Tom & Jerry triples and the 4-intersection unprojection formats

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Assumption: All rings are commutative and with unit.

Definition

Assume $A = [a_{ij}]$ is an $m \times m$ skewsymmetric matrix, (i.e., $a_{ji} = -a_{ij}$ and $a_{ii} = 0$) with entries in a ring R.

- If $\mathbf{m = 2\ell}$ then $\det A = f(a_{ij})^2$. The polynomial $f(a_{ij})$ is called the **Pfaffian** of the matrix A and is denoted by $Pf(A)$.
- If $\mathbf{m = 2\ell + 1}$ by Pfaffians of A we mean the set

$$\{Pf(A_1), Pf(A_2), \dots, Pf(A_m)\},$$

where A_i denotes the skewsymmetric submatrix of A obtained by deleting the ith row and ith column of A.

Example

- For $\mathbf{m = 2}$:

$$Pf(\begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix}) = a_{12}$$

- For $\mathbf{m = 5}$:

$$Pf(\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 \end{pmatrix}) =$$

$$= \{Pf(A_1), Pf(A_2), \dots, Pf(A_5)\}$$

where

$$Pf(A_1) = a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34},$$
$$Pf(A_2) = a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34},$$
$$Pf(A_3) = a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24},$$
$$Pf(A_4) = a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23},$$
$$Pf(A_5) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Definition

A Noetherian local ring R is a **Gorenstein** ring if $\operatorname{inj}\dim_R R < \infty$. More generally, a Noetherian ring R is called **Gorenstein** if for every maximal ideal $\mathfrak{m}$ of R the localization $R_{\mathfrak{m}}$ is Gorenstein.

Examples of Gorenstein rings

- The anticanonical ring $R = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(-mK_X))$ of a (smooth) Fano n-fold.
- The canonical ring $R = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$ of a (smooth) regular surface of general type.
- The Stanley-Reisner ring of a simplicial sphere over any field.

Theorem

Let $R = k[x_1, \dots, x_m]/I$ be the polynomial ring in n variables divided by a homogeneous ideal I.

- (**Serre**) If **codim I = 1 or 2** then

  $$R \textit{ is Gorenstein} \Leftrightarrow I \textit{ is a complete intersection.}$$

- (**Buchsbaum-Eisenbud (1977)**) If **codim I = 3** then
  R is Gorenstein $\Leftrightarrow$ I is generated by the $2n \times 2n$ Pfaffians of a skewsymmetric $(2n+1) \times (2n+1)$ matrix with entries in $k[x_1, \dots, x_m]$.

Question

Is there a structure theorem for **codim I $\geq$ 4** ?

- **A.Kustin & M.Milller (1983)** introduced a procedure which constructs more «complicated» Gorenstein rings from simpler ones by increasing codimension. This procedure is called **Kustin-Miller unprojection**.

- **M.Reid (1995)** rediscovered what was essentially the same procedure working with Gorenstein rings arising from K3 surfaces and 3-folds.

Assumptions of Kustin-Miller unprojection:

- J $\subset$ R codimension 1 ideal
- R Gorenstein
- R/J Gorenstein.

Codimension: increasing by one.

Denote by i the canonical injection.
Under the assumptions above Reid proves that there exists ϕ such that $\mathrm{Hom}_R(J, R)$ is generated by i, ϕ as R-module.

Definition (M.Reid)

Unpr(J,R)= «graph of ϕ» $= \frac{R[T]}{(T\alpha - \phi(\alpha) \colon \alpha \in J)}$

Theorem (Kustin-Miller, Reid-Papadakis)

The ring Unpr(J,R) is Gorenstein.

Remarks

- Unpr(J,R) has typically more complicated structure than both R, R/J.
- Unpr(J,R) is useful to construct/analyse Gorenstein rings in terms of simpler ones.

Kustin-Miller unprojection can be used many times over an inductive way to produce Gorenstein rings of arbitrary codimensions, whose properties are nevertheless controlled by just a few equations as a number of new unprojection variables are adjoined.

Applications

- Construction of new interesting algebraic surfaces and 3-folds.
- Explicit Birational Geometry.
  (That is, writing down explicitly varieties, morphisms and rational maps that Minimal Model Program says they exist.)
- Algebraic Combinatorics.

Neves and Papadakis (2013) develop a theory, which is called **parallel Kustin-Miller unprojection**.

They set sufficient conditions on a positively graded Gorenstein ring R and a finite set of codimension 1 ideals which ensure the series of unprojections.

Furthermore, they give a simple and explicit description of the end product ring which corresponds to the unprojection of the ideals.

This theory applies when all the unprojection ideals of a series of unprojections correspond to ideals already present in the initial ring.

Assume J is a codimension 4 complete intersection ideal and M is a 5×5 skewsymmetric matrix.

Definition

1. Assume $1 \leq i \leq 5$. The matrix M is called Tom_i in J if after we delete the i-th row and i-th column of M the remaining entries are elements of the codimension 4 ideal J.
2. Assume $1 \leq i < j \leq 5$. The matrix M is called $Jerry_{ij}$ in J if all the entries of M that belong to the i-th row or i-th column or j-th row or j-th column are elements of J.

Remark

In both cases the Pfaffian ideal of M is a subset of J.

Papadakis' Calculation for Tom (2004)

Let $R = k[x_k, z_k, m_{ij}^k]$, where $1 \leq k \leq 4$, $2 \leq i < j \leq 5$, be a polynomial ring. Set $J = (z_1, z_2, z_3, z_4)$. Denote by

$$N = \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 \\ -x_1 & 0 & m_{23} & m_{24} & m_{25} \\ -x_2 & -m_{23} & 0 & m_{34} & m_{35} \\ -x_3 & -m_{24} & -m_{34} & 0 & m_{45} \\ -x_4 & -m_{25} & -m_{35} & -m_{45} & 0 \end{pmatrix},$$

where

$$m_{ij} = \sum_{k=1}^4 m_{ij}^k z_k.$$

Let I be the ideal generated by the Pfaffians P_0, P_1, P_2, P_3, P_4 of N. It holds that $I \subset J$.

Papadakis using multilinear and homological algebra calculates the equations of the codimension 4 ring which occurs as unprojection of the pair $I \subset J$.
More precisely, calculates 4 polynomials g_i for i $= 1, \dots, 4$ and defines the map ϕ by

$$\phi\colon J/I \to R/I, \quad z_i + I \mapsto g_i + I.$$

Moreover, he proves that $\mathrm{Hom}_{R/I}(J/I, R/I)$ is generated as R/I-module by the inclusion map i and ϕ. From the theory it follows that the ideal

$$(P_0, P_1, P_2, P_3, P_4, Tz_1 - g_1, Tz_2 - g_2, Tz_3 - g_3, Tz_4 - g_4)$$

of the polynomial ring $R[T]$ is Gorenstein of codimension 4.

We will now define Tom & Jerry triples.

Let

$$M = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} & m_{15} \\ -m_{12} & 0 & m_{23} & m_{24} & m_{25} \\ -m_{13} & -m_{23} & 0 & m_{34} & m_{35} \\ -m_{14} & -m_{24} & -m_{34} & 0 & m_{45} \\ -m_{15} & -m_{25} & -m_{35} & -m_{45} & 0 \end{pmatrix}$$

be a 5×5 skewsymmetric matrix and J_1, J_2, J_3 be three complete intersection ideals of codimension 4.

Definition

We say that M is a $\text{Tom}_1 + \text{Tom}_2 + \text{Tom}_3$ in J_1, J_2, J_3 if the entries of M satisfy the following conditions:

$$m_{12} \in J_3, \ \ m_{13} \in J_2, \ \ m_{14}, m_{15} \in J_2 \cap J_3, \ \ m_{23} \in J_1,$$
$$m_{24}, m_{25} \in J_1 \cap J_3, \ \ m_{34}, m_{35} \in J_1 \cap J_2, \ \ m_{45} \in J_1 \cap J_2 \cap J_3.$$

Remark

Equivalently, the matrix M is Tom_1 in J_1, Tom_2 in J_2 and Tom_3 in J_3.

Similarly, we set conditions in the entries of M such that M is

- Jerry_{ij} in J_1, Jerry_{kl} in J_2 and Jerry_{mn} in J_3.
- Tom_i in J_1, Tom_j in J_2 and Jerry_{kl} in J_3.
- Tom_i in J_1, Jerry_{jk} in J_2 and Jerry_{lm} in J_3.

We work over the polynomial ring $R = k[z_i, c_j]$, where $1 \leq i \leq 7$ and $1 \leq j \leq 25$. Denote by **Tom(1,2,3)** the following 5×5 skewsymmetric matrix

$$\begin{pmatrix} 0 & c_1z_1+c_2z_2+c_3z_3+c_4z_6 & c_5z_1+c_6z_2+c_7z_4+c_8z_5 & c_9z_1+c_{10}z_2 & c_{11}z_1+c_{12}z_2 \\ & 0 & c_{13}z_2+c_{14}z_3+c_{15}z_5+c_{16}z_7 & c_{17}z_2+c_{18}z_3 & c_{19}z_2+c_{20}z_3 \\ & & 0 & c_{21}z_2+c_{22}z_5 & c_{23}z_2+c_{24}z_5 \\ & -Sym & & 0 & c_{25}z_2 \\ & & & & 0 \end{pmatrix}$$

which is $\mathrm{Tom}_1+\mathrm{Tom}_2+\mathrm{Tom}_3$ matrix in the ideals

$$J_1 = (z_2, z_3, z_5, z_7), \quad J_2 = (z_1, z_2, z_4, z_5), \quad J_3 = (z_1, z_2, z_3, z_6).$$

Let I be the ideal generated by the Pfaffians of Tom(1,2,3).

Proposition

(i) For all t with $1 \leq t \leq 3$, the ideal J_t/I is a codimension 1 homogeneous ideal of R/I with Gorenstein quotient.
(ii) For all t, s with $1 \leq t < s \leq 3$, it holds that

$$codim_{R/I}(J_t/I + J_s/I) = 3.$$

Aim: Compution of $\phi_t \colon J_t/I \to R/I$ for all t with $1 \leq t \leq 3$.

Strategy: We combine Papadakis' Calculation for Tom_1 with the fact that a Tom_i matrix in an ideal J is related to Tom_1 matrix in the ideal J via a sequence of elementary row and column operations.

Proposition

For all t with $1 \leq t \leq 3$, the R/I-module $\text{Hom}_{R/I}(J_t/I, R/I)$ is generated by the two elements i_t and ϕ_t.

Proposition

For all t, s with $1 \leq t, s \leq 3$ and $t \neq s$, it holds that

$$\phi_s(J_s/I) \subset J_t/I.$$

.

Proposition

For all t, s with $1 \leq t, s \leq 3$ and $t \neq s$, there exists a homogeneous element A_{st} such that

$$\phi_s(\phi_t(p)) = A_{st} p$$

for all p$\in J_t/I$.

Let T_1, T_2, T_3 be three new variables of degree 6.

Definition

We define as I_{un} the ideal

$$(I) + (T_1z_2 - \phi_1(z_2),\ T_1z_3 - \phi_1(z_3),\ T_1z_5 - \phi_1(z_5),\ T_1z_7 - \phi_1(z_7),$$

$$T_2z_1 - \phi_2(z_1),\ T_2z_2 - \phi_2(z_2),\ T_2z_4 - \phi_2(z_4), T_2z_5 - \phi_2(z_5),$$

$$T_3z_1 - \phi_3(z_1),\ T_3z_2 - \phi_3(z_2),\ T_3z_3 - \phi_3(z_3),\ T_3z_6 - \phi_3(z_6),$$

$$T_1T_2 - A_{12},\ T_1T_3 - A_{13},\ T_2T_3 - A_{23})$$

of the polynomial ring $R[T, S, W]$. We set $R_{un} = R[T, S, W]/I_{un}$.

Proposition

The homogeneous ideal I_{un} is a codimension 6 ideal with a minimal generating set of 20 elements.

Theorem (P.)

The ring R_{un} is Gorenstein.

Assume that J is a fixed codimension 3 complete intersection ideal.

Question

Define a codimension 2 complete intersection ideal I such that I is a subset of J.

One answer on this Question is given by the following computation.

Papadakis' Calculation for Unprojection of a complete intersection inside a complete intersection, (2004)

Let $R = k[a_i, b_i, x_j]$, where $1 \leq i \leq 3$ and $j \in \{1, 3, 5\}$ be the standard graded polynomial ring. Fix $J = (x_1, x_3, x_5)$. Denote by A the following 2×3 matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

and by A_i the 2×2 submatrix of A which is obtained by removing the ith column.

Consider the ideal $I = (a_1x_1 + a_2x_3 + a_3x_5, b_1x_1 + b_2x_3 + b_3x_5)$ of R. It holds that $I \subset J$.

Denote by h_i for i$= 1 \dots 3$, the polynomial which is equal to the determinant of the submatrix A_i and define the map ϕ by

$$\phi\colon J/I \to R/I,$$

$$\phi(x_1 + I) = h_1 + I, \quad \phi(x_3 + I) = -h_2 + I, \quad \phi(x_5 + I) = h_3 + I$$

Papadakis proved that $\mathrm{Hom}_{R/I}(J/I, R/I)$ is generated as R/I-module by the inclusion map i and ϕ. From the theory it follows that the ideal

$$I + (Tx_1 - h_1, Tx_3 - (-h_2), Tx_5 - h_3)$$

of the polynomial ring $R[T]$ is Gorenstein of codimension 3.

Let J_1, J_2, J_3, J_4 be four codimension 3 complete intersection ideals and I is a codimension 2 complete intersection ideal.

Definition

We say that I is a 4-intersection ideal with respect to the ideals J_1, J_2, J_3, J_4 if I is subset of each of the ideals J_1, J_2, J_3, J_4.

An example of a 4-intersection unprojection format is the following: We work over the standard graded polynomial ring $R = k[c_i, x_i]$, where $1 \leq i \leq 6$. We set

$$I_{1234} = (c_1x_1x_2 + c_2x_3x_4 + c_3x_5x_6, c_4x_1x_2 + c_5x_3x_4 + c_6x_5x_6).$$

Then, I_{1234} is a 4-intersection ideal in the ideals

$$J_1 = (x_1, x_3, x_5), J_2 = (x_1, x_4, x_6), J_3 = (x_2, x_3, x_6), J_4 = (x_2, x_4, x_5).$$

We proved that this initial data satisfies the conditions for parallel Kustin-Miller unprojection.

For all t, with $1 \leq t \leq 4$, denote by $i_t \colon J_t/I_{1234} \to R/I_{1234}$ the inclusion map.

Aim: Definition of $\phi_t \colon J_t/I_{1234} \to R/I_{1234}$ for all t with $1 \leq t \leq 4$.

Strategy: We combine Papadakis' Calculation for a complete intersection I inside a complete intersection J with the fact that I_{1234} is a complete intersection in J_t.

Let T_1, T_2, T_3, T_4 be four new variables of degree 3.

Definition

We define as $\overline{I_{un}}$ the ideal

$$(I) + (T_1x_1\text{-}\phi_1(x_1),\ T_1x_3\text{-}\phi_1(x_3),\ T_1x_5\text{-}\phi_1(x_5),\ T_2x_1\text{-}\phi_2(x_1),$$

$$T_2x_4\text{-}\phi_2(x_4),\ T_2x_6\text{-}\phi_2(x_6),\ T_3x_2\text{-}\phi_3(x_2),\ T_3x_3\text{-}\phi_3(x_3),\ T_3x_6\text{-}\phi_3(x_6),$$

$$T_4x_2\text{-}\phi_4(x_2), T_4x_4\text{-}\phi_4(x_4),\ T_4x_5\text{-}\phi_4(x_5),\ T_2T_1\text{-}A_{21},\ T_3T_1\text{-}A_{31},$$

$$T_4T_1\text{-}A_{41},\ T_3T_2\text{-}A_{32},\ T_4T_2\text{-}A_{42},\ T_4T_3\text{-}A_{43})$$

of the polynomial ring $R[T_1, T_2, T_3, T_4]$. We set
$\overline{R_{un}} = R[T_1, T_2, T_3, T_4]/\overline{I_{un}}$.

Theorem (P., 2021)

$\overline{I_{un}}$ is a codimension 6 Gorenstein ideal with 20 generators.

Applications using Tom & Jerry triples unprojection format

We now give two applications of the construction of R_{un}.

Theorem (P.)

There exists a family of quasismooth, projectively normal and projectively Gorenstein Fano 3-folds $X \subset \mathbb{P}(1^3, 2^7)$, nonsingular away from eight quotient singularities $\frac{1}{2}(1,1,1)$, with Hilbert series of the anticanonical ring

$$\frac{1 - 20t^4 + 64t^6 - 90t^8 + 64t^{10} - 20t^{12} + t^{16}}{(1-t)^3(1-t^2)^7}.$$

Theorem (P.)

There exists a family of quasismooth, projectively normal and projectively Gorenstein Fano 3-folds $X \subset \mathbb{P}(1^3, 2^5, 3^2)$, nonsingular away from four quotient singularities $\frac{1}{2}(1,1,1)$, and two quotient singularities $\frac{1}{3}(1,1,2)$, with Hilbert series of the anticanonical ring

$$\frac{1-11t^4-8t^5+23t^6+32t^7-13t^8-48t^9-13t^{10}+32t^{11}+23t^{12}-8t^{13}-11t^{14}+t^{18}}{(1-t)^3(1-t^2)^5(1-t^3)^2}.$$

Construction of the first family:

Denote by $k = \mathbb{C}$ the field of complex numbers.
Let R_{un} be the ring and I_{un} the ideal which were defined above.
Substitute the variables $(c_1, \dots, c_{25})$ with a general element of k^{25}.
$\hat{R}_{un}$: the ring which occurs from R_{un} after this substitution.
$\hat{I}_{un}$: the ideal which obtained by the ideal I_{un} after this substitution.
In what follows we set

$$degree\ z_i = degree\ T_1 = degree\ T_2 = degree\ T_3 = 2,$$

for all i with $1 \leq i \leq 7$.
Since R_{un} is Gorenstein, Proj $\hat{R}_{un} \subset \mathbb{P}(2^{10})$ is a projectively Gorenstein 3-fold.

Let $A = k[w_1, w_2, w_3, z_1, z_2, z_3, z_5, T_1, T_2, T_3]$ be the polynomial ring over k with w_1, w_2, w_3 variables of degree 1. Consider the graded k-algebra homomorphism

$$\psi\colon \hat{R}_{un}[T_1, T_2, T_3] \to A$$

with

$$\psi(z_1) = z_1, \quad \psi(z_2) = z_2, \quad \psi(z_3) = z_3, \quad \psi(z_4) = f_1,$$
$$\psi(z_5) = z_5, \quad \psi(z_6) = f_2, \quad \psi(z_7) = f_3, \quad \psi(T_1) = T_1,$$
$$\psi(T_2) = T_2, \quad \psi(T_3) = T_3$$

where

$f_1 = l_1 z_1 + l_2 z_2 + l_3 z_3 + l_4 z_5 + l_5 T_1 + l_6 T_2 + l_7 T_3 + l_8 w_1^2 + l_9 w_1 w_2 + l_{10} w_1 w_3 + l_{11} w_2^2 + l_{12} w_2 w_3 + l_{13} w_3^2$,

$f_2 = l_{14} z_1 + l_{15} z_2 + l_{16} z_3 + l_{17} z_5 + l_{18} T_1 + l_{19} T_2 + l_{20} T_3 + l_{21} w_1^2 + l_{22} w_1 w_2 + l_{23} w_1 w_3 + l_{24} w_2^2 + l_{25} w_2 w_3 + l_{26} w_3^2,$

$f_3 = l_{27} z_1 + l_{28} z_2 + l_{29} z_3 + l_{30} z_5 + l_{31} T_1 + l_{32} T_2 + l_{33} T_3 + l_{34} w_1^2 + l_{35} w_1 w_2 + l_{36} w_1 w_3 + l_{37} w_2^2 + l_{38} w_2 w_3 + l_{39} w_3^2$

and $(l_1, \dots, l_{39}) \in k^{39}$ are general.

Denote by Q the ideal of the ring A generated by the subset $\psi(\hat{I}_{un})$. Let $X = V(Q) \subset \mathbb{P}(1^3, 2^7)$. Then X is a codimension 6 projectively Gorenstein 3-fold.

Proposition

The ring A/Q is an integral domain.

Proposition

Consider $X = V(Q) \subset \mathbb{P}(1^3, 2^7)$. Denote by $X_{cone} \subset \mathbb{A}^{10}$ the affine cone over X. The scheme X_{cone} is smooth outside the vertex of the cone.

Proposition

Consider the singular locus $Z = V(w_1, w_2, w_3)$ of the weighted projective space $\mathbb{P}(1^3, 2^7)$. The intersection of X with Z consists of exactly eight points which are quotient singularities of type $\frac{1}{2}(1,1,1)$ for X.

Proposition

The minimal graded resolution of A/Q as A-module is equal to

$$0 \to A(-16) \to A(-12)^{20} \to A(-10)^{64} \to A(-8)^{90} \to A(-6)^{64}$$

$$\to A(-4)^{20} \to A$$

Moreover, the canonical module of A/Q is isomorphic to $(A/Q)(-1)$ and the Hilbert series of A/Q as graded A-module is equal to

$$\frac{1 - 20t^4 + 64t^6 - 90t^8 + 64t^{10} - 20t^{12} + t^{16}}{(1-t)^3(1-t^2)^7}.$$

Application using 4-intersection unprojection format

As an application of the construction of $\overline{R_{un}}$ we proved the following theorem.

Theorem (P.)

There exists a family of quasismooth, projectively normal and projectively Gorenstein Fano 3-folds $X \subset \mathbb{P}(1^8, 2, 3)$, nonsingular away from eight quotient singularities $\frac{1}{3}(1,1,2)$, with Hilbert series of the anticanonical ring

$$\frac{1 - 6t^2 + 15t^4 - 20t^6 + 15t^8 - 6t^{10} + t^{12}}{(1-t)^8(1-t^2)(1-t^3)}.$$

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Thank you!!!