Integrable systems with S^1-actions and the associated polygons

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University of Nottingham geometry seminar

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Symplectic manifolds and integrable systems

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- Denote by $\mathcal{X}_f$ the Hamiltonian vector field of f, which satisfies

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- Fixed point or rank zero point is $p \in M$ such that $dF(p) = 0$.

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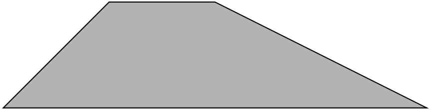
- i.e. a global Hamiltonian T^n-action.
- $F\colon M \to \mathbb{R}^n$, Atiyah, Guillemin-Sternberg (1982) showed that in this case $F(M)$ is the convex hull of the images of the fixed points.

Toric integrable systems: the classification



Theorem (Delzant, 1988)

Given any "Delzant polytope" $\Delta \subset \mathbb{R}^n$*, there exists a unique (up to isomorphism) toric integrable system* (M, ω, F) *such that* $F(M) = \Delta$.

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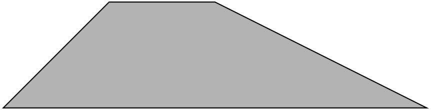

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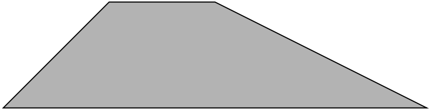

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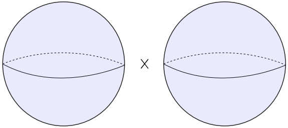
- System can be recovered by symplectic reduction on $\mathbb{C}^d$.

Example



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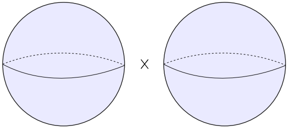

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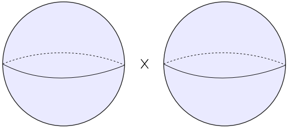

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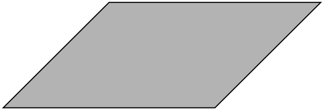

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- Simple = at most one focus-focus point in each level set of J.

Semitoric integrable systems: fibers

Points in simple semitoric systems:

- regular points;
- rank one: elliptic-regular points;
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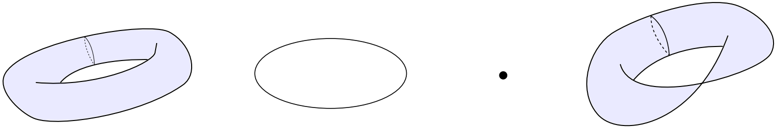

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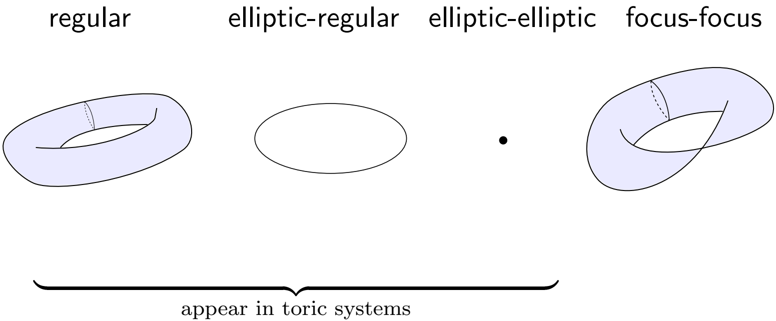

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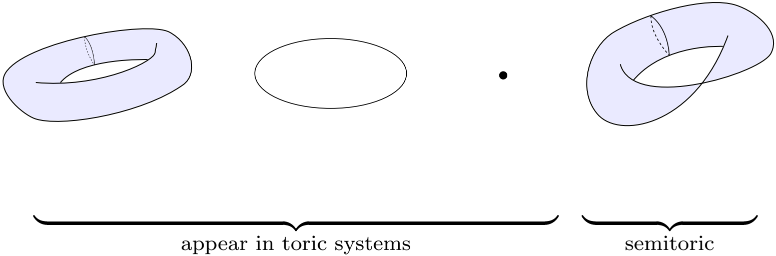

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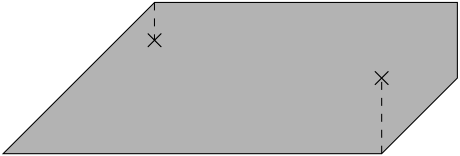

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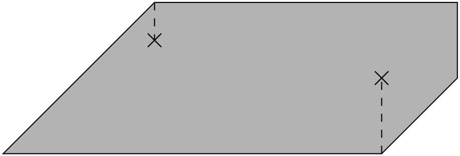

- Each interior point is labeled with an integer and a Taylor series in two variables.

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(1) the number of focus-focus points invariant;

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1. *Two simple semitoric systems are isomorphic if and only if they have the same invariants (1)-(5);*
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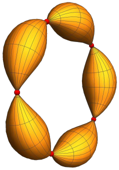

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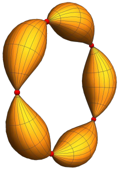

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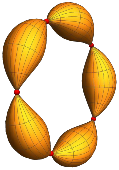

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Goal

Given specified semitoric polygon invariant try to find an explicit system with that invariant (forgetting about the other invariants).

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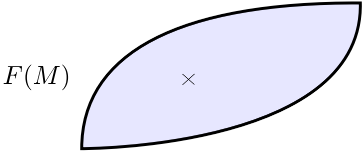

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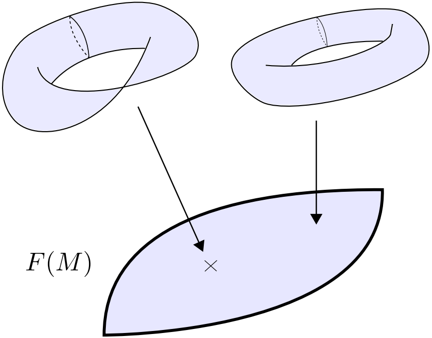

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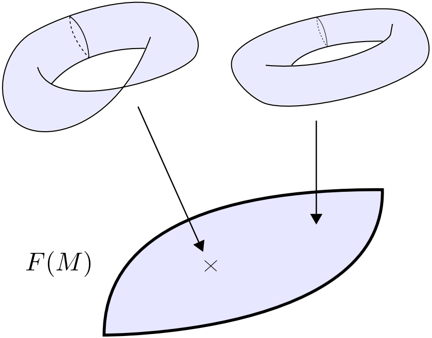

- Torus fibration $\to$ integral affine structure on $(F(M))_{\text{regular}}$.
  - NOT equal to integral affine structure inherited from $\mathbb{R}^2$.

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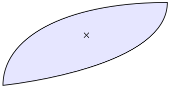

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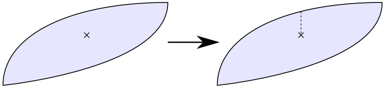

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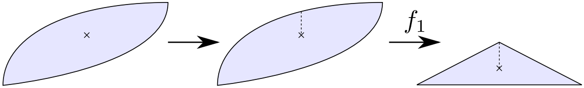

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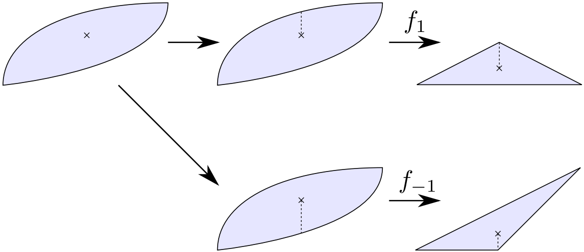

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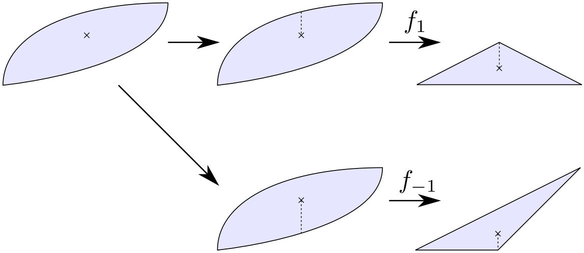

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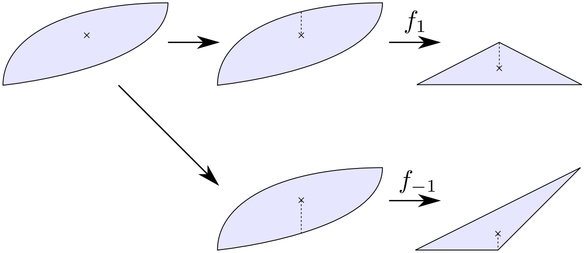

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- Note: $f_\varepsilon \circ F$ is a the momentum map for a Hamiltonian T^2-action away from the cuts.

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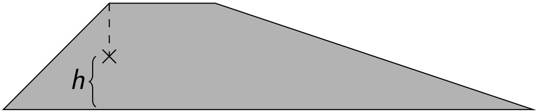

Semitoric invariants: 1./2./3.: the marked polygon

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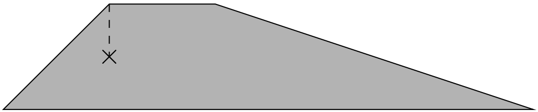

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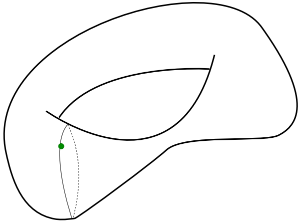

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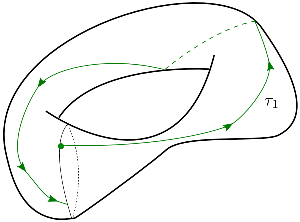

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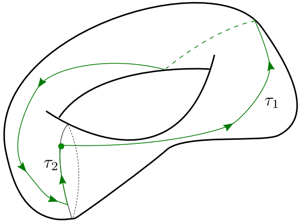

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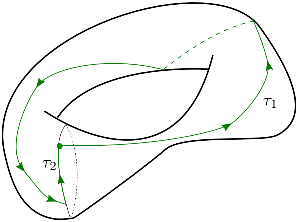

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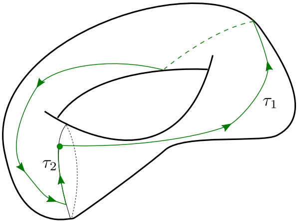

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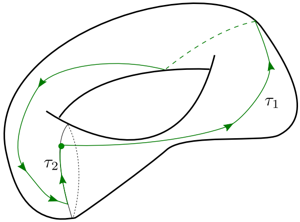

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- **Notice:** this construction only sees where the trajectory "lands" - it can't detect a twist.

Semitoric invariants: 5. Twisting index invariant

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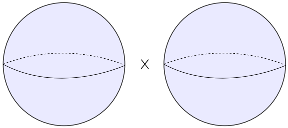
- This freedom is specified by the twisting index, an integer for each marked point in each choice of polygon.
- Can think of it as discrete freedom in how to glue neighborhood of focus-focus point into the system.

Example: Coupled angular momenta

[Sadovskií and Zĥilinskií, 1999]



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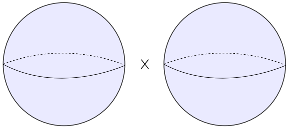

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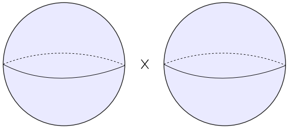

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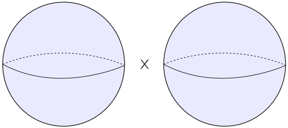

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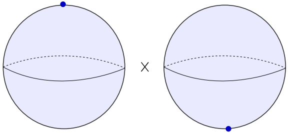

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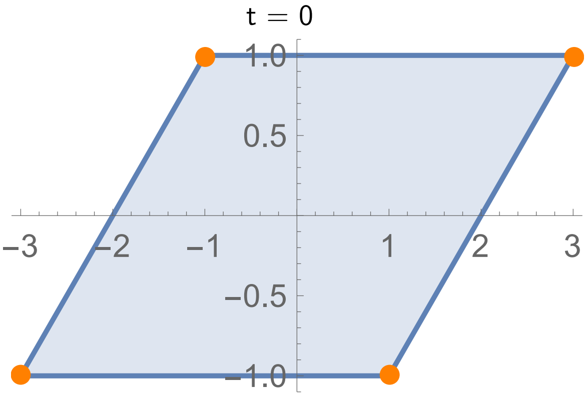
- In particular, $(J, H_{1/2})$ is semitoric.

Coupled angular momenta: moment map image



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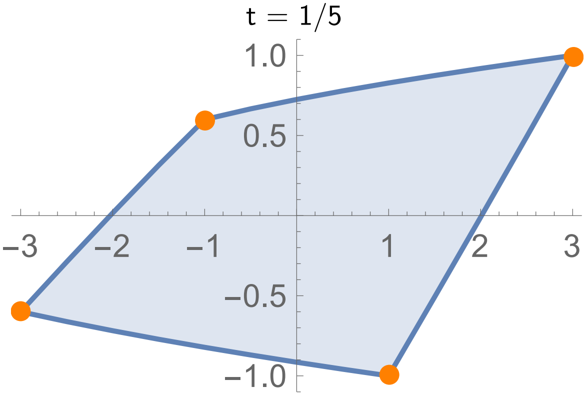

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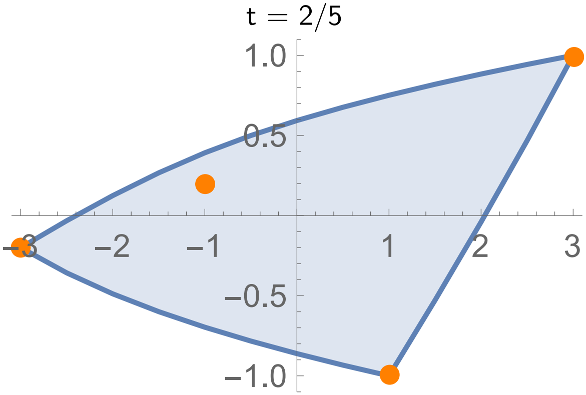

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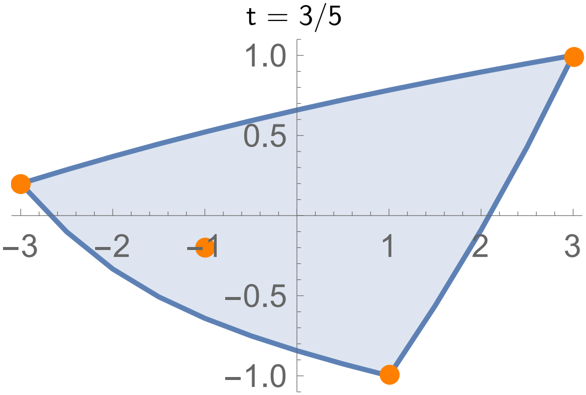

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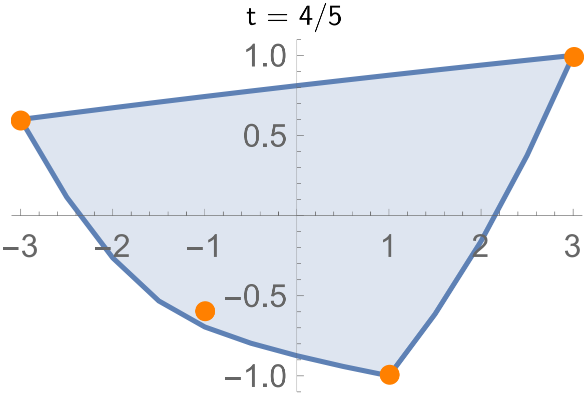

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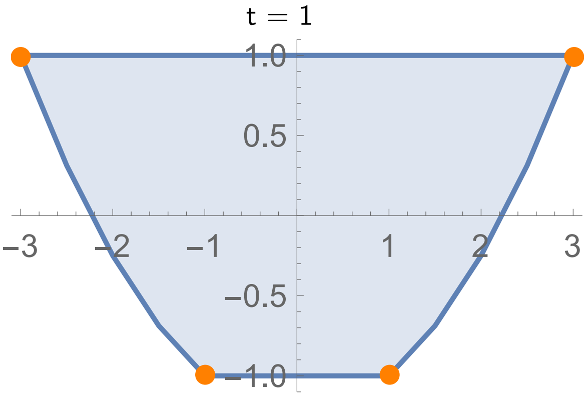

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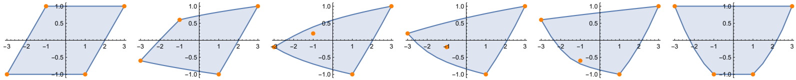
Coupled angular momenta: semitoric polygon

The image of the momentum map for (J, H_t):



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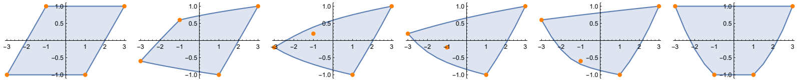

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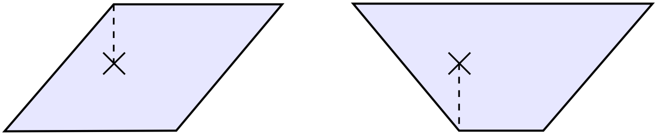

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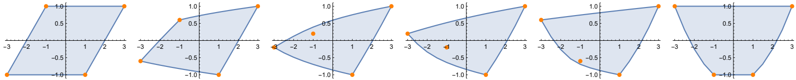

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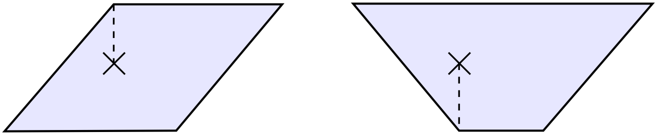

Idea

Interpolate between systems "related to the semitoric polygons" to find desired semitoric system.

Semitoric families: definition

Definition (Le Floch-P., 2018)

A semitoric family is a family of integrable systems (M, ω, F_t), $0 \leq t \leq 1$, where

- $\dim(M) = 4$;
- $F_t = (J, H_t)$;
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- The behavior at the degenerate times can be very complicated!

Semitoric transition families

Definition (Le Floch-P.)

A semitoric transition family with transition point p is a semitoric family with exactly two degenerate times t^-, t^+ such that $0 < t^- < t^+ < 1$ and

- p is of elliptic-elliptic type for $t < t^-$;
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- First example: coupled angular momenta

Polygons in a semitoric family

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Lemma (Le Floch-P.)

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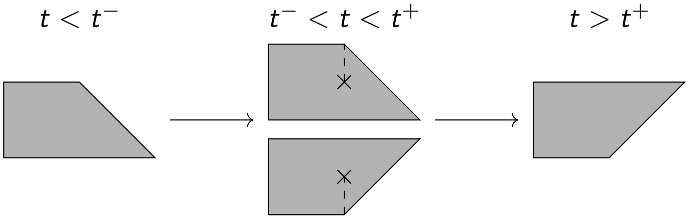
Lemma (Le Floch-P.)

Let $(M, \omega, (J, H_t))$ be a semitoric transition family with transition point p. Roughly, the set of semitoric polygons for $t^- < t < t^+$ is the union of the ones for $t < t^-$ and $t > t^+$.

Polygons in a semitoric family



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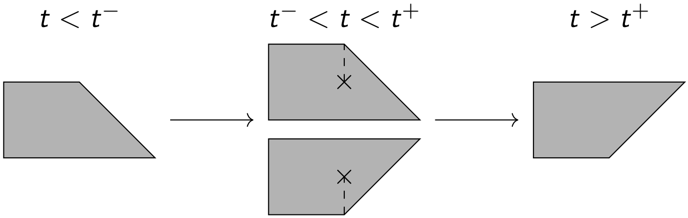
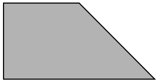


- To construct a system with certain semitoric polygons can try to transition between "toric type" systems corresponding to the semitoric polygons.

The first Hirzebruch surface



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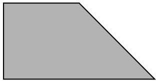

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$$N = (1/2)\left(|u_1|^2 + |u_2|^2 + |u_3|^2, |u_3|^2 + |u_4|^2\right) \text{ at } (2,1).$$

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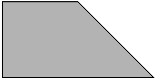

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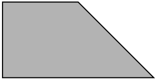

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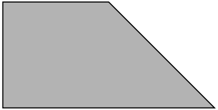

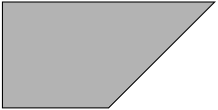

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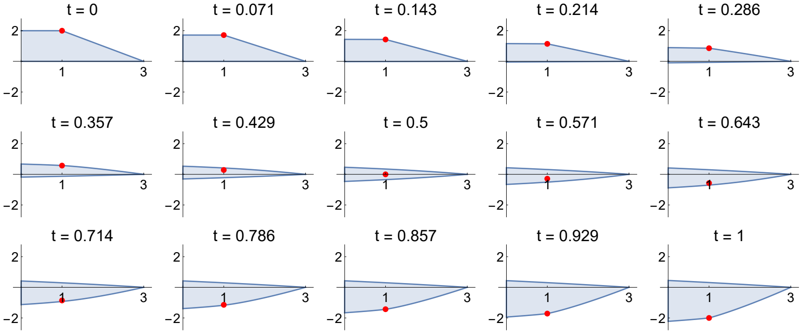

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Theorem (Le Floch-P.)

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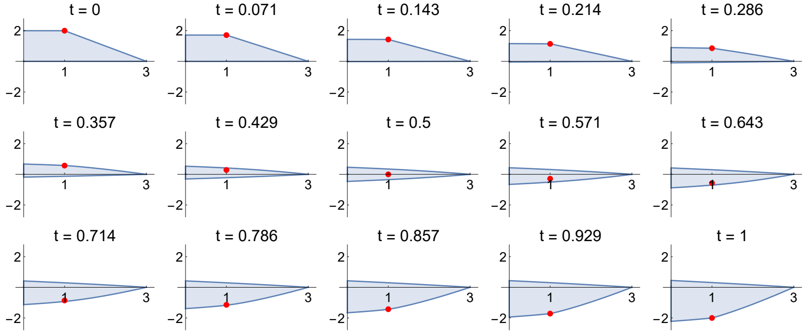

- Fixed points move on sphere $S = J^{-1}(0)$.

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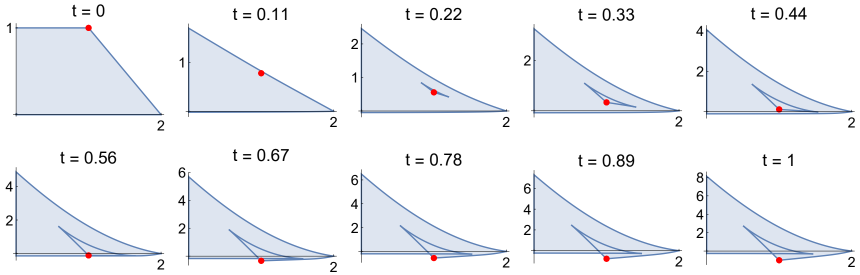

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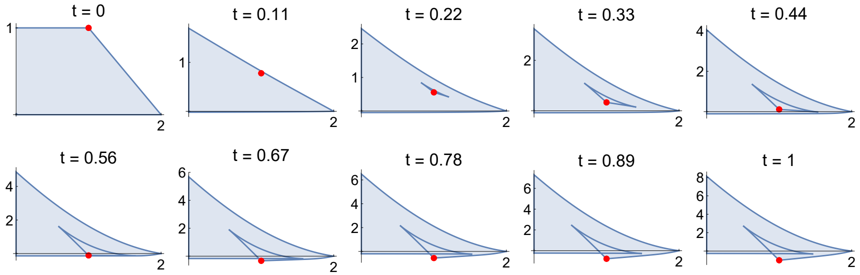

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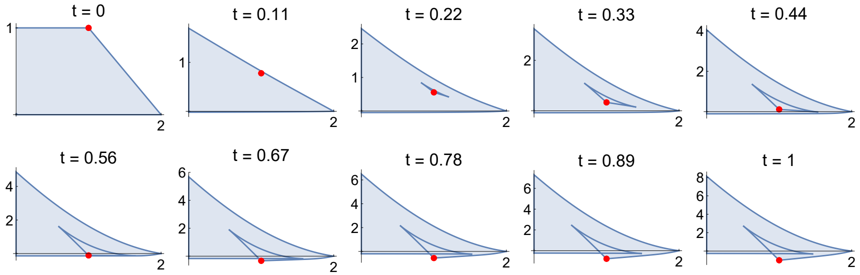

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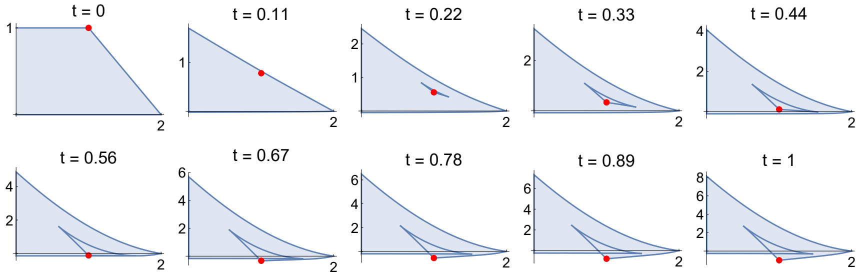

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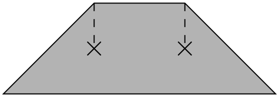
  - Similar to Dullin-Pelayo (2016).

A system with two focus-focus points

- Another semitoric polygon:



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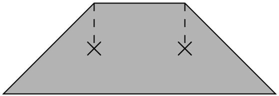

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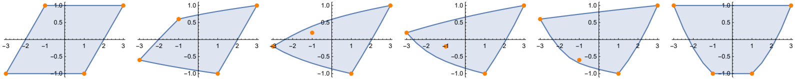

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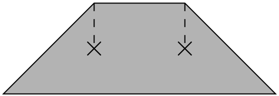

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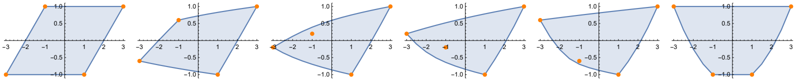

- The point NS passes through the interior and becomes focus-focus, can we do this with SN as well?

A two parameter family

Let $J = R_1 z_1 + R_2 z_2$ and

$$\begin{cases} H_{0,0} & = x_1 x_2 + y_1 y_2 + z_1 z_2 \\ H_{1,0} & = z_1 \\ H_{0,1} & = z_2 \\ H_{1,1} & = x_1 x_2 + y_1 y_2 - z_1 z_2 \end{cases}$$

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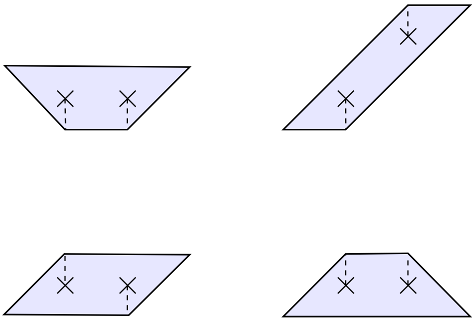
Theorem (Hohloch-P., 2018)

Let $R_1 = 1$ and $R_2 = 2$. Then $(J, H_{\frac{1}{2},\frac{1}{2}})$ is a semitoric integrable system with exactly two focus-focus points (and so is every system in an open neighborhood of these parameters).

The semitoric polygons

The semitoric polygons for $(J, H_{\frac{1}{2},\frac{1}{2}})$ (minimal polygons of type (2)):



The momentum map image

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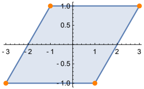

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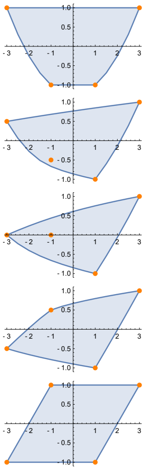

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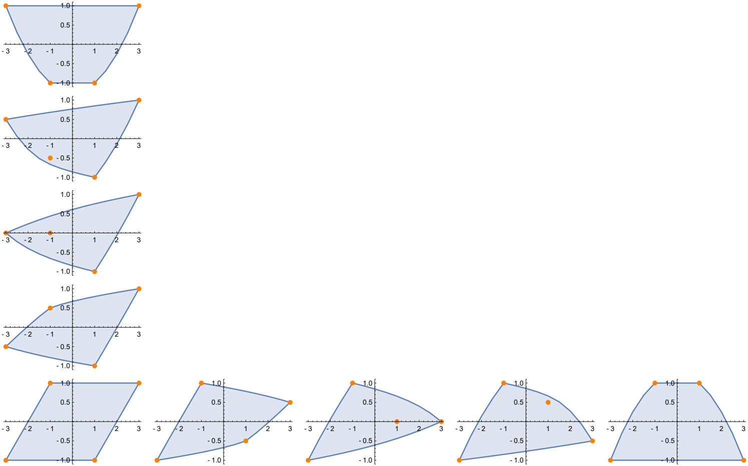

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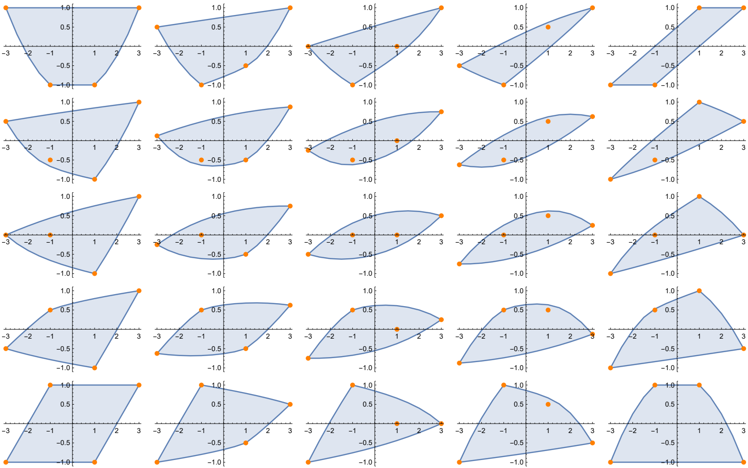

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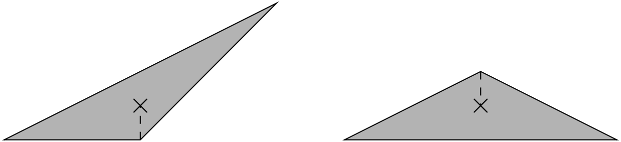

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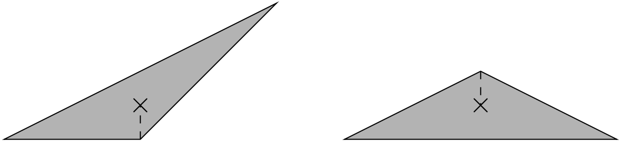

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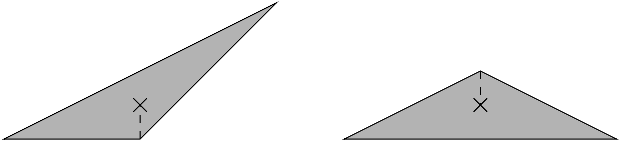

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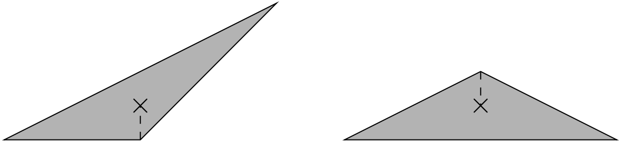

- The right polygon does not correspond to a toric system!
  - This means we cannot use a semitoric transition family in the same way (when the focus-focus point collides with the boundary)
- But there are more difficulties too...

Z_k-spheres

- $F_t = (J, H_t)$ means that the underlying S^1-manifold (M, ω, J) is fixed.

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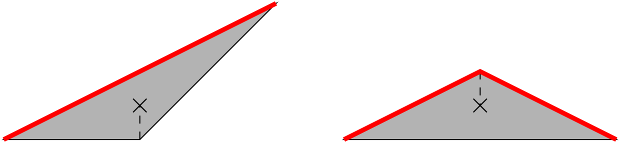

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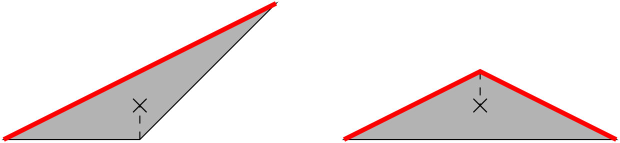

- Points in Z_k-spheres are automatically singular points of the integrable system, but in toric and semitoric systems *lines of singular points cannot enter the interior of $F(M)$.*

A system on $\mathbb{CP}^2$

- λ, δ, γ are parameters satisfying $0 < \gamma < \frac{1}{4\lambda}$ and $\delta > \frac{1}{2\gamma\lambda}$.
- Let $M = \mathbb{CP}^2 = N^{-1}(0)/S^1$ where

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- last term "pushes" the Z_2-sphere to keep it on the boundary.

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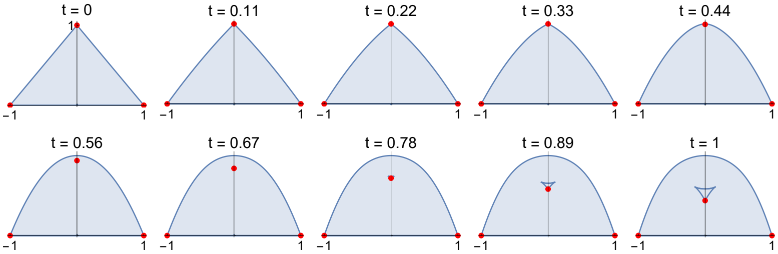

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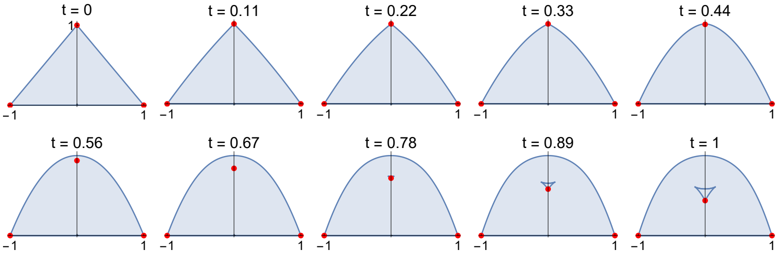

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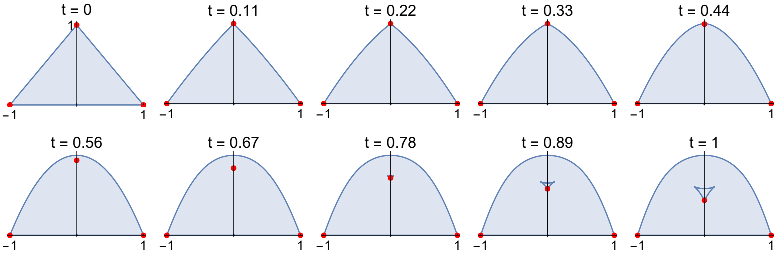
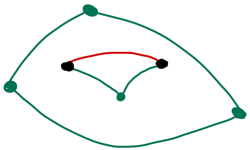


- For large t the system develops a flap, including hyperbolic-regular points and parabolic points
- the transition point still changes EE to FF to EE, but it can't merge with the bottom boundary (the Z_2-sphere) so instead it forms a flap.

Flaps in integrable systems

Fibers in a "flap"



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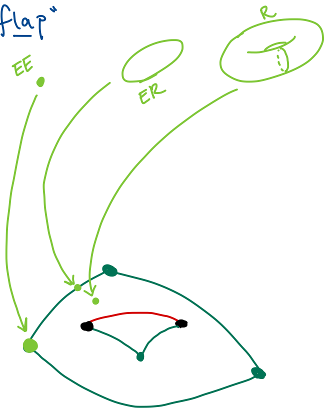

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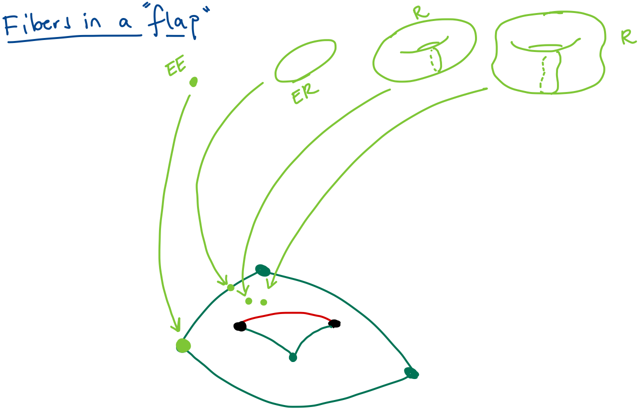

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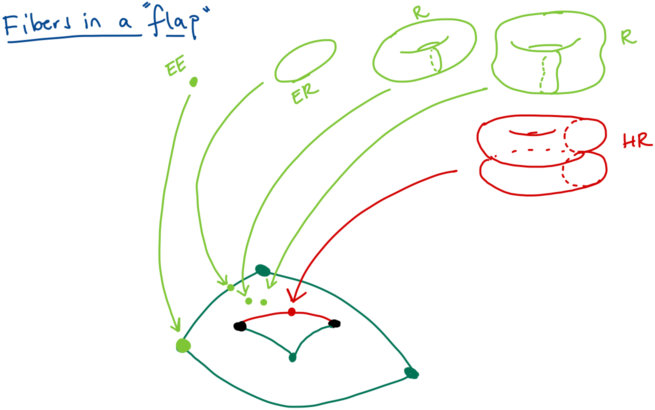

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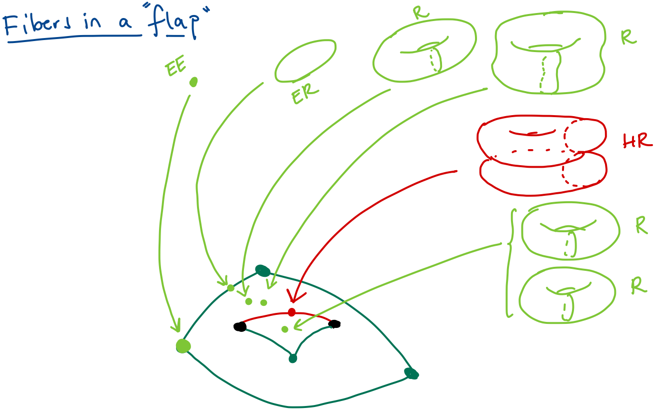

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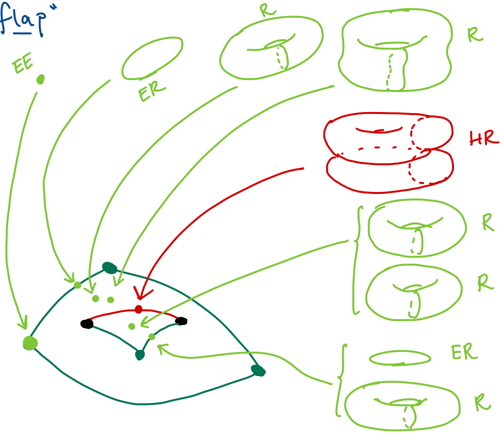

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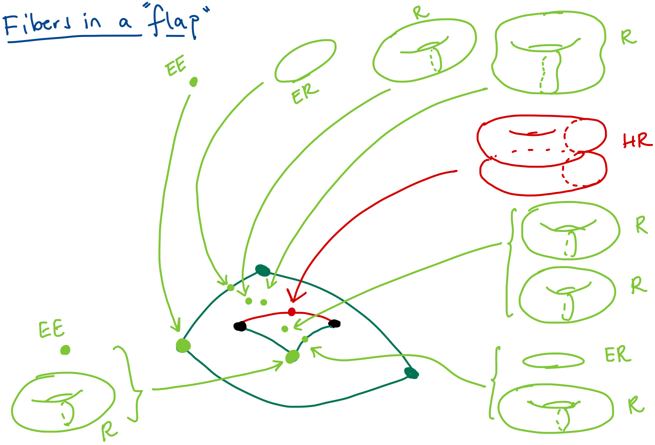

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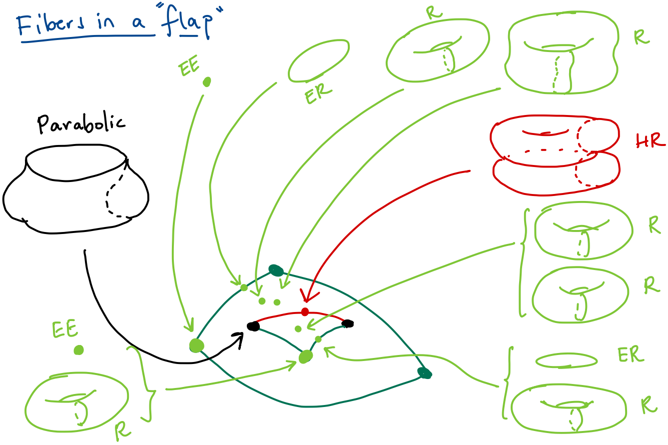

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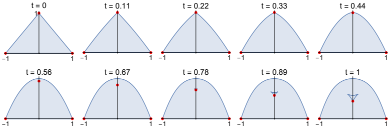

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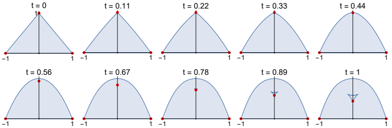

Theorem (Le Floch-P., "2022")

The family $(\mathbb{CP}^2, n\omega_{FS}, F_t = (J, H_t))_{0 \leq t \leq 1}$ *is*

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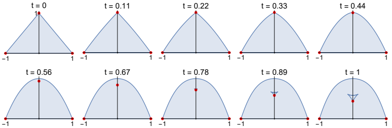

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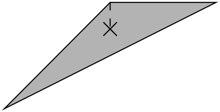

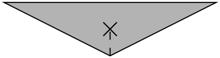

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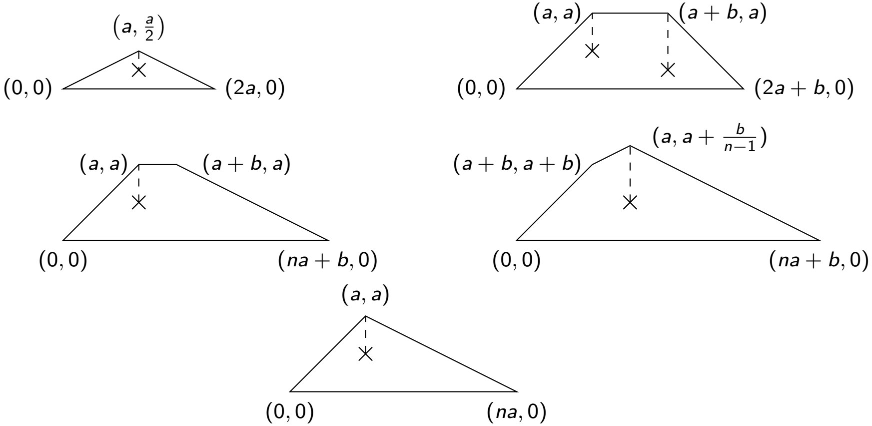

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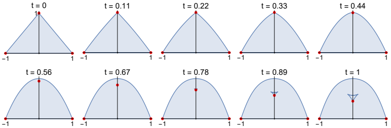
- Study various properties of the fibers (non-displacible?, Hamiltonian isotopic?, heavy or superheavy?, etc...)

Thanks!



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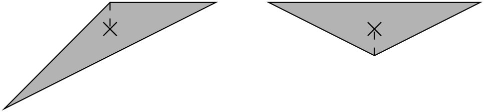

"Appendix"

Some extra slides

Minimal models

- Around an elliptic-elliptic point can perform a blowup of toric type, by performing a $\mathbb{T}^2$-equivariant blow up with respect to $f_\varepsilon \circ F$ (blowing down is the inverse operation)

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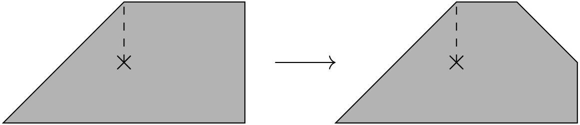
- Then all systems can be obtained from these by performing a sequence of blowups.

Minimal models: blowups and corner chops

- Blowups correspond to a corner chop of the semitoric polygon.



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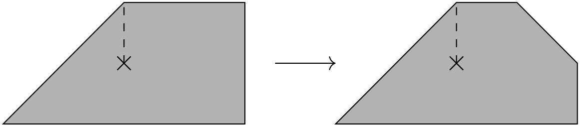
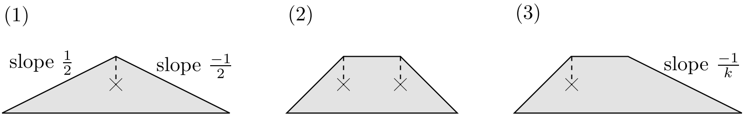


- Semitoric minimal models were classified in terms of the semitoric helix invariant into types (1) - (7) in [P.-Pelayo-Kane, 2018]

Minimal models: minimal polygons (1), (2), (3)

The polygons of minimal systems of types (1), (2), and (3):



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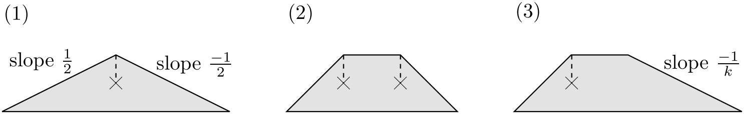

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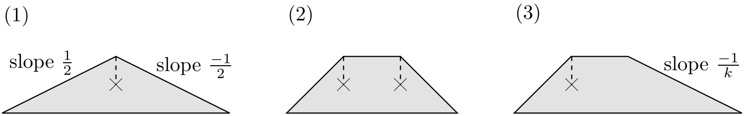

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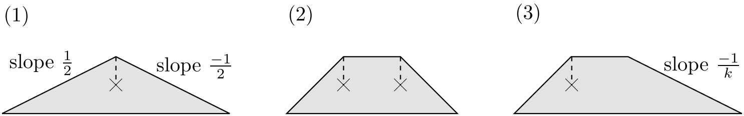

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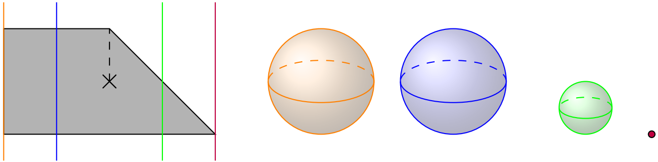

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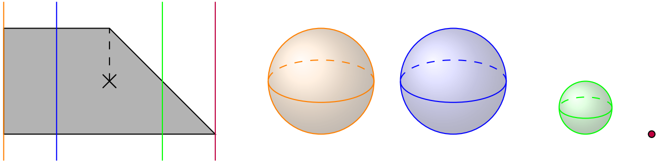
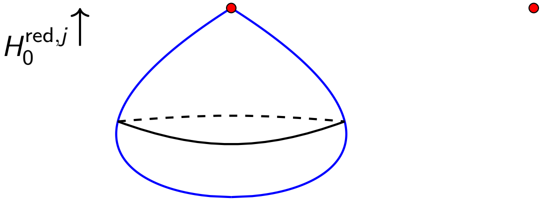


- If $dJ_j = 0$ get a 'teardrop' or 'pinched sphere' singular space.

Coupled angular momentum: reduction

M_j^{red}

singular fiber



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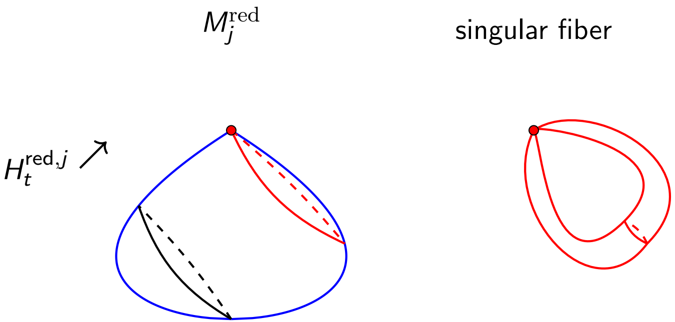

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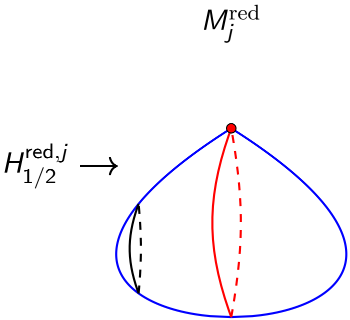

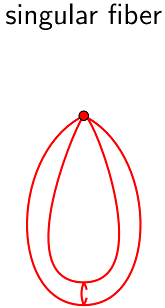

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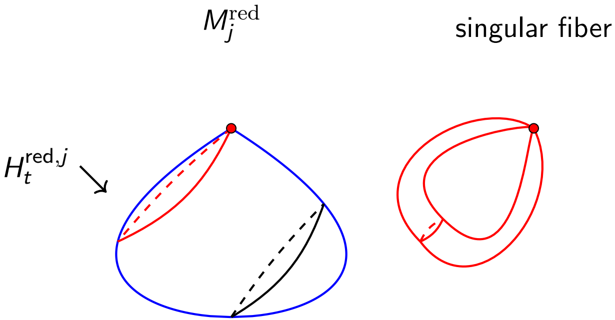

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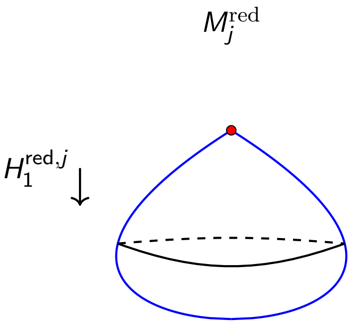

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