The geometry of Weyl orbits on blow-ups of projective spaces

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Blow-ups of projective spaces at points in general position

Let $p_1, \dots, p_s \subset \mathbb{P}^n_{\mathbb{C}}$ points in general position. Consider the blow-up

$$\pi : X^n_s \to \mathbb{P}^n$$

$$E_i \mapsto p_i$$

If $H = \pi^* \mathcal{O}_{\mathbb{P}^n}(1)$ is the general hyperplane class, then

$$\mathrm{Pic}(X^n_s) = \langle E_1, \dots, E_s, H \rangle$$

- *Motivation (polynomial interpolation):*

$$\left\{\begin{array}{l}\text{Degree-}d \text{ hypersurfaces of } \mathbb{P}^n \\ \text{vanishing with multiplicity} \geq m_i \text{ at } p_i, \\ i = 1 \dots, s\end{array}\right\} \leftrightarrow |D| = \left| dH - \sum_{i=1}^{s} m_i E_i \right|$$

How many?
Any/how many *unexpected* ones? $\leftrightarrow$ $\dim H^0(X, \mathcal{O}(D)) = h^0(D)$
$\dim H^1(X, \mathcal{O}(D))$

Virtual dimension and Base locus

The virtual dimension of $|D|$ is

$$\mathrm{vdim}(D) = \chi(\mathcal{O}_X(D)) - 1 = \binom{n+d}{n} - \sum_{i=1}^{s} \binom{n+m_i-1}{n} - 1$$

- If $h^0(D) = \chi(D) \rightsquigarrow D$ non-special
- $\chi(D) = h^0(D) - \underbrace{h^1(D)}_{\text{measure of speciality}}$, $h^i(D) = 0 \quad i \geq 2$

- If $\mathrm{Bs}|D| = \emptyset$, we expect that it is non-special.
- If $\mathrm{Bs}|D| \neq \emptyset$, then it might be special (in this case we talk about special effect subvariety).

Can we say when it actually is special?

Special effect plane curves: examples

$\mathrm{Bl}_s\,\mathbb{P}^2$

1. $\underbrace{|6H-4E_1-4E_2|}_{D} = 2\underbrace{(H-E_1-E_2)}_{\text{fixed part}} + \underbrace{|4H-2E_1-2E_2|}_{\text{mef}},$ $\quad h^1(D)=1$

   $\chi(D)=8 \qquad h^0=9$

2. $|7H-5E_1-5E_2| = 3(H-E_1-E_2) + |4H-2E_1-2E_2|,$ $\quad h^1(D)=3$

   $\chi(D)=6$ special effect line $h^0=9$

3. $|4H-2\sum_{i=1}^5 E_i| = 2(2H-\sum_{i=1}^5 E_i)$ $\quad h^1(D)=1$

   $\chi(D)=0$ special effect conic

4. $|6H-2\sum_{i=1}^9 E_i| = 2(3H-\sum_{i=1}^9 E_i)$ $\quad h^1(D)=0$

Rmk
- (-1)-curves : irred. rat'l $C^2=-1$ $(CK_X=-1)$
- cubic is not

Conjectures for $\mathbb{P}^2$

B. Segre

Conjecture (SHGH) $Bl_s \mathbb{P}^2$

Special effect curves for nonempty linear systems on X^2_s are all and only the (-1)-curves (contained at least twice in the base locus).

True for $s \leq 9$ (Castelnuovo)

$s \leq 8$ finitely many (-1)-curves

$s = 9$ ∞ many (-1)-curves

Conjecture (Nagata)

The divisor $|dH - m\sum_i^s E_i| = \emptyset$ if $s \geq 9$ and $d \leq \sqrt{s}m$.

Mori dream blow-ups of projective spaces

Theorem (Mukai '01; Castravet, Tevelev '06)

X^n_s is a Mori dream space i.e. Cox(X) f.g. if and only if

1. $n = 2$ & $s \leq 8$, ✓ Castelnuovo
2. $n = 3$ & $s \leq 7$, ✓ Today's talk
3. $n = 4$ & $s \leq 8$, In progress (C. Santana Sánchez)
4. $n \geq 5$ & $s \leq n + 3$. ✓ Today's talk

Dimensionality

Linear systems on del Pezzo surfaces

Assume $s \leq 8$ and consider $X = X^2_8$.

Theorem (Castelnuovo)

Consider divisors $D = dH - \sum_i^s m_i E_i$. Then

$\max\{0, -C.D\}$

$$h^0(X, D) = \chi(D) + \underbrace{\sum_{C} \binom{\mathrm{mult}_C(D)}{2}}_{h^1(D)}$$

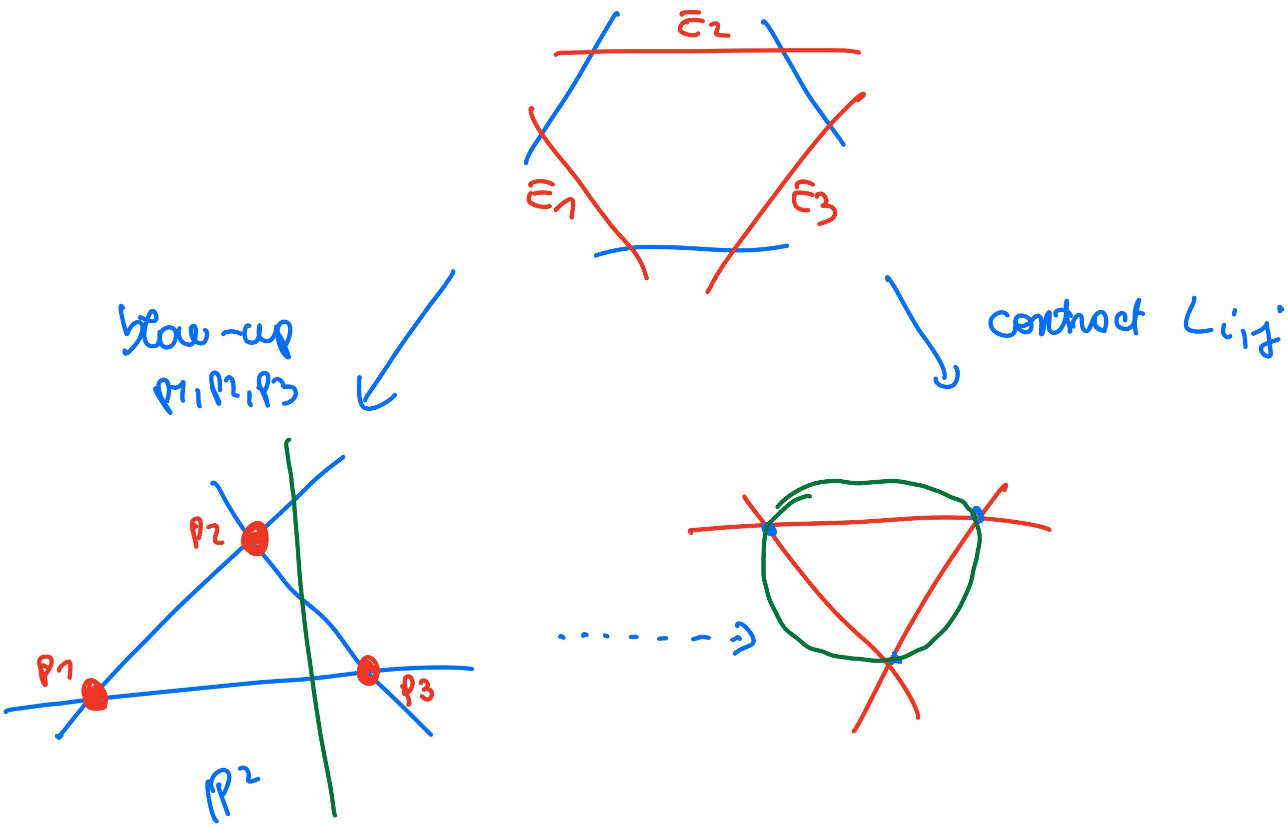
(-1)-curves

Moreover,

- (-1)-curves generate the effective cone of X.
- (-1)-curves are related to one another by a sequence of Cremona transformations

Standard Cremona involutions of $\mathbb{P}^2$

$$\mathrm{Cr} : \mathbb{P}^2 \to \mathbb{P}^2, \quad [x_0 : x_1 : x_2] \to [x_0^{-1} : x_1^{-1} : x_2^{-1}],$$



Standard Cremona involutions of $\mathbb{P}^2$

Cr lifts to an automorphism of $\mathrm{Pic}(X_3^2)$, that extends to $\mathrm{Pic}(X_s^2)$

$$
\begin{array}{rlcl}
 & \mathrm{Pic}(X_s^2) & \longrightarrow & \mathrm{Pic}(X_s^2) \\
 & H & \longmapsto & 2H - E_1 - E_2 - E_3 \\
\{i,j,k\} = \{1,2,3\} & E_i & \longmapsto & H - E_j - E_k \\
j \notin \{1,2,3\} & E_j & \longmapsto & E_j
\end{array}
$$

Action is transitive with

- finite orbit if $s \leq 8$
- infinite orbit if $s \geq 9$

Standard Cremona involutions of $\mathbb{P}^n$

$$\mathrm{Cr} : \mathbb{P}^n \to \mathbb{P}^n, \quad [x_0 : \cdots : x_n] \to [x_0^{-1} : \cdots : x_n^{-1}],$$

Action on $\mathrm{Pic}(X_s^n)$:

$$\begin{aligned}
H &\longmapsto nH - (n-1)\sum_{i \in I} E_i \\
E_i &\longmapsto H - \sum_{j \in I \setminus \{i\}} E_j & i \in I \\
E_i &\longmapsto E_i & i \notin I
\end{aligned}$$

Definition (Dolgachev '83)

The Weyl group W_s^n acting on $\mathrm{Pic}(X_s^n)$ is the group generated by the standard Cremona involutions with the operation of composition.

(-1)-divisors: Dolgachev-Mukai pairing

$\langle\cdot,\cdot\rangle : Pic(X_s^n) \times Pic(X_s^n) \to \mathbb{Z}$

$$\langle H, H\rangle = n-1$$
$$\langle H, E_i\rangle = 0$$
$$\langle E_i, E_j\rangle = -\delta_{i,j}.$$

D is a (-1)-divisor if

$$\langle D, D\rangle = -1, \quad \frac{1}{n-1}\langle D, -K_X\rangle = 1$$

If X_s^n is a MDS $\iff$ finitely many (-1)-divisors
They form a single orbit for the W_s^n action.
They generate $Eff(X)$

Weyl cycles

(-1) divisor

Definition (Brambilla, Dumitrescu, P)

- We say that an effective divisor $D \in \mathrm{Pic}(X^n_s)$ is a Weyl divisor if it belongs to the Weyl orbit of an exceptional divisor E_i.
- A non-trivial effective cycle $S \in A^{n-r}(X^n_s)$ is a Weyl cycle of dimension r if it is an irreducible component of the intersection of pairwise orthogonal Weyl divisors.

w.r.t Mukai pairing

Examples of Weyl cycles

Example

$n = 3$

$\langle D, F \rangle = 0$

$$D = H - E_1 - E_2 - E_3$$
$$F = H - E_1 - E_2 - E_4$$

$$D \cap F = \boxed{L_{1,2}}$$

Example

$n = 4$

$$D = H - E_1 - E_2 - E_3 - E_4$$
$$F = H - E_1 - E_2 - E_3 - E_5$$
$$G = H - E_1 - E_2 - E_4 - E_5$$

$$D \cap F = \boxed{L_{1,2,3}}$$
$$D \cap F \cap G = \boxed{L_{1,2}}$$

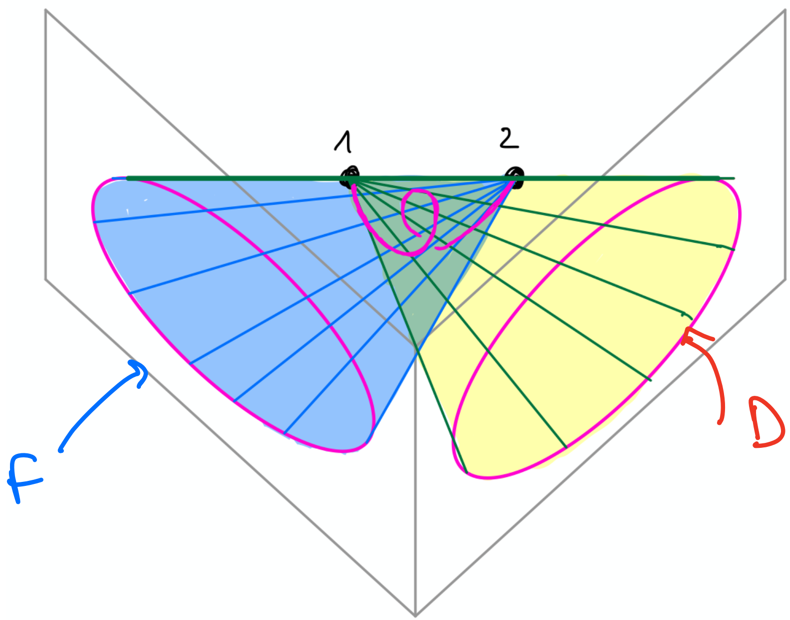
In general: (Strict transforms of) linear spans of points are Weyl cycles

Weyl cycles

Example $n = 3$

$$D = 2H - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6$$

$$F = 2H - E_1 - 2E_2 - E_3 - E_4 - E_5 - E_6$$



$$D \cap F = (h - e_1 - e_2) + (3h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6)$$

Weyl line and Weyl twisted cubic

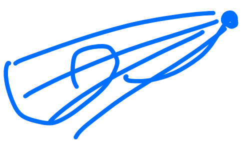
Weyl cycles on X^n_{n+3}

Theorem (Brambilla, Dumitrescu, Laface, P, Santana Sánchez)

The following are Weyl cycles:

- L_I, $I = \{i_0, \dots, i_r\}$, *linear spans of points*
- C, *the rational normal curve of degree n through $n+3$ points*
- $\sigma_t(C)$, *the secant varieties of C*
- $\mathrm{Join}(\sigma_t(C), L_I)$

$$\sigma_t(C) = \overline{\bigcup \{t\text{-secant } (t-1)\text{-planes}\}}$$



•) all Weyl cycles in the list of dim r belong to Weyl orbit of r-plane

•) divisorial ones generate $\mathrm{Eff}(X^n_{n+3})$

Weyl cycles on X^3_7

The Weyl cycles on X^3_7 are all and only the following:

- Curves (28 classes):
  1. $L_{i,j}$ — 21 lines through two points
  2. C_i — 7 twisted cubic through six points

- Surfaces (126 classes):
  1. E_1 — 7 exceptional divisors
  2. $H - E_1 - E_2 - E_3$ — 35 planes through 3 points
  3. $2H - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6$ — 42 quadric cones
  4. $3H - 2(E_1 + E_2 + E_3 + E_4) - E_5 - E_6 - E_7$ — 35 Cayley surfaces
  5. $4H - 3E_1 - 2(E_2 + E_3 + E_4 + E_5 + E_6 + E_7)$ — 7 quartic surfaces

Weyl curves and surfaces on X^4_8

(Up to the S_8 action)

- Curves (35 classes):

1. $L_{i,j}$ — 28 lines through 2 points
2. C_i — 7 quartic normal curve through 7 points

- Surfaces (196 classes):

1. $h - e_1 - e_4 - e_5$ — 56 planes through 3 points
2. $3h - 3e_1 - \sum_{i=2}^{7} e_i$ — 48 pointed cones
3. $6h - 3\sum_{i=1}^{5} e_i - \sum_{i=6}^{8} e_i$ — 56 sextic surfaces
4. $10h - 6e_1 - 6e_2 - \sum_{i=3}^{8} 3e_i$ — 28 degree 10 surfaces
5. $15h - \sum_{i=1}^{7} 6e_i - 3e_8$ — 8 degree 15 surfaces

Single Weyl orbit

Weyl divisors on X_8^4

- Divisors (2160 classes):

1. E_1, — exceptional
2. $H - \sum_{i=1}^{4} E_i$ — hyperplane through 4 points
3. $2H - 2E_1 - 2E_1 - \sum_{i=3}^{7} E_i$ — quadric cone over a RNC
4. $3H - \sum_{i=1}^{7} 2E_i$ — secant-line variety to a RNC
5. $3H - 3E_1 - \sum_{i=2}^{5} 2E_i - \sum_{i=6}^{8} E_i$ — pointed cone over a Cayley surface
6. $4H - \sum_{i=1}^{4} 3E_i - \sum_{i=5}^{7} 2E_i - E_8$
7. $4H - 4E_1 - 3E_2 - \sum_{i=3}^{8} 2E_i$
8. $5H - 4E_1 - 4E_2 - \sum_{i=3}^{6} 3E_i - 2E_7 - 2E_8$
9. $6H - 5E_1 - \sum_{i=2}^{4} 4E_i - \sum_{i=5}^{8} 3E_i$
10. $6H - \sum_{i=1}^{6} 4E_i - 3E_7 - 2E_8$
11. $7H - \sum_{i=1}^{3} 5E_i - \sum_{i=4}^{7} 4E_i - 3E_8$
12. $7H - 6E_1 - \sum_{i=2}^{8} 4E_i$
13. $8H - 6E_1 - \sum_{i=2}^{6} 5E_i - 4E_7 - 4E_8$
14. $9H - \sum_{i=1}^{4} 6E_i - \sum_{i=5}^{8} 5E_i$
15. $10H - 7E_1 - \sum_{i=2}^{8} 6E_i$

Dimensionality for X^n_{n+3} and X^3_7

Theorem (Brambilla, Dumitrescu, Laface, P, Santana Sánchez)

For $X = X^n_{n+3}$ or $X = X^3_7$, then

- *the special effect varieties are all and only the above Weyl cycles.*
- *the dimension formula is*

$$h^0(D) := \chi(D) + \underbrace{\sum_{W} (-1)^{r+1} \binom{n + mult_W(D) - \overbrace{\dim(W)}^{\sim} - 1}{n}}_{h^1(D)}.$$

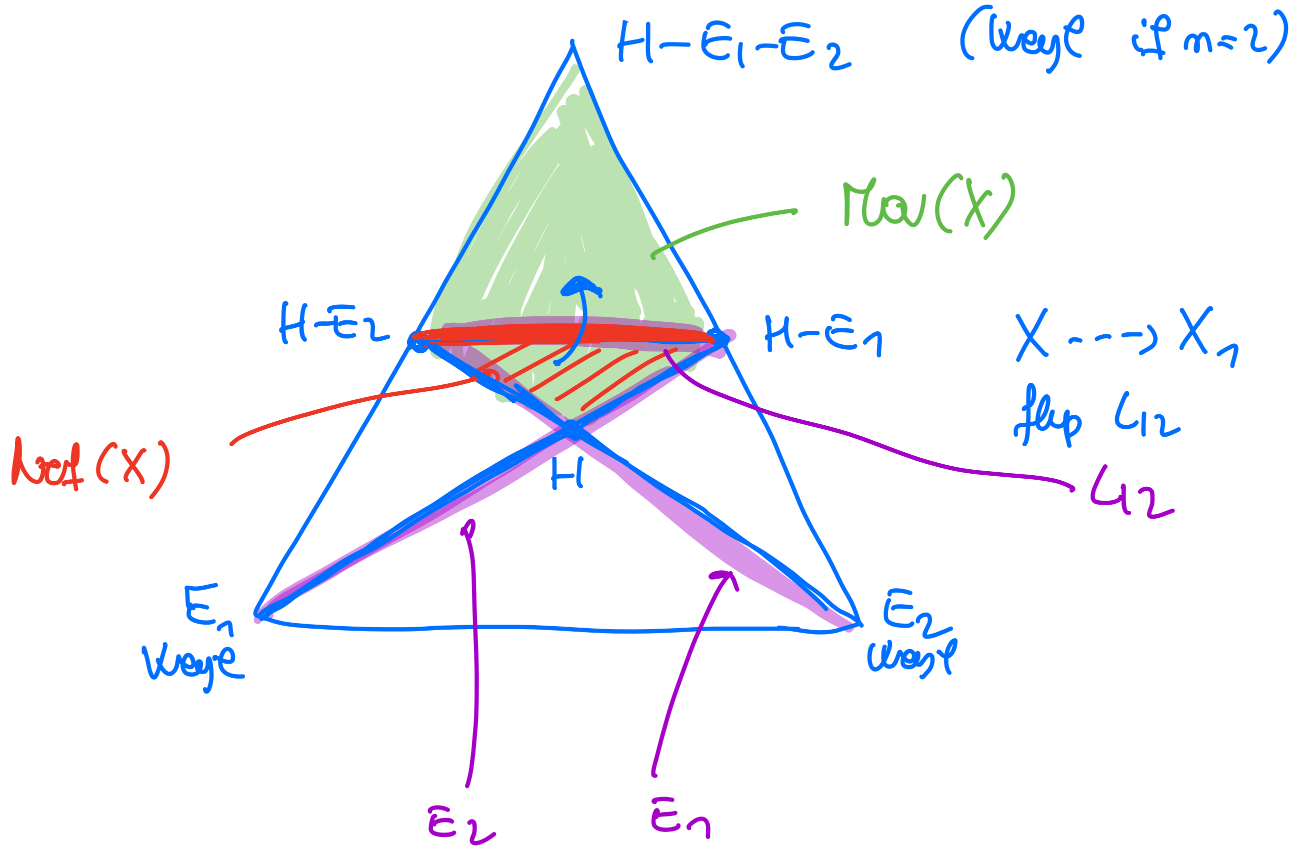
Weyl cycles

Open case: X^4_8

Chamber decompositions of the effective cone of divisors

If X_s^n is a MDS, then $Eff(X)_{\mathbb{R}}$ and $Mov(X)_{\mathbb{R}}$ are closed polyhedral cones, and $Mov(X)_{\mathbb{R}}$ has finite nef chamber decomposition.

X_2^n



Chamber decompositions of the effective cone of divisors

Lemma (Brambilla, Dumitrescu, P)

For $X = X^n_s$ MDS, if W is a Weyl cycle, then

$$mult_W(D) = \max\{0, -D \cdot \gamma_W\},$$

for $(\exists!)$ γ_W in $N_1(X)_{\mathbb{R}}$ that sweeps out W.

Theorem (Mukai; Casagrande-Codogni-Fanelli; B-D-P-S)

For $X = X^n_{n+3}$ and for $X = X^4_8$, the hyperplane arrangement in $N^1(X)_{\mathbb{R}}$:

$$\bigcup_i \{m_i = 0\} \cup \bigcup_W \{D \cdot \gamma_W = 0\},$$

induce the Mori chamber decomposition (and the stable base locus decomposition).