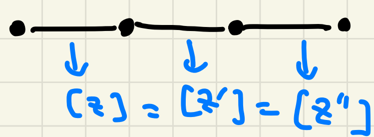
- consider $[z] =: (a, b) \in \mathbb{Z}^2 \rightsquigarrow (a,b)/\sqrt{a^2+b^2} \in S^1$

  $\rightsquigarrow$ Define the cyclic order of slopes of z.z. paths along S^1

- define the <u>zigzag polygon</u> $\Delta(\Gamma)$ satisfying the following:

  - {outer normal vectors of side segments of $\Delta(\Gamma)$}

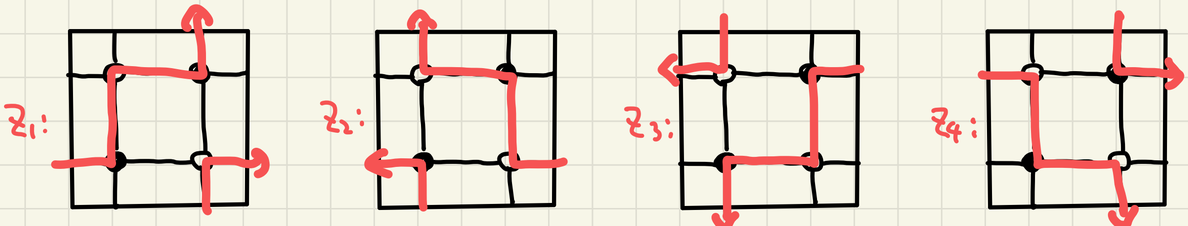
    = {slopes of zigzag paths of Γ}



  - The cyclic order of the slopes along $\Delta(\Gamma)$

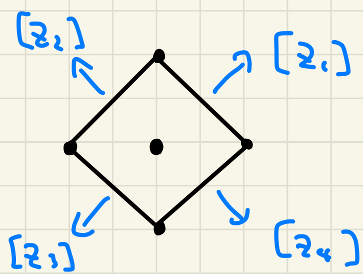
    = the above cyclic order.

Ex.



$[z_1] = (1,1)$ $[z_2] = (-1,1)$ $[z_3] = (-1,-1)$ $[z_4] = (1,-1)$

$\Delta(\Gamma_A)$:



- $\Delta(\Gamma)$ is determined up to translations.

Deformations of dimer models

Def A zigzag path z is called <u>type I</u> if

$\tilde{z}$ intersects with any other zigzag paths on $\mathbb{R}^2$ at most once.

On the universal cover.

Def (Deformation data)

(1) $\mathcal{Z}_v = \{z_1, \cdots, z_n\}$: type I zigzag paths with $[z_1] = \cdots = [z_n] =: v$

$\overset{\text{Lem}}{\Longrightarrow} \ell(z_1) = \cdots = \ell(z_n)$

(2) Choose $r > 0$ zigzag paths in $\mathcal{Z}_v$ with $h := \dfrac{\ell(z_i)}{2} - r > 0$

(3) Take non-negative integers $q = (q_1, \cdots, q_r) \in \mathbb{Z}_{\geq 0}^r$

s.t. $h = q_1 + \cdots + q_r$

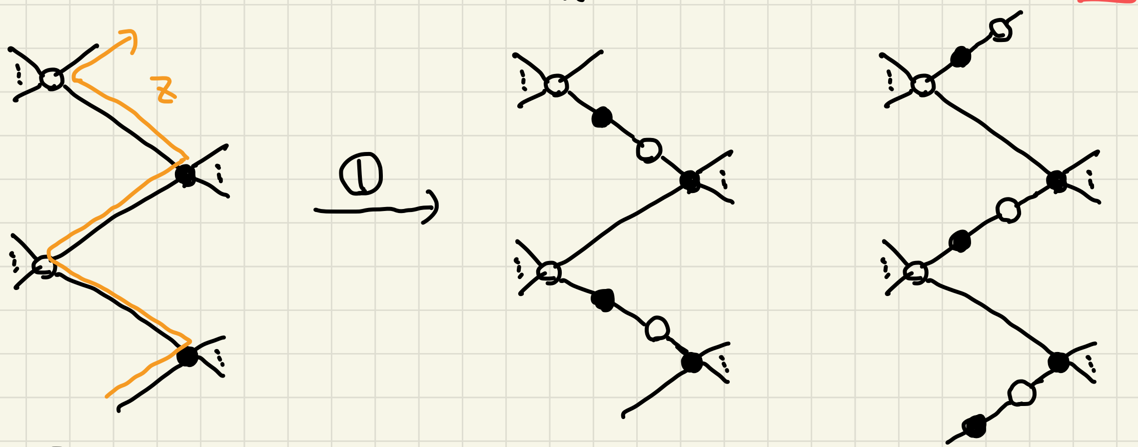
deformation weight.

Today, I will discuss the case of $r = 1$, thus $h = \frac{\ell(z)}{2} - 1 = q$

Def (The zig-deformation and zag-deformation)

For the chosen zigzag path $z \in \mathcal{Z}_v$ on a cosistent dimer model Γ,

① insert q black nodes and q white nodes in each zig (resp. zag) of z



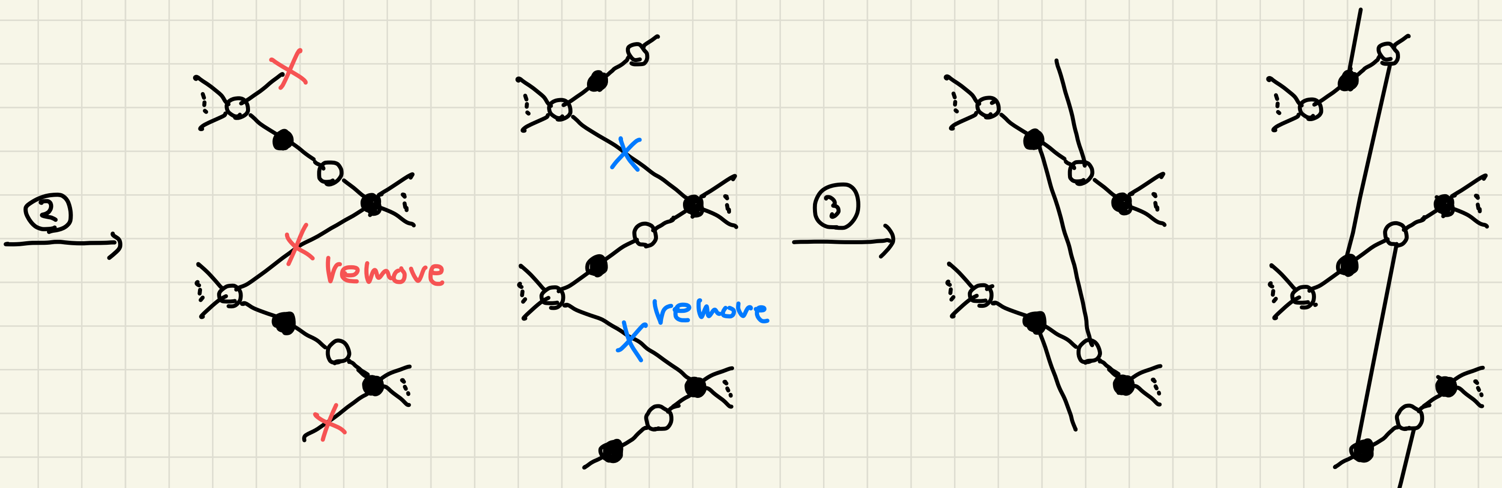
$[z] = v$

zig ver. (if $q = 1$)

zag ver. (if $q = 1$)

② remove all zags (resp. zigs) of Z

③ Create the new zigzag path Z' so that $[Z'] = -\nu$ and zigzag paths passing through zigs (resp. zags) of Z as zag (resp. zig) will be preserved.



zig ver. (if $q=1$) | zag ver. (if $q=1$) | zig ver. (if $q=1$) | zag ver. (if $q=1$)

④ If $\exists$ 2-valent nodes, then I contract them.

We will denote the resulting dimer model by $V_{\hbar}^{zig}(\Gamma, z)$ (resp. $V_{\hbar}^{zag}(\Gamma, z)$)

and call the zig-deformation (resp. zag-deformation) of Γ

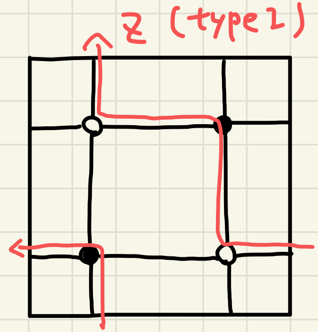
Thm (Higashitani - N.)

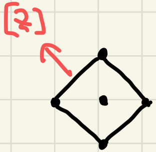
The dimer models $V_{\hbar}^{zig}(\Gamma, z)$ and $V_{\hbar}^{zag}(\Gamma, z)$ are consistent.

Rem If $r > 1$, then we have to consider additional operations on $V_{\hbar}^{zig}(\Gamma, \{z_1, \cdots, z_r\})$ (or $V_{\hbar}^{zag}(\Gamma, \{z_1, \cdots, z_r\})$) to obtain the same result.

Example (zig version)

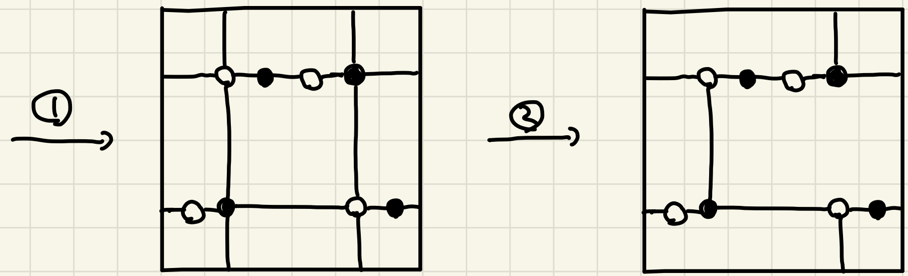
Γ_A

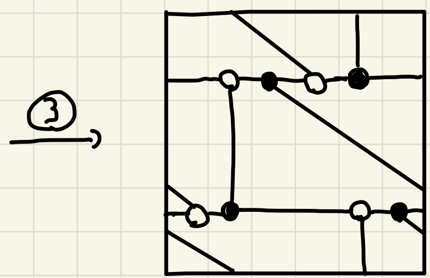




$\ell(z) = 4, \quad h = 1 = \hbar$

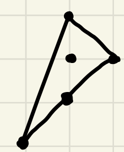
$v = [z] = (-1, 1)$





$V^{zig}_{\hbar}(\Gamma_A, z)$

zigzag polygon

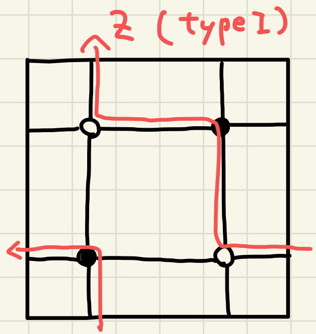


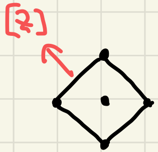
$\|$

$\mathrm{mut}_{(-v)}(P, F)$

Example (zag version)

Γ_A

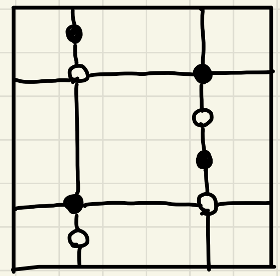




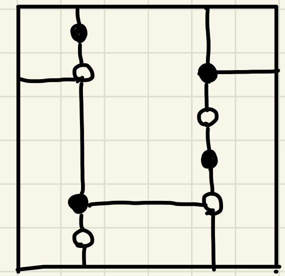
$\ell(z) = 4, \quad h = 1 = \mathfrak{h}$

$v = [z] = (-1, 1)$

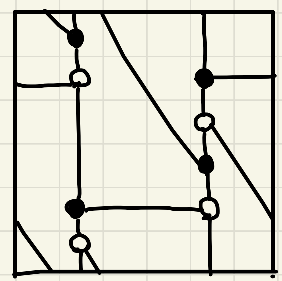
$\xrightarrow{①}$



$\xrightarrow{②}$

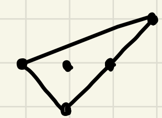


$\xrightarrow{③}$



$V^{zag}_{\mathfrak{h}}(\Gamma_A, z)$

zigzag polygon



$\parallel$

$\mathrm{mut}_{(-v)}(P, -F)$

Thm (Higashitani - N.)

Γ: consistent dimer model. z: type I zigzag path

We can fix the origin O so that $O \in \Delta(\Gamma)$ and

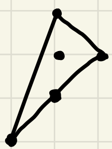
Deformation data	Mutation data
$[z] = v$	$w = -v$
$r = 1$	$h_{min} = -r$
$q = \ell(z)/2 - 1$	$h_{max} = q$

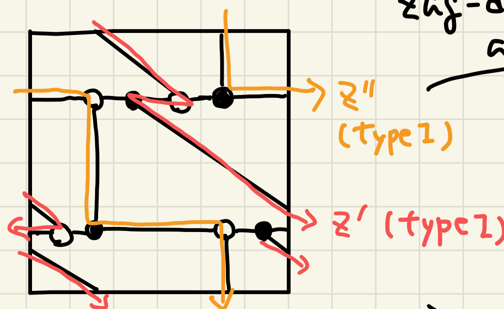
Then, we have
$$\Delta(V_q^{zig}(\Gamma, z)) = \mathrm{mut}_w(\Delta(\Gamma), F)$$
$$\Delta(V_q^{zag}(\Gamma, z)) = \mathrm{mut}_w(\Delta(\Gamma), -F)$$

Rem As before, if $r > 1$, we need additional operations to obtain the same result.

Rem

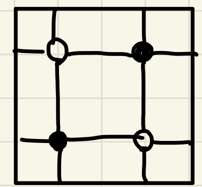
If we take an appropriate deformation data, then we see that these deformations are mutually inverse.





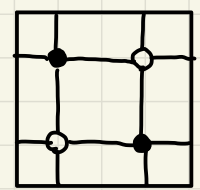
$V_{h}^{\mathrm{zig}}(\Gamma_A, Z)$

zag-deformation at z'



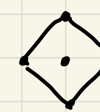
Γ_A

zag-deformation at z''



Γ_A^{op}

$\Delta(\Gamma_A) = \Delta(\Gamma_A^{op})$



$Z_{(1,-1)} = \{z', z''\}$