Construction of non-Kähler Calabi-Yau manifolds by log deformations

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§ Intro

X : CY mfd (in the strict sense) $\overset{\text{def}}{\Longleftrightarrow}$
$$\begin{cases} X : \text{compact complex mfd.} \\ \omega_X \simeq \mathcal{O}_X \\ H^i(X, \mathcal{O}_X) = 0 = H^0(X, \Omega_X^i) \quad (0 < {}^\forall i < \dim X) \end{cases}$$

classification

- $\dim X = 1$: ell. curve. · $\dim X = 2$: K3 surface
- $\dim X \geq 3$: not classified (≥ 10000 top types of proj. CY3) — finiteness is not known. important open problem.

rem $\exists\, \infty$-many top. types of non-Kähler CY3.

(Clemens, Friedman) ${}^\exists X_5 \subseteq \mathbb{P}^4_{\mathbb{C}}$: sm. quintic 3-fold with ∞-many disj. $(-1,-1)$-curves $C_1, C_2, \cdots$

${}^\forall m > 0$

$$X_5 \downarrow \text{(analytic contraction of } C_1, \cdots, C_m\text{)}$$

$$\overline{X}_5 \xrightarrow{\text{smoothing}} Y_m : \text{non-Kähler CY3}$$

$\overline{X}_5 \ni p_1, \cdots, p_m$: ODP

$\rightsquigarrow \begin{cases} b_2(Y_m) = 0 & \text{(2nd Betti \#)} \to \text{non-Kähler} \\ e(Y_m) = -200-2m & \text{(top. Euler \#)} \to \infty\text{-many top types.} \end{cases}$

- (Miyaoka, Friedman, Oguiso)

  analytic flop of a $(-1,-1)$-curve on a proj. CY3 $\rightsquigarrow$ $\exists\, \infty$-many top. types of Moishezon CY3.

- (Hashimoto–S.)

  $\forall a>0$ $\exists X(a)$: non-Kähler CY3 w/ $\begin{cases} b_2(X(a)) = a+3 \\ a(X(a)) = 1 \quad \text{(algebraic dimension)} \\ \text{w/ } X(a) \to \mathbb{P}^1 : K3 \text{ fibration.} \end{cases}$

  (constructed by smoothing simple normal crossing var.)

<u>Q</u> : Fix $N \geq 4$.
Can we construct ∞-many top. types of non-Kähler CY N-folds?

Main Thm (S.) $\forall N \geq 4$: fix, $\forall m > 0$.

$\Rightarrow \exists X(m)$: non-Kähler CY N-fold s.t.

- $b_2(X(m)) = \begin{cases} m+10 & (N=4) \\ m+2 & (N \geq 5) \end{cases}$ $\nearrow \infty$ $(m \to \infty)$
- $a(X(m)) = N-2$ &

$\exists f : X(m) \to T$: K3 fibration over T

($T \to \mathbb{P}^{N-2}$: blow-up along sm. codim 2. subvar)

(**Construction**

Use log deformation theory due to Kawamata-Namikawa)

rem:

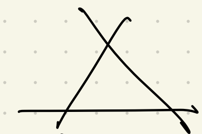
- **N-H. Lee** : $\exists$ non-Kähler CY 4-fold. (smoothing SNC variety)
- X, Y : CY mfd $\Rightarrow$ $X \times Y$: "not strict CY." ($\because H^0(\Omega^{\dim X}_{X \times Y}) \neq 0$)

$\begin{pmatrix} \dim X > 0 \\ \dim Y > 0 \end{pmatrix}$

§ log deformation theory of SNC CY varieties

X: <u>SNC. CY variety</u> $\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} X : \text{proper SNC } \mathbb{C}\text{-scheme}. \left(X = \bigcup_{i=1}^{N} X_i \ (X_i : \text{sm. proper var.}) \right) \\ \omega_X \cong \mathcal{O}_X \end{cases}$

<u>e.g.</u> (1) $X := (z_0 \cdots z_n = 0) \subseteq \mathbb{P}^n_{\mathbb{C}}$: SNC. CY $(n-1)$-fold.

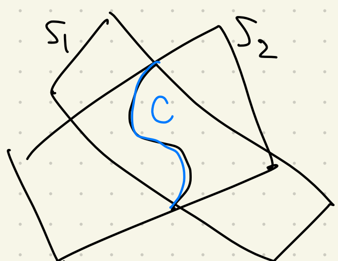


$(n=2)$

(2) $S = (q_1 q_2 = 0) \subseteq \mathbb{P}^3_{\mathbb{C}}$ $\quad$ $[q_i \in |\mathcal{O}_{\mathbb{P}^3}(2)| \text{ : general }]$

$\;\;\; \shortparallel$

$S_1 \cup S_2 \quad (S_i := (q_i = 0) \supseteq C := (q_1 = q_2 = 0))$

$S =$ 

Thm (Friedman, Kawamata-Namikawa, Chan-Leung-Ma, Felten-Petracci [Felten-Filip-Ruddat])

X: SNC CY var, Assume X: "d-semistable"

$\Rightarrow$ X has a semistable smoothing, i.e.

$\exists \phi : \mathcal{X} \to \Delta^1$: deformation of X s.t. $\begin{cases} \mathcal{X} : \text{smooth} \\ \mathcal{X}_t : \text{smooth } (t \neq 0) \\ (\& \ \omega_{\mathcal{X}_t} \cong \mathcal{O}_{\mathcal{X}_t}) \end{cases}$

$$\begin{array}{ccc} \mathcal{X} & \to & \Delta^1 \\ \cup & & \cup \\ \mathcal{X}_0 & \longrightarrow & 0 \\ \wr\Vert & & \\ X & & \end{array}$$

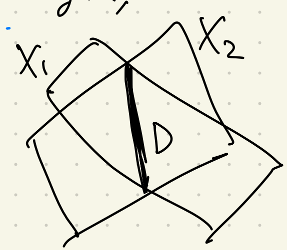
(Δ^1: unit disk)

rem (1) d-semistability ⋯ necessary condition for $\exists$ce of s.s. smoothing. (Friedman)

(2) When $X = X_1 \cup X_2$: SNC. var & $D := X_1 \cap X_2 = \mathrm{Sing}\, X$,

- X: d-s.s. $\Longleftrightarrow$ $N_{D/X_1} \otimes N_{D/X_2} \cong \mathcal{O}_D$

(N_{D/X_1}, N_{D/X_2}: normal bundles.)



e.g. (i) $S_0 := (q_1 q_2 = 0) \subseteq \mathbb{P}^3_{\mathbb{C}}$ as before.

$S_0 = S_1 \cup S_2 \supseteq C = (q_1 = q_2 = 0) \quad (S_i := (q_i = 0) \subseteq \mathbb{P}^3)$

$\Rightarrow S_0$: not d-semistable. $\left(\because N_{C/S_1} \otimes N_{C/S_2} \cong \mathcal{O}_C(2) \otimes \mathcal{O}_C(2) \cong \mathcal{O}_C(4) \not\cong \mathcal{O}_C\right)$

(ii) In (i), take $p_1 + \cdots + p_{16} \in |\mathcal{O}_C(4)|$: distinct 16 points.

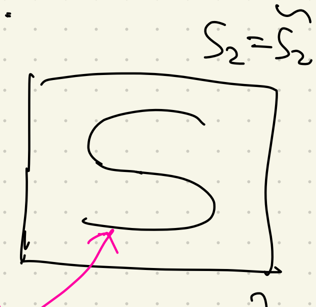
$\mu : \tilde{S}_1 \to S_1$: blow-up at $p_1, \cdots, p_{16}$, $\tilde{S}_2 := S_2$

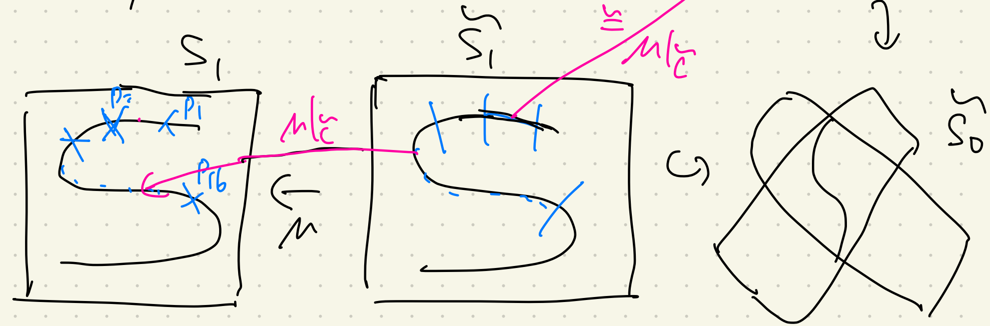
$|-K_{\tilde{S}_1}| \ni \tilde{C} \xrightarrow[\mu|_{\tilde{C}}]{\cong} C \subset S_1$, $\quad \tilde{S}_1 \supset \tilde{C}$, $\quad S_2 \supset C \in |-K_{S_2}|$

$\Rightarrow \tilde{S}_0 := \tilde{S}_1 \cup \tilde{S}_2$: glued by $\mu|_{\tilde{C}} : \tilde{C} \to C$

$\Rightarrow \tilde{S}_0$: d-s.s. SNC CY.

$\because \left(N_{\tilde{C}/\tilde{S}_1} \otimes \mu|_{\tilde{C}}^{*} N_{C/S_2} \cong \mathcal{O}_{\tilde{C}}(-2) \otimes \mathcal{O}_{\tilde{C}}(2) \cong \mathcal{O}_{\tilde{C}} . \right)$





We used:

<u>Fact</u> (Anantharaman, Ferrand, ~)

X_1, X_2 : sm. proper var, $D_i \subseteq X_i$: smooth divisor $(i=1,2)$ with $\varphi : D_1 \xrightarrow{\simeq} D_2$.

$\Rightarrow \exists!\ X_0$: SNC proper variety with $\iota_i : X_i \hookrightarrow X_0$ (cld. imm.) &

$$\begin{array}{ccc} D_1 & \hookrightarrow & X_1 \\ {\scriptstyle \varphi}\downarrow\wr & \square & \downarrow \\ D_2 \hookrightarrow X_2 & \hookrightarrow & X_0 \end{array}.$$

$X_0 = X_1 \cup_{\varphi} X_2$ ← depends on φ!

<u>rem</u>: If $D_i \in |-K_{X_i}|$ & D_i : connected $(i=1,2)$, then X_0 : SNC. CY.

We shall construct examples by several isomorphisms of rational elliptic surfaces & CY mfds of Schoen' type.

§. Rational elliptic surfaces & their quadratic transformations

Prop. $S \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(3,1)| \; (= |pr_1^* \mathcal{O}_{\mathbb{P}^2}(3) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}(1)|)$: general smooth.

$\pi_1 : S \to \mathbb{P}^2$, $\pi_2 : S \to \mathbb{P}^1$: projections.

$\Rightarrow$ S : rational elliptic surface s.t. $\pi_2 : S \to \mathbb{P}^1$: elliptic fib.

$\pi_1 : S \to \mathbb{P}^2$: blow-up at 9 points $p_1, \cdots, p_9$ $= (F_1 = F_2 = 0)$.

(Proof.)

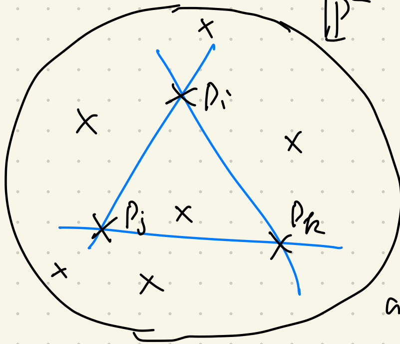
Follows from : $S = (sF_1 + tF_2 = 0) \subseteq \mathbb{P}^2 \times \mathbb{P}^1$ $\ni ([z_0 : z_1 : z_2], [s:t])$ $\left(F_i \in |\mathcal{O}_{\mathbb{P}^2}(3)| \text{ : general } (i = 1, 2) \right)$

∎

rem (Manin, Totaro)

- $p_1, \cdots, p_9$: "Cremona general position" (i.e. $\forall$ 3 points are not collinear even after any quadratic transformations) e.g.

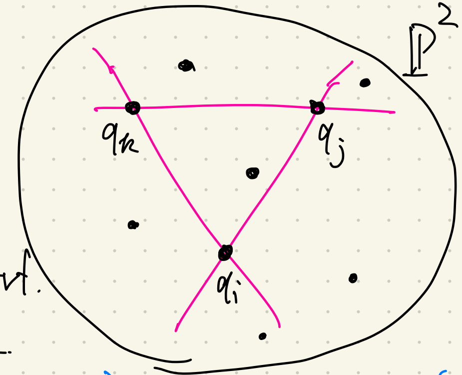
$\mathbb{P}^2$



$\overset{\chi_{ijk}}{\dashrightarrow}$ quadratic trf. at p_i, p_j, p_k.

(e.g. $[z_0 : z_1 : z_2] \mapsto \left[\frac{1}{z_0} : \frac{1}{z_1} : \frac{1}{z_2}\right]$)

$\mathbb{P}^2$



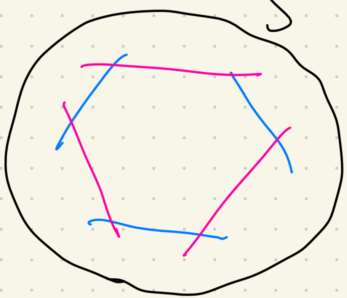
$\dashrightarrow \cdots$ quad. trf at q_i, q_j, q_k, etc

(we can perform $\forall$ times if $p_1, \cdots, p_9$: Cremona general)

bl-up $p_1, \cdots, p_9$ $\uparrow$

bl-up $q_1, \cdots, q_9$ $\uparrow$

S

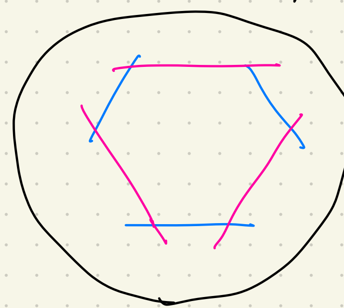


$\cap$

$\mathbb{P}^2 \times \mathbb{P}^1$

$\overset{\phi_{ijk}}{\underset{\cong}{\longrightarrow}}$

$\chi_{ijk} \times id$ $\dashrightarrow$



$\cap$

$\mathbb{P}^2 \times \mathbb{P}^1$

$\exists S_{ijk} \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(3,1)|$ general sm.

$\xrightarrow{\phi_{i'j'k'}} \cdots\cdots$

Notation

For $R \in |\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(3,1)|$ general sm., put

$\mu_R := \pi_2 : R \to \mathbb{P}^2$: birat.

$H_R := \mu_R^* \mathcal{O}_{\mathbb{P}^2}(1)$: nef & big on R, ($H_S, H_{S_{ijk}}, \cdots$)

Note : $\phi_{ijk}^*(H_{S_{ijk}}) = 2H_S - E_i - E_j - E_k$ ($E_\ell := \mu_S^{-1}(p_\ell)$ $(\ell = 1, \cdots, 9)$: exc. curves on S.)

$\rightsquigarrow$ $\forall m > 0$ $\exists \phi_m : S \xrightarrow{\cong} \exists S_m \in |\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(3,1)|$.

(composition of $6m$ quadratic trf.)

s.t. $H_S + \phi_m^*(H_{S_m}) - f_1 - \cdots - f_m$: ample & free

"sufficiently positive"!

remarkable property!

($f_i \in |-K_S|$: ell. curves $(i = 1, \cdots, m)$)

(e.g.

$\phi_1 = (\quad " \quad) \circ (\phi_{789} \circ \phi_{456} \circ \phi_{123})$.)

Prop. $n \geq 2$. $T \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(1, n+1)|$: general sm, $\left.\begin{array}{l}\pi_1 : T \to \mathbb{P}^1 \\ \pi_2 : T \to \mathbb{P}^n\end{array}\right\}$ projections.

$\Rightarrow \pi_2 : T \to \mathbb{P}^n$: blow-up along a sm. subvar of codim $= 2$.
$(G_1 = G_2 = 0)$

(∵ $T = (sG_1 + tG_2 = 0) \subseteq \mathbb{P}^1 \times \mathbb{P}^n$, $G_i \in |\mathcal{O}_{\mathbb{P}^n}(n+1)|$: general.)
$[s:t]$

Prop. $\left.\begin{array}{l}\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^n \supseteq S \times \mathbb{P}^n =: D_S \\ \supseteq \mathbb{P}^2 \times T =: D_T\end{array}\right\}$ general, $D_{ST} := D_S \cap D_T \in |-K_{D_T}|$

$\Rightarrow D_{ST}$: proj. CY mfd. & $D_{ST} \simeq S \times_{\mathbb{P}^1} T$ $\left(\begin{array}{l}S \to \mathbb{P}^1 \\ T \to \mathbb{P}^1\end{array} \text{ : projections}\right)$

(**rem** Schoen, Hosono-Saito-Stienstra : $n = 2$. (Schoen's CY3) .)
D_{ST} :

Summarize: $\exists\, \phi_m : S \xrightarrow{\cong} S_m \in |\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(3,1)|$

& $T \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1,n+1)|$ $\rightsquigarrow$ induces

$$\left(\begin{array}{ccccccc} & & \mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^n & \overset{\chi_m\times id}{\dashrightarrow} & \mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^n & & \\ & & \cup & & \cup & & \\ S \xleftarrow{\pi_S} & \Phi_m : & D_{ST} & \xrightarrow{\cong} & D_{S_mT} & \xrightarrow{\pi_{S_m}} & S_m \\ & & \cong & \circlearrowright & \cong & \nearrow & \\ & & S\times T & \xrightarrow{\phi_m\times id} & S_m\times T & & \\ & & \mathbb{P}^1 & & \mathbb{P}^1 & & \end{array}\right).$$

(use this for gluing.)

Prop. $\operatorname{Pic} D_{ST} \cong \operatorname{Pic} S \oplus \operatorname{Pic} T \,/\, \mathbb{Z}(-K_S, K_T) \cong \mathbb{Z}^{9+\rho_T} = \begin{cases} \mathbb{Z}^{19} & (n=2) \\ \mathbb{Z}^{11} & (n\geq 3) \end{cases}$

$\forall \mathcal{L} = [(\mathcal{L}_S, \mathcal{L}_T)]$. $(\exists \mathcal{L}_S \in \operatorname{Pic} S, \exists \mathcal{L}_T \in \operatorname{Pic} T)$

$\rightsquigarrow$ $\Phi_m^* : \operatorname{Pic} D_{S_mT} \to \operatorname{Pic} D_{ST}$ is described by:

$$\Phi_m^*(\underbrace{[\mathcal{L}_{S_m}, \mathcal{L}_T]}_{\in \operatorname{Pic} D_{S_mT}}) = [\phi_m^* \mathcal{L}_{S_m}, \mathcal{L}_T].$$

(e.g. $\Phi_m^* \pi_{S_m}^*(H_{S_m}) = \pi_S^*(\phi_m^*(H_{S_m}))$.)

§ Construction of examples

Recall:
$$\left(\begin{array}{ccccc} \mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^n & \supseteq & D_T & =: & \mathbb{P}^2\times T \\ \cup & & \cup & & \\ S\times\mathbb{P}^n = D_S & \supseteq & D_{ST} & = & D_S\cap D_T \end{array}\right) \quad \pi_S : D_{ST}\to S \text{ : projection.}$$

$$\begin{array}{ccc} Y_1 := \mathbb{P}^2\times T, & & Y_2 := \mathbb{P}^2\times T, \quad \forall m>0. \\ \cup & & \cup \\ D_1 := D_{ST} & \underset{\Phi_m}{\xrightarrow{\ \simeq\ }} & D_2 := D_{S_m T} \end{array}$$

$\Rightarrow$ · $Y_0(m) := Y_1 \overset{\Phi_m}{\cup} Y_2$: SNC CY ($\because D_i \in |-K_{Y_i}|$)

· but not d-semistable.

$$\left(\begin{array}{l} \because\ N_{D_1/Y_1}\otimes\Phi_m^* N_{D_2/Y_2} \simeq \pi_S^*\mathcal{O}_S(3H_S+(-K_S))\otimes\Phi_m^*\pi_S^*\mathcal{O}_S(3H_{S_m}+(-K_S)) \\ \simeq \pi_S^*\mathcal{O}_S(\underbrace{3H_S+\phi_m^*(3H_{S_m})+(-2K_S)}_{\substack{\| \\ L_m\text{ : ample!}}}) \neq \mathcal{O} \end{array}\right)$$

<u>Key</u> : $L_m + mK_S$: still ample & free ! (by remarkable property)

$$\begin{array}{c} L_m+mK_S \\ \| \\ |L_m - f_1 - \cdots - f_m| \ni C_m \text{ : smooth irred.} \end{array} \quad \left(\begin{array}{c} f_i \in |-K_S| \text{ : general} \\ (i=1,\ldots,m) \end{array} \right)$$

Let $F_i := \pi_S^{-1}(f_i)$ $(i=1,\cdots,m)$, $\Gamma_m := \pi_S^{-1}(C_m) \subseteq D_1$: sm. divisors $\left(\Rightarrow F_i, \Gamma_m \subseteq Y_1 \text{ codim}=2\right)$

$$\left(D_1 = D_{ST} \xrightarrow{\pi_S} S \right)$$

$$\mu : X_1 \xrightarrow[\varphi]{} Y_1' \xrightarrow[\psi]{} Y_1 : \text{proj. birat.}, \qquad \tilde{\Gamma}_m \subset Y_1',\ \tilde{\Gamma}_m \xrightarrow{\cong} \Gamma_m \subset Y_1 = \mathbb{P}^2 \times T$$

φ : bl-up $\tilde{\Gamma}_m$, ψ : blow-up $F_1, \cdots, F_m$,

$$X_2 := Y_2 = \mathbb{P}^2 \times T \Rightarrow$$

$$\begin{array}{ccccc} X_1 & \xrightarrow{\mu} & Y_1 & & X_2 \\ \cup & & \cup & & \cup \\ \tilde{D}_1 & \xrightarrow[\mu|_{\tilde{D}_1}]{\cong} & D_1 & \xrightarrow[\Phi_m]{\cong} & D_2 \end{array} \qquad \tilde{\Phi}_m : \tilde{D}_1 \to D_2$$

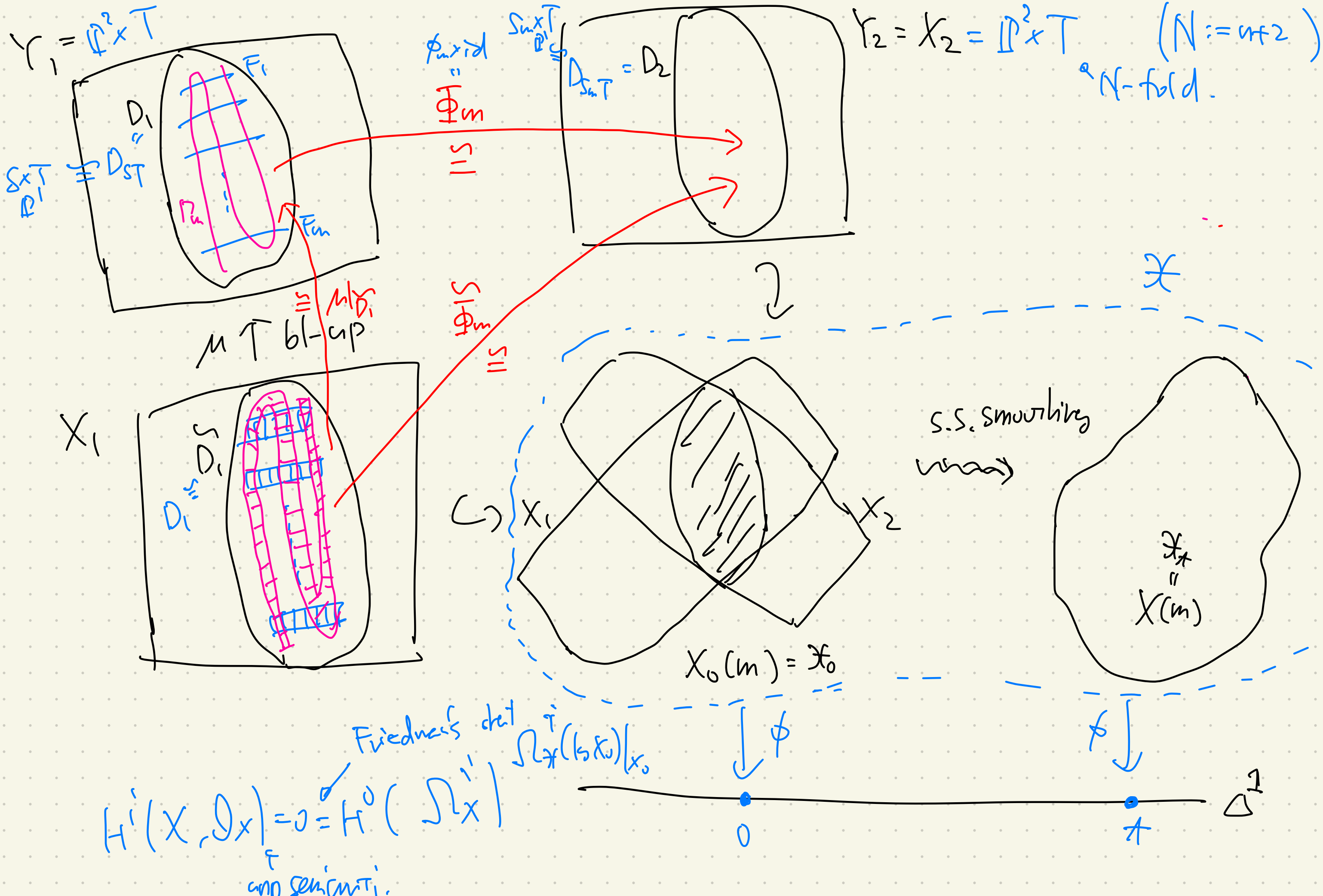
$$\Rightarrow \cdot\ X_0(m) := X_1 \overset{\tilde{\Phi}_m}{\cup} X_2 : \text{SNC CY} \left(\because \begin{array}{l} \tilde{D}_1 \in |-K_{X_1}| \\ D_2 \in |-K_{X_2}| \end{array} \right)$$

& d-semistable

$$\left(\because N_{D_1/Y_1} \otimes \Phi_m^* N_{D_2/Y_2} \cong \mathcal{O}_{D_1}(F_1 + \cdots + F_m + \Gamma_m) \right)$$

$\rightsquigarrow$ $X_0(m)$ can be deformed to $X(m)$: CY N-fold.
(not nec. proj.)

Thm (KN)


Y_1 = P^2 × T
F_1
D_1
S×T / P^1 ≅ D_ST
Γ_m
F_m
φ_m × id = Φ_m ≅
S_m×T / P^1 ⊂ D_{S_m T} = D_2
Y_2 = X_2 = P^2 × T
(N := m+2)
N-fold.
μ|_{D_1} ≅
μ T bl-up
≅ Φ_m ≅
X_1
D_1
D_1
𝒳
⊂ X_1
X_2
s.s. smoothing
X_0(m) = 𝒳_0
𝒳_t = X(m)
φ
φ
0
t
Δ^1
Friedman's def
Ω^1_𝒳(log X_0)|_{X_0}
H^1(X, O_X) = 0 = H^0(Ω^1_X)
upp semicont.

Properties of $X(m)$ (i). $b_2(X(m)) = m+\rho_T$ $\left(\rho_T := \operatorname{rank} \operatorname{Pic} T = \begin{cases} 10 & (n=2) \\ 2 & (n \geq 3) \end{cases} \quad (T \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1,n+1)| \text{ general sm.}) \right)$

(ii). $a(X(m)) = N-2$ (alg. dim.)

$\Rightarrow \varphi_m : X(m) \to T$: K3 fibration. $\leftarrow$ ($X(m)$: fiber over $t \in \Delta^1$: very general)

prf (i). First: $b_2(X_0(m)) = m+\rho_T+1$ by the exact sequence: $(X_{12} := X_1 \cap X_2)$

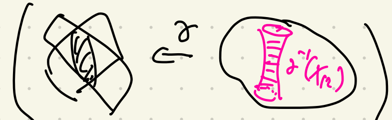
$$0 \to \operatorname{Pic} X_0(m) \to \operatorname{Pic} X_1 \oplus \operatorname{Pic} X_2 \xrightarrow{R} \operatorname{Pic} X_{12} \cong (\operatorname{Pic} S \oplus \operatorname{Pic} T)/\mathbb{Z}(-K_S, K_T)$$

(restriction; X_2 = "$\mathbb{P}^2 \times T$")

$$0 \to \mathbb{Z}^{\rho_T+m+1} \to \mathbb{Z}^{m+\rho_T+2} \oplus \mathbb{Z}^{1+\rho_T} \to \operatorname{Im} R \cong \mathbb{Z}^{\rho_T+2} \to 0$$

$\leftarrow$ (gen'd by $\operatorname{Pic} T, H_S, \varphi_m^*(H_{S_m})$)

$\rightsquigarrow$ use Clemens map $X(m) \xrightarrow{\gamma} X_0(m)$: $\begin{cases} \text{diffeo outside } X_{12} \\ S^1\text{-bdle over } " \end{cases}$



$\rightsquigarrow$ Leray's spec. seq.

$\operatorname{Pic} X_0(m) \longrightarrow \operatorname{Pic} X(m)$ $\qquad H^1(R^1\gamma_* \mathbb{Z}_{X(m)})$

$$H^0(R^1\gamma_*\mathbb{Z}) \to H^2(X_0(m), \mathbb{Z}) \to H^2(X(m), \mathbb{Z}) \to 0"$$

$H^0(R^1\gamma_*\mathbb{Z}) \cong \mathbb{Z}$, $\neq 0$; $\operatorname{Pic} X_0(m) \cong H^2(X_0(m), \mathbb{Z}) \cong \mathbb{Z}^{\rho_T+m+1}$; $\operatorname{Pic} X(m) \cong H^2(X(m), \mathbb{Z}) \cong \mathbb{Z}^{\rho_T+m}$

(ii). $\mathcal{L}_1 := \mu^* \pi_T^* (H_T) \in \mathrm{Pic}\, X_1$, $\mathcal{L}_2 := \pi_T^* H_T \in \mathrm{Pic}\, X_2$

(· H_T : very ample on T
· $\pi_T : \mathbb{P}^2 \times T \to T$: proj.
$$\begin{array}{c} \mathbb{P}^2 \times T \\ \uparrow \mu \\ X_1 \end{array}$$
)

⇝ $\mathcal{L}_0 \in \mathrm{Pic}\, X_0(m)$ s.t. $\Phi_{|\mathcal{L}_0|} : X_0(m) \to T$: fiber space.

⇝ $\mathcal{L}_A \in \mathrm{Pic}\, X(m)$ s.t. $\Phi_{|\mathcal{L}_A|} : X(m) \to T$: K3 fibration. $\Rightarrow a(X(m)) \geq N-2$.

- Suppose $\exists \mathcal{M}_A \in \mathrm{Pic}\, X(m)$ w/ $\kappa(\mathcal{M}_A) \geq N-1$.

⇝ $\exists \mathcal{M}_0 \in \mathrm{Pic}\, X_0(m)$ s.t. $\begin{cases} \kappa(\mathcal{M}_0) \geq N-1 \\ \mathcal{M}_0|_{X_i} : \text{effective } (i=1,2). \end{cases}$ — (*)

upp. semi-conti
+
some argument

<u>claim</u> $\nexists \mathcal{M}_0 \in \mathrm{Pic}\, X_0(m)$ as in (*).

(· by calculation.
· $H_S, \phi_m^*(H_{S_m}) \in \mathrm{Pic}\, S / (\mathbb{Z} K_S)$: lin. indep)

<u>rem</u> $X(m)$: simp. conn'd.

□

§ Further problem

Q : Does $X(m)$ satisfy $\partial\bar{\partial}$-lemma & Hard Lefschetz property?

$\Updownarrow$

(Hodge decomposition (with Hodge symmetry))

rem: The Hodge to de Rham spec. seq

$$H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C}) \quad : \text{degen at } E_1.$$

Q. $\forall N \geq 3$: fix. Are there ∞-many non-Kähler CY N-fold X with $a(X) = N-1$?

($X \xrightarrow{\exists} S$: ell. fibration

Reid's fantasy: Can we connect projective CY 3-folds via geometric transitions?

Q : $\exists$ geometric relation between proj. CY mfds & $X(m)$?