Extra-twisted connected sum G_2-manifolds

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These slides available at
`http://people.bath.ac.uk/jlpn20/xxtcs2.pdf`

Overview

Topic: Riemannian metrics on 7-manifolds with holonomy group G_2

Use a construction starting from complex algebraic geometry to exhibit examples with interesting topological and geometric properties, such as disconnected moduli space of holonomy G_2 metrics.

1. Background
2. Main results
3. Twisted connected sums
4. The matching problem
5. Extra-twisted connected sums
6. Invariants

1. Context
Berger's list

The *holonomy group* of a Riemannian n-manifold is the subgroup of $O(n)$ generated by parallel transport around closed loops.

Theorem (Berger)

The only possible holonomy groups of complete, simply connected Riemannian manifolds that are neither a product nor a symmetric space are

Holonomy group	dim	*Parallel spinors*	*Type*
$SO(n)$	n		*generic*
$U(k)$	$2k$		*Kähler*
$SU(k)$	$2k$	2	*Calabi-Yau (Ricci-flat)*
$Sp(\ell)$	4ℓ	$\ell + 1$	*hyper-Kähler (Ricci-flat)*
$Sp(\ell) \cdot Sp(1)$	4ℓ		*Quat. Kähler (Einstein)*
G_2	7	1	*exceptional (Ricci-flat)*
$Spin(7)$	8	1	*exceptional (Ricci-flat)*

Holonomy G_2 and G_2-structures

$G_2 = \mathrm{Aut}(\mathbb{O})$, automorphisms of the 8-dimensional octonion algebra.

$G_2 \subset SO(7)$ can also be defined as the stabiliser of a definite 3-form

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \in \Lambda^3(\mathbb{R}^7)^*.$$

Therefore G_2-structure on $M^7 \ \leftrightarrow \ \varphi \in \Omega^3(M)$ pointwise equivalent to φ_0 (an open condition on φ).

A G_2-structure induces a metric. A metric has Hol $\subseteq G_2$ if and only if it is induced by a G_2-structure that is torsion-free, *ie* satisfies

$$d\varphi = d^*\varphi = 0.$$

In particular, φ represents a de Rham cohomology class

$$[\varphi] \in H^3(M).$$

The G_2 moduli space

Let M be a closed 7-manifold. The moduli space

$$\mathcal{M} = \{\text{torsion-free } G_2\text{-structures on } M\}/\mathrm{Diff}(M)$$

is a locally finite quotient of the "Teichmüller space"

$$\mathcal{M}_0 = \{\text{torsion-free } G_2\text{-structures on } M\}/(\text{id component of } \mathrm{Diff}(M))$$

Theorem (Joyce)
$\varphi \to [\varphi]$ *induces a local homeomorphism* $\mathcal{M}_0 \to H^3(M)$.

So $\mathcal{M}$ is an orbifold, but little is understood about its *global* properties.

Existing constructions of closed G_2-manifolds to some extent map out neighbourhood of boundary components whose points correspond to singular or otherwise degenerate G_2-manifolds.

Constructions of closed G_2-manifolds

Closed G_2-manifolds cannot admit continuous symmetries (because $\mathrm{Ric} = 0$). All known examples come from gluing constructions.

- **Joyce (1995)** Orbifold construction
  Resolve singularities of T^7/Γ using QALE Calabi-Yau spaces.
  Many examples, topology slightly complicated to pin down.
- **Kovalev (2003)**, **Corti-Haskins-N-Pacini (2014)**
  Twisted connected sums: glue asymptotically cylindrical Calabi-Yaus$\times S^1$.
  Many examples, topology computable and classification results often applicable, but limited variation.
- **Joyce-Karigiannis (2018)**
  Resolve singularities of $(CY^3 \times S^1)/\mathbb{Z}_2$.
  Not yet any new topological types.
- **Crowley-Goette-N (2018)** Extra-twisted connected sums
  Glue quotients of ACyl CY $\times S^1$ by finite groups.
  Limited (thousands?) but quite varied supply of examples.

2. Main results
Some interesting phenomena

- There exist smooth closed 7-manifolds M_1, M_2 both admitting holonomy G_2 metrics such that M_1 is homeomorphic but not diffeomorphic to M_2.
- There exist smooth closed 7-manifolds M that admit two holonomy G_2 metrics such that the associated torsion-free G_2-structures are not homotopic (*ie* they cannot be connected by any path of G_2-structures, torsion-free or otherwise), even after applying a diffeomorphism of M to one of the G_2-structures.
  In particular, the G_2 moduli space is disconnected.
- There exist smooth closed 7-manifolds M with two holonomy G_2 metrics in different components of the G_2-moduli space, such that the associated torsion-free G_2-structures are nevertheless homotopic.

Ingredients

- Invariants to distinguish
- Supply of examples
- Classification results for 2-connected 7-manifolds

Invariants

For a 2-connected 7-manifold M with $H^4(M;\mathbb{Z})$ torsion-free, the "obvious" algebraic-topological invariants boil down to

- $b_3(M) = b_4(M)$
- The greatest divisor d of the first Pontrjagin class, *ie* $p_1(M) = dx$ for a primitive $x \in H^4(M;\mathbb{Z})$. (Set $d = 0$ if $p_1(M) = 0$, but that never happens for G_2-manifolds.)

The more subtle invariants we need are

- Generalisation of the Eells-Kuiper invariant $\mu(M) \in \mathbb{Z}/N$, where N is the greatest common divisor of 28 and $\frac{1}{4}\mathrm{lcm}(8, d)$.
- $\nu(\varphi) \in \mathbb{Z}/48$ and $\xi \in \mathbb{Z}$ invariant under both diffeomorphisms and homotopies of a G_2-structure φ
- $\bar{\nu}(\varphi) \in \mathbb{Z}$, a refinement of ν in the sense that $\nu(\varphi)$ is the mod 48 reduction of $\bar{\nu}(\varphi) + 24$ for any torsion-free G_2-structure φ.

Classification theorems

We can generate many examples of closed G_2-manifolds using the twisted connected sum and extra-twisted connected sum constructions, and compute many of their topological invariants. In many cases one can arrange that they are 2-connected, and apply the following classification theorems (simplified form when H^4 is torsion-free).

Theorem (Wilkens, Crowley-N)

Let M_1 and M_2 be closed smooth 2-connected manifolds with $H^4(M_i)$ torsion-free, and let φ_i be a G_2-structure on M_i. Then

- *M_1 is homeomorphic to M_2 if and only if $b_3(M_1) = b_3(M_2)$ and $d(M_1) = d(M_2)$.*
- *M_1 is diffeomorphic to M_2 if and only if in addition $\mu(M_1) = \mu(M_2)$ (vacuous if the numerator of $\frac{d(M_i)}{8}$ is coprime to 28).*
- *There is a diffeomorphism $f : M_1 \to M_2$ such that $f^*\varphi_2$ is homotopic to φ_1 if in addition $\nu(\varphi_1) = \nu(\varphi_2)$ and $\xi(\varphi_1) = \xi(\varphi_2)$.
(The condition on ξ is vacuous if $d(M_i)$ divides 224.)*

3. Twisted connected sums
Basic outline

Kovalev (2003), **Corti-Haskins-N-Pacini (2014)**.
Ingredients:

- Closed simply-connected Kähler 3-folds Z_+, Z_-
- $\Sigma_\pm \subset Z_\pm$ anticanonical K3 divisors ($[\Sigma_\pm] = c_1(Z_\pm)$) with trivial normal bundle
- $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ diffeomorphism

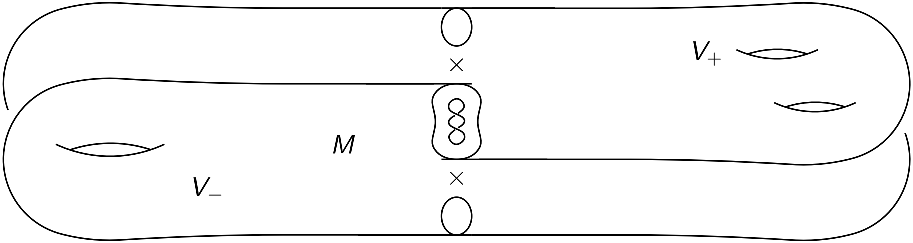
Let $V_\pm := Z_\pm \setminus$ tubular neighbourhood $\Sigma_\pm \times \Delta$; so $\partial V_\pm = \Sigma_\pm \times S^1$.
Form simply-connected M^7 by gluing boundaries of $V_+ \times S^1$ to $V_- \times S^1$ by

$$\Sigma_+ \times S^1 \times S^1 \to \Sigma_- \times S^1 \times S^1,$$
$$(x, u, v) \mapsto (\mathfrak{r}(x), v, u)$$

Tian-Yau, **Haskins-Hein-N**:
$V_\pm$ admits asymptotically cylindrical Calabi-Yau metrics

$\rightsquigarrow$ metrics on $V_\pm \times S^1$ with holonomy $SU(3) \subset G_2$.
For carefully chosen $\mathfrak{r}$, these metrics glue to a holonomy G_2 metric on M.

Diagram of twisted connected sum



V_+, V_- asymptotically cylindrical Calabi-Yau threefolds
with ends asymptotic to $\Sigma_\pm \times S^1 \times \mathbb{R}$, where $\Sigma_\pm$ are K3 surfaces.

Truncate ends and glue $V_- \times S^1$ to $V_+ \times S^1$, flipping the circles.

For large neck length, the G_2-structure on M obtained by gluing has $d\varphi = 0$ and $d^*\varphi$ "small", and can be perturbed to a torsion-free one.

Building blocks from Fano 3-folds

Example

Let Z be the blow-up of $\mathbb{P}^3$ in the complete intersection C of two quartic K3s (or in other words a smooth base locus of an anticanonical pencil). Let $\Sigma \subset Z$ be the proper transform of one of those quartics.

The blow-up ensures that the normal bundle of Σ in Z is trivial, so that $Z \setminus \Sigma$ admits asymptotically cylindrical Calabi-Yau metrics.

Moreover, clearly *any* quartic K3 Σ (*ie* any smooth non-hyperelliptic K3 with ample class of degree 4) appears in a "building block" (Z, Σ) of this form.

One can construct building blocks like this starting from all but 2 of the 105 families in the classification of smooth Fano 3-folds, and from thousands of weak Fano 3-folds.

But to apply the construction, we in addition need appropriate diffeomorphisms $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ for *pairs* of such building blocks.

Hyper-Kähler rotations

The Calabi-Yau structure on V can be encoded in terms of a pair (Ω, ω), where ω is the Kähler form, and Ω is a holomorphic 3-form.
Along the cylindrical end $\mathbb{R} \times S^1 \times \Sigma$, they can be written as

$$\omega \sim \omega^I + dt \wedge du, \quad \Omega \sim (du - idt) \wedge (\omega^J + i\omega^K),$$

where u is the S^1 coordinate and $(\omega^I, \omega^J, \omega^K)$ is a "hyper-Kähler triple".

The torsion-free G_2-structure φ on $V \times S^1$ is defined by $\varphi = dv \wedge \omega + \mathrm{Re}\,\Omega$, for v the "external" S^1 coordinate.
For a pair of ACyl Calabi-Yau manifolds $(V_\pm, \omega_\pm, \Omega_\pm)$, the condition that

$$\Sigma_+ \times S^1 \times S^1 \times \mathbb{R} \to \Sigma_- \times S^1 \times S^1 \times \mathbb{R}, \;\; (x, u, v, t) \mapsto (\mathfrak{r}(x), v, u, -t)$$

identifies the asymptotic limits of the G_2-structures on $V_+ \times S^1$ and $V_- \times S^1$ reduces to

$$\mathfrak{r}^*\omega^I = \omega^J, \quad \mathfrak{r}^*\omega^J = \omega^I, \quad \mathfrak{r}^*\omega^K = -\omega^K.$$

Call such $\mathfrak{r}$ "hyper-Kähler rotations".

4. The matching problem
Set-up

In practice, it is fruitful to ask:

Given two sets $\mathcal{Z}_+$ and $\mathcal{Z}_-$ of building blocks, can we find some $(Z_+, \Sigma_+) \in \mathcal{Z}_+$ and $(Z_-, \Sigma_-) \in \mathcal{Z}_-$ such that $V_\pm := Z_\pm \setminus \Sigma_\pm$ admit ACyl Calabi-Yau structures, and there is a hyper-Kähler rotation $\mathfrak{r} : \Sigma_+ \to \Sigma_-$?

For control on the topology of the resulting G_2-manifold we need

- all elements of $\mathcal{Z}_\pm$ to have same topology (*eg* $\mathcal{Z}_\pm$ a deformation type)
- to prescribe the action $\mathfrak{r}^* : H^2(\Sigma_-; \mathbb{Z}) \to H^2(\Sigma_+; \mathbb{Z})$.

In particular, all $(Z_\pm, \Sigma_\pm) \in \mathcal{Z}_\pm$ should have the same "polarising lattice"

$$N_\pm := \mathrm{Im}(H^2(Z_\pm; Z) \to H^2(\Sigma_\pm; \mathbb{Z}),$$

and to *eg* apply Mayer-Vietoris we need to know $N_+ \cap \mathfrak{r}^* N_-$.

Other significance of $N_\pm$: always $N_\pm \subseteq \mathrm{Pic}\,\Sigma_\pm := H^2(\Sigma_\pm; \mathbb{Z}) \cap H^{1,1}(\Sigma_\pm)$, giving an a priori constraint on what K3 surfaces can appear in $\mathcal{Z}_\pm$.

Configurations of polarising lattices

Given blocks $(Z_\pm, \Sigma_\pm)$ and an isomorphism $F : H^2(\Sigma_+;\mathbb{Z}) \to H^2(\Sigma_-;\mathbb{Z})$, we can identify both $H^2(\Sigma_+;\mathbb{Z})$ and $H^2(\Sigma_-;\mathbb{Z})$ with a fixed copy L of the K3 lattice (*ie* unimodular lattice of signature $(3,19)$) in a compatible way.

This way we obtain a pair of embeddings $N_+, N_- \hookrightarrow L$, defined up to simultaneous action of isometry group $O(L)$ on both embeddings.

All topological invariants we can compute for twisted connected sums depend only on data about the two blocks and the configuration, so we reformulate the matching problem as:

Given two sets $\mathcal{Z}_+$ and $\mathcal{Z}_-$ of blocks, which configurations $N_+, N_- \hookrightarrow L$ of their polarising lattices are realised by hyper-Kähler rotations?

Nikulin

If the sum of images $N_+ + N_-$ is primitive in L (*ie* no cotorsion) and has rank ≤ 11, then it is determined up to $O(L)$ by the form on $N_+ + N_-$.

Most configurations we care about are described simply by a form on $N_+ \oplus N_-$.

$$\left(\begin{array}{c|c} N_+ & A^T \\ \hline A & N_- \end{array}\right)$$

Non-metric translation in terms of periods

Given a configuration $N_+, N_- \hookrightarrow L$ of a pair of blocks $(Z_\pm, \Sigma_\pm)$, in $L \otimes \mathbb{R}$ we have distinguished

- period 2-planes $\Pi_\pm$, spanned by real/imag parts of holomorphic 2-form
- open cones $\mathcal{K}_\pm \subset N_\pm \otimes \mathbb{R}$, the images of the Kähler cones of $Z_\pm$

For the configuration to be realised by a hyper-Kähler rotation $\mathfrak{r}$ of the asymptotic limits of some ACyl Calabi-Yau structures on $V_\pm := Z_\pm \setminus \Sigma_\pm$, we must have

$$[\omega_+^I] = [\omega_-^J] \in \mathcal{K}_+ \cap \Pi_-, \quad [\omega_+^I] = [\omega_-^J] \in \Pi_+ \cap \mathcal{K}_-, \quad [\omega_+^K] = -[\omega_-^K] \in \Pi_+ \cap \Pi_-.$$

Proposition

If $(Z_\pm, \Sigma_\pm)$ are building blocks and $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ is a diffeomorphism such that $\mathcal{K}_+ \cap \Pi_-$, $\Pi_+ \cap \mathcal{K}_-$ and $\Pi_+ \cap \Pi_-$ are non-empty, then $V_\pm := Z_\pm \setminus \Sigma_\pm$ admit ACyl Calabi-Yau structures so that $\mathfrak{r}$ is a hyper-Kähler rotation.

Proof.

Calabi-Yau theorem allows to realise any Kähler class by Ricci-flat
$+$ Torelli theorem $\square$

Genericity

Given a configuration $N_+, N_- \hookrightarrow L$, let

- $N'_- \subseteq N_-$ the orthogonal complement to N_+, and
- $\Lambda_+ \subseteq N_+ + N_-$ the orthogonal complement to N'_-, so
$$N_+ \subseteq \Lambda_+ \subset N_+ + N_- \subset L.$$

The period $\Pi_\pm \cong H^{2,0}(\Sigma_\pm)$ is always orthogonal to $N_\pm \subset H^{1,1}(\Sigma_\pm; \mathbb{R})$.
If there is a hyper-Kähler rotation $\mathfrak{r}$ with the given configuration, then

$$[\omega_+^J] \in \Pi_+ \cap \mathcal{K}_- \subset N'_- \quad \text{and} \quad [\omega_+^K] \in \Pi_+ \cap \Pi_- \subset (N_+ + N_-)^\perp$$

are both orthogonal to Λ_+. Thus $\Lambda_+ \subseteq \operatorname{Pic} \Sigma_+$.

Punchline: To find a matching among $\mathcal{Z}_+$ and $\mathcal{Z}_-$ with given configuration, there must be at least some $(Z_\pm, \Sigma_\pm) \in \mathcal{Z}_\pm$ with $\Lambda_\pm \subseteq \operatorname{Pic} \Sigma_\pm$.

Approximate converse: if a generic K3 $\Sigma_\pm$ with $\Lambda_\pm \subseteq \operatorname{Pic} \Sigma_\pm$ appears in $\mathcal{Z}_\pm$, then matchings exist.

Mass-production vs hand-crafting

Theorem (Beauville, Corti-Haskins-N-Pacini)

Let Y be a smooth Fano 3-fold, $\Sigma \subset Y$ an anticanonical K3 divisor, and let N be the image of $H^2(Y;\mathbb{Z})$ in $H^2(\Sigma;\mathbb{Z})$. Then a generic K3 Σ' with $N \subseteq \operatorname{Pic}\Sigma'$ embeds an anticanonical K3 divisor in a deformation of Y. Same holds for smooth "semi-Fano" 3-folds.

So for configurations with $\Lambda_{\pm} = N_{\pm}$, any sets of blocks produced by blowing up semi-Fanos in a curve have the desired genericity property.

That is the case if the configuration is defined by orthogonal sum $N_{+} \perp N_{-}$

$\rightsquigarrow 10^8$ examples

Many are 2-connected so that it is easy to apply smooth classification results, and many different constructions yield the same smooth manifold.

Example (Crowley-N)

Some twisted connected sums $\cong$ total spaces of S^1-bundles.

But μ, ν, $\bar{\nu}$ always take the same value for such configurations...

Handcrafting

Actually, we can never use ν or $\bar{\nu}$ to distinguish twisted connected sums.

Theorem (Crowley-Goette-N)

Any twisted connected sum has $\bar{\nu} = 0$ (and $\nu = 24$).

If one uses non-perpendicular matchings then $\Lambda_\pm \neq N_\pm$ so one has do more work (to some extent "by hand") to prove improved genericity results.

Example

The set of blocks obtained by blowing up $\mathbb{P}^3$ in the intersection of two quartics has the Λ-genericity property if Λ contains a degree 4 class H, and no class v such that $H.v = 0$ and $v^2 = -2$, or $H.v = 2$ and $v^2 = 0$.

Pay-off: μ and ξ can take different values $\rightsquigarrow$ examples of

- G_2-manifolds that are homeomorphic, but diffeomorphism types distinguished by μ (**Crowley-N**)
- G_2-manifolds where components of moduli space are distinguished by ξ. (**Wallis**)

5. Extra-twisted connected sums
Tori

Recall:

From a building block (Z, Σ) we get an ACyl Calabi-Yau 3-fold $V := Z \setminus \Sigma$ with cylindrical end $\mathbb{R} \times S^1 \times \Sigma$. Think of this circle factor as "internal".

Now suppose the building block (Z, Σ) has a cyclic automorphism group Γ that fixes Σ pointwise.
Then the action of Γ on V acts trivially on the Σ factor in the asymptotic end while rotating the S^1_{int} factor.

Next choose a free action of Γ on "external" circle S^1_{ext}.
Then $(S^1_{ext} \times V)/\Gamma$ is a smooth ACyl G_2-manifold. Its asymptotic end is of the form $\mathbb{R} \times T^2 \times \Sigma$, but the torus $T^2 := (S^1_{ext} \times S^1_{int})/\Gamma$ need *not* be a metric product of two circles.
The geometry of T^2 depends on the circumferences of S^1_{ext} and S^1_{int}, which can be chosen freely.

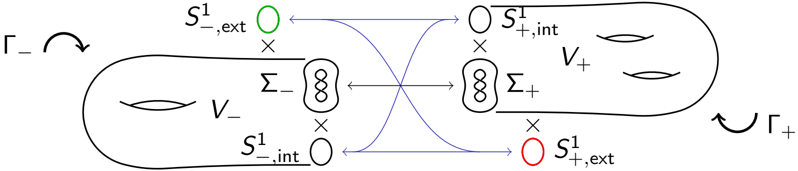
Adding the extra twist

To make an extra-twisted connected sum

- Find some building blocks $(Z_\pm, \Sigma_\pm)$ with automorphism groups $\Gamma_\pm$
- Choose circumferences so that there is an isometry $\mathfrak{t} : T^2_+ \to T^2_-$
- Find ACyl Calabi-Yau metrics so that there is $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ that makes

$$(-1) \times \mathfrak{t} \times \mathfrak{r} : \mathbb{R} \times T^2_+ \times \Sigma_+ \to \mathbb{R} \times T^2_- \times \Sigma_-$$

an isomorphism of the asymptotic limits of the G_2-structures.

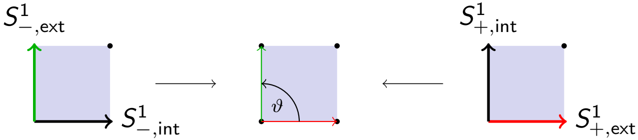


Inflexibility of the gluing angle for TCS

In the twisted connected sum construction we identify the asymptotic cross-sections $S^1_{+,ext} \times S^1_{+,int} \times \Sigma_+$ and $S^1_{-,ext} \times S^1_{-,int} \times \Sigma_-$ by the product of an isometry $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ and the "flip" isometry

$$S^1_{+,ext} \times S^1_{+,int} \to S^1_{-,ext} \times S^1_{-,int}, \quad (u, v) \mapsto (v, u).$$

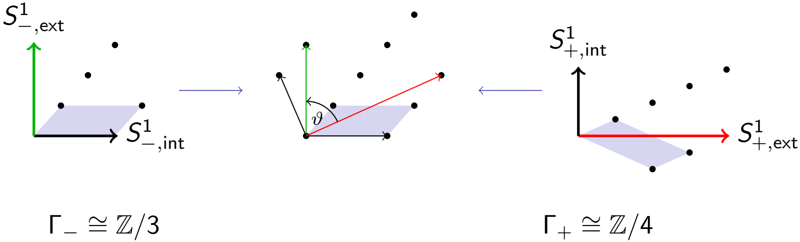
We can choose the circumferences of $S^1_{+,ext} = S^1_{-,int}$, and $S^1_{-,ext} \cong S^1_{+,int}$, but the angle ϑ between the external circle direction will *always* be $\frac{\pi}{2}$.



This "gluing angle" ϑ turns out to play a key role in the calculation of $\bar{\nu}$.

More exciting torus isometries

As soon as at least one of the tori T^2_+ and T^2_- is not simply an isometric product $S^1_{ext} \times S^1_{int}$, there are other possibilities for the gluing angle ϑ.



eg $\qquad \vartheta = \dfrac{3\pi}{4}, \ \dfrac{2\pi}{3}$ or $\arccos\left(\dfrac{1}{\sqrt{6}}\right)$.

The matching problem and configurations

Given the size of Γ_+ and Γ_-, it is essentially a combinatorial problem to determine all possible torus isometries $\mathfrak{t} : T^2_+ \to T^2_-$.

Eg for $\Gamma_+ = \mathbb{Z}/3$ and $\Gamma_- = \mathbb{Z}/4$ there are 28 possibilities (up to symmetries).

Once we have chosen a torus isometry $\mathfrak{t}$, we need to find blocks (Z_+, Σ_+) and (Z_-, Σ_-) with those automorphism groups and diffeomorphism

$$\mathfrak{r} : \Sigma_+ \to \Sigma_-$$

satisfying a condition that depends only on the gluing angle ϑ of $\mathfrak{t}$, and on the periods and Kähler cones of $\Sigma_\pm$.

The topology of the resulting extra-twisted connected sum G_2-manifold depends the topology of Z_+ and Z_-, the choice of $\mathfrak{t}$, and on configuration of polarising lattices $N_+, N_- \hookrightarrow L$ defined by $\mathfrak{r}^* : H^2(\Sigma_-;\mathbb{Z}) \to H^2(\Sigma_+;\mathbb{Z})$.

Difference from before:

"Easy case" of matching problem is now when N_+ is "at angle ϑ" to N_-.

If $\vartheta = \frac{\pi}{2}$ one can use perpendicular direct sum, but for $\vartheta \neq \frac{\pi}{2}$ it is a non-trivial arithmetic problem whether any such configuration exists.

A building block with $\mathbb{Z}/4$ action

Let

- $Q \subset \mathbb{P}^3$ any quartic K3 surface.
- $Y \to \mathbb{P}^3$ the fourfold cover branched over Q.
- $C \subset Q$ a hyperplane section ($=$ divisor of normal bundle of Q in Y).
- $Z \to Y$ the blow-up along C
- $\Sigma \subset Z$ the proper transform of Q (which is isomorphic to Q).

The blow-up ensures that the normal bundle of Σ in Z is trivial.
The deck transformation action of $\mathbb{Z}/4$ on Y lifts to Z.
Thus (Z, Σ) is a building block with $\mathbb{Z}/4$ action.

Like in example before, any smooth quartic K3 appears in a block of this form.

Survey of "low-hanging fruit"

Using 25 similar blocks with involution (and rk $N \leq 2$) and 6 blocks with automorphisms of order 3 to 6 (and rk $N = 1$), one can make (at least)

- 305 matchings using only involutions blocks

  In many cases classifying diffeomorphism invariants can be worked out completely, and used to exhibit disconnected G_2 moduli space.

  $\bar{\nu}$ is always divisible by 3.

- 192 matchings using at least one block with automorphism of order ≥ 3 (thanks to greater variety of choices for the torus isometry)

  Greater variety in

  - topology, *eg* can get fundamental groups of order 2, 3, 4, 5, 6, 7, 8, 9, 10, 15 and 21 (but harder to work out full invariants)
  - values of $\bar{\nu}$ realised.

6. Coboundary defect invariants
Milnor's λ-invariant

Milnor: Define invariants of closed manifolds as "defect" of coboundaries.
E.g. for oriented 8-manifolds W whose boundary M has $p_1(M) = 0$ consider

- signature $\sigma(W)$ of intersection form on $H^4(W, M)$
- $p_1(W)^2 \in \mathbb{Z}$ ($p_1(M) = 0 \Rightarrow p_1(W)$ has a preimage in $H^4(W, M)$, whose square is independent of choice)

These are additive under gluing boundaries. Therefore linear combinations that vanish for closed manifolds are invariants of the boundary M.
Eg Hirzebruch signature theorem gives

$$45\sigma(X) + p_1(X)^2 = 7p_2(X)$$

for any closed oriented 8-manifold X, so that

$$3\sigma(X) + p_1(X)^2 \equiv 0 \mod 7.$$

Therefore

$$\lambda(M) := 3\sigma(W) + p_1(W)^2 \in \mathbb{Z}/7$$

depends only on the smooth manifold M, and not on W.
Used by Milnor (1956) to detect non-standard smooth structures on S^7.

The Eells-Kuiper invariant

For a closed spin 8-manifold X, the Atiyah singer index theorem for the index of the Dirac operator $\not{D}_X$

$$\mathrm{ind}\,\not{D}_X = \frac{7p_1^2 - 4p_2}{45 \cdot 2^7}$$

combined with the Hirzebruch signature theorem gives

$$\frac{p_1(X)^2 - 4\sigma(X)}{32} = 28\,\mathrm{ind}\,\not{D}_X.$$

For a closed spin 7-manifold M with $p_1(M) = 0$ and spin coboundary W

$$\mu(M) = \frac{p_1(W)^2 - 4\sigma(W)}{32} \in \mathbb{Z}/28$$

is thus a well-defined diffeomorphism invariant.
It distinguishes all 28 classes of smooth structures on S^7.

Generalised Eells-Kuiper invariant

If $p_1(M) \neq 0$ then we cannot interpret $p_1(W)^2$ as a well-defined element of $\mathbb{Z}$.
But if $H^4(M)$ is torsion-free and $p_1(M)$ is divisible by d, then

$$p_1(W)^2 \in \mathbb{Z}/4\tilde{d}.$$

is well-defined, where $\tilde{d} := \mathrm{lcm}(8, d)$. Therefore

$$\mu(M) := \frac{p_1(W)^2 - 4\sigma(W)}{32} \in \mathbb{Z}/\gcd\left(28, \tfrac{\tilde{d}}{8}\right)$$

is a well-defined diffeomorphism invariant of M.

Crowley-N:
$\mu(M)$ detects all equivalence classes of smooth structures on such manifolds.

G_2-structures and spinors

To define defect invariants of G_2-structures on a closed 7-manifold, first relate them to spinors.

The spinor representation of $Spin(7)$ is real of rank 8.

The stabiliser in $Spin(7)$ of any non-zero spinor is isomorphic to G_2. Therefore

$$G_2\text{-structure on } M^7 \leftrightarrow$$

metric g + spin structure + nowhere vanishing spinor field s modulo scale

The positive spinor bundle of a spin 8-manifold is also real of rank 8.

If W is a compact 8-manifold with boundary M, then we can consider $s_+ \in \Gamma(S_W^+)$ with transverse zeros such that the restriction $s \in \Gamma(S_M)$ of s_+ to M defines a given G_2-structure.

Then $\# s_+^{-1}(0)$ (counted with signs) depends only on W and s.

Invariants of G_2-structures

If X is closed and $s_+ \in \Gamma(S_X^+)$ has transverse zeros, then $\# s_+^{-1}(0)$ equals the Euler class $e(S_X^+)$, related to Euler characteristic $\chi(X)$ by

$$-3\sigma(X) + \ \ \chi(X) - \ \ 2\,\# s_+^{-1}(0) = -48 \operatorname{ind} D_X,$$

$$\frac{3p_1(X)^2 - 180\sigma(X)}{8} + 7\chi(X) - 14\,\# s_+^{-1}(0) = 0.$$

Therefore, for W compact spin 8-manifold with boundary M and a transverse $s_+ \in \Gamma(S_W^+)$ with $s := s_{+|M} \in \Gamma(S_M)$

$$\nu(M,s) := 3\sigma(X) + \ \ \chi(X) - \ \ 2\,\# s_+^{-1}(0) \in \mathbb{Z}/48,$$

$$\xi(M,s) := \frac{3p_1(W)^2 - 180\sigma(W)}{8} + 7\chi(W) - 14\,\# s_+^{-1}(0) \in \mathbb{Z}/\tfrac{3}{2}\tilde{d}$$

are well-defined diffeomorphism invariants of (M,s), *ie* of M equipped with a G_2-structure.
Also clear that ν and ξ are invariant under continuous deformation of a G_2-structure.

Invariants of twisted connected sums

For "ordinary" twisted connected sums, μ, ν and ξ can be computed by finding an explicit coboundary.

But even though computing the cobordism group $\Omega_7^{Spin} = 0$ shows that coboundaries always exist, there is no algorithm for constructing them.

Crowley-Goette-N:

It is possible to define a diffeomorphism invariant $\bar{\nu} \in \mathbb{Z}$ of a G_2-structure on a closed 7-manifold in terms of eta invariants of elliptic operators, so that

$$\nu = \bar{\nu} + 24 \mod 48$$

for any torsion-free G_2-structure.

For extra-twisted connected sums, $\bar{\nu}$ can be computed by a cut-and-paste formula.

If the gluing angle is $\frac{\pi}{2}$ then all terms vanish, but otherwise there is a range of possible values depending on the configuration and (if $|\Gamma_\pm| \geq 3$) on the isolated fixed points of $\Gamma_\pm$ in $Z_\pm$.