# Constant scalar curvature and Kähler manifolds with nef canonical bundle

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# Constant scalar curvature and Kähler manifolds with nef canonical bundle

$(X, \omega)$ cpct Kähler mfld, $n := \dim_{\mathbb{C}} X$

$\omega \underset{loc}{=} i \sum_{j,k} \underline{g_{j\bar{k}}} \, dz_j \wedge d\bar{z}_k$ fund. 2-form of $(g_{j\bar{k}})$ assoc. herm. metric.

$\left( \text{e.g. } X \hookrightarrow \mathbb{CP}^N, \quad \omega = \omega_{FS} \big|_X \right)$

$d\omega = 0$ (Kähler)

$\hookrightarrow$ $[\omega] \in H^{1,1}(X, \mathbb{R}) \subseteq H^2(X, \mathbb{R})$ assoc. Kähler class.

(e.g. $(X, L)$ smooth polarized variety, $L \to X$ ample l. b.

$[\omega] = c_1(L)\,)$.

$\mathrm{Ric}(\omega) := -\frac{i}{2\pi}\partial\bar\partial \log \omega^n$ Ricci curvature form.

norm. s.t. $[\mathrm{Ric}(\omega)] = c_1(X) = c_1(-K_X)$.

**Question:** Existence of a canonical metric on $X$,
(Calabi) in the given class $[\omega]$ ?

Canonical: • KE : $\mathrm{Ric}(\omega) = \lambda\omega$, $\lambda \in \mathbb{R}$.

$\Rightarrow c_1(X) = \lambda[\omega]$

• cscK: $\boxed{\mathrm{Scal}(\omega) = \bar{s}}$ $\quad\equiv$

$\mathrm{Scal}(\omega) = \mathrm{Tr}_\omega \mathrm{Ric}(\omega) = n\frac{\mathrm{Ric}(\omega)\wedge\omega^{n-1}}{\omega^n} \in C^\infty(X)$

$\bar{S}$ coh. cst $= n \dfrac{c_1(X) \cdot [\omega]^{n-1}}{[\omega]^n} \in \mathbb{R}$.

- KE $\Rightarrow$ cscK $\Rightarrow$ extremal
- They do not always exist! //

(obstructions: $\rightsquigarrow$ Donaldson-Futaki invariant, K-stability YTD conj.).

**Recall:**

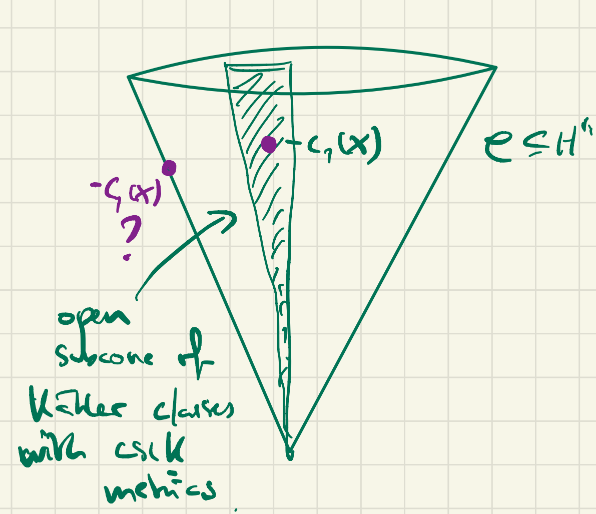
**Thm:** (Aubin, Yau '78) If $K_X$ ample, then $X$ admits a KE metric. (in $-c_1(X)$)

$\Rightarrow$ $X$ carries cscK metric, if $K_X$ ample.

**Q:** What if $K_X$ nef? ($K_X \cdot C \geq 0$ $\forall C$ curves.)

↳ KE can only exist if
$c_1(X) = \lambda [\omega]$.

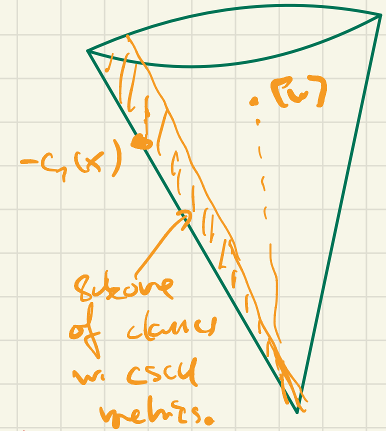
→ In gen^l look for cscK metrics!
Do they exist?




**Main Theorem:** (–) **YES!**

Suppose $X$ is a cpt Kähler mfld w. $K_X$ nef. ($-c_1(X)$ nef).
Then for any Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R})$, there is
$\varepsilon_{X,[\omega]} > 0$, s.t. $\forall \varepsilon \in (0, \varepsilon_{X,[\omega]})$, there exists a cscK
metric in the Kähler class $-c_1(X) + \varepsilon [\omega]$.

Special case: If $K_X$ nef, for any $L \to X$ ample l.b., there is $\varepsilon_{X,L}$ s.t. $\forall \varepsilon \in (0, \varepsilon_{X,L})$, there is a cscK metric in $c_1(K_X) + \varepsilon c_1(L)$.




Note: · In particular smooth minimal models always admit cscK metrics.

· Motivated by MMP.

· Proven by Jian-Shi-Song '18 for $K_X$ semi-ample.

Abundance conj: $K_X$ nef $\Rightarrow$ $K_X$ semi-ample.

· Proven indep. by Song.

**Applications:** Idea of the proof is to show that

$K_X$ nef $\Longrightarrow$ K-energy functional is **proper** $\Longrightarrow$

$\exists$ cscK metrics

by Chen-Cheng '18.

**Cor 1:** $K_X$ nef $\Longrightarrow$ $Bl_{p_1 \dots p_m} X$ admits cscK metrics.

[Main Thm + Arezzo-Pacard]

**Cor 2:** $K_X$ nef $\Longrightarrow$ $\mathrm{Aut}_0(X)$ is a cplx torus (or a pt).

## § Proof: the variational approach.

$(X, \omega)$ cpt Kähler mflds.

$$\mathcal{H} := \mathcal{H}(X, \omega) := \{ \varphi \in C^\infty(X, \mathbb{R}) : \underline{\omega_\varphi := \omega + i\partial\bar\partial\varphi} > 0 \}$$

sp. of Kähler potentials.

$\mathcal{H}/\mathbb{R} \simeq$ sp. of Kähler metrics $\omega'$, s.t. $[\omega'] = [\omega]$.

**Mabuchi '85**: There is a functional $M : \mathcal{H} \to \mathbb{R}$ whose minimizers $\varphi \in \mathcal{H}$ are precisely the solu's to

$$\mathrm{Scal}(\omega_\varphi) = \bar{S}$$

$$\frac{d}{dt} M(\varphi_t) = -\frac{1}{[\omega]^n} \int_X \dot\varphi_t \, \underbrace{(\mathrm{Scal}(\omega_{\varphi_t}) - \bar{S})} \, \omega_{\varphi_t}^n$$

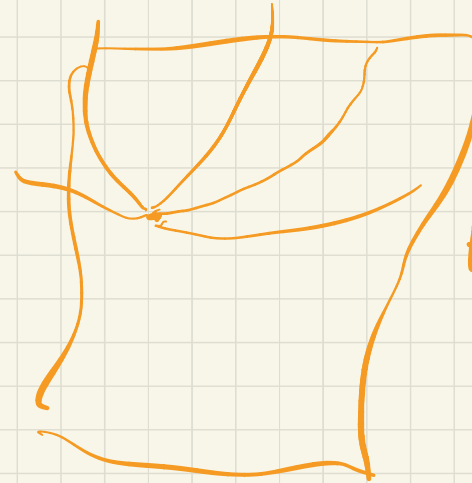
<u>Darvas:</u> $(\mathcal{H}, d_1)$ metric space. $\quad T_\varphi \mathcal{H} \simeq C^\infty(X)$.

2013.

<u>Thm:</u> (Chen-Cheng '18) Suppose that $M$ is <u>proper</u> on $\mathcal{H}(X,\omega)$, i.e. $\exists\, \delta, C > 0$ s.t.

$$\boxed{M(\varphi) \geq \delta\, d_1(0,\varphi) - C}$$

Then $\exists$ cscK metrics in $[\omega]$.



$\mathcal{H}$

How determine when $M$ is proper on $\mathcal{H}(X,\omega)$?

<u>Idea:</u> Relate existence of cscK metrics to solutions of the "J-equation" (Donaldson) / J-stability.

$V = [\omega]^n > 0$

<u>Chen-Tian def:</u> $\quad M = \underline{J_{-Ric(\omega)}} + \underline{H}$

where $\quad H(\varphi) = \frac{1}{V} \int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right) \omega_\varphi^n$

<u>rel. entropy</u>

and $\quad \underline{J_\theta}$ is the functional whose min.

$(1,1)$-form smooth closed

e.g. $\theta = -Ric(\omega)$

$\Longleftrightarrow$ satisfies $\varphi \in H(X,\omega)$ of $\theta \wedge \omega_\varphi^{n-1} = c\,\omega_\varphi^n$ ($\underline{J\text{-eq.}}$). $\quad \underline{\underline{\geq 0}}.$

$\Updownarrow$ $J$-stability. of $(X, L)$ if $c_1(L) = [\omega]$.

## Numerical invariants :

$\Delta(\omega) > 0 \iff \exists$ cscK metrics.

$$\Delta(\omega) := \sup\{\delta \in \mathbb{R} : \exists C_\delta > 0 : M(\varphi) \geq \delta\, d_1(0,\varphi) - C_\delta \quad \forall \varphi \in \mathcal{H}(X,\omega)\}$$

$$\Delta_\theta(\omega) := \sup\{\delta \in \mathbb{R} : \text{—''—}\ \mathcal{J}_\theta(\varphi) \geq \delta\, d_1(0,\varphi) - C \text{ —''—}\}$$

$$\Delta(\omega) \geq \Delta_\theta(\omega)$$

Notation: $\Delta_{-Ric(\omega)}(\omega) = \Delta_K(\omega)$.

**Prop** (_)

M is proper on $\mathcal{H}(X,\omega)$ $\longleftrightarrow$ $\mathcal{J}_{\theta_\omega}$ proper on $\mathcal{H}(X,\omega)$, where

$$\theta_\omega := -Ric(\omega) + (\Delta(\omega) - \Delta_K(\omega))\,\omega.$$

i.e. $[\theta_\omega] = c_1\{K_X + \underbrace{(\Delta(\omega) - \Delta_K(\omega))}_{\geq 0} L\}$.

YTD: "K-stable" $\overset{?}{\longleftrightarrow}$ "J-stable" $(X, [\omega], [\theta_\omega])$.

Apply numerical criteria J-eq. (Weinkove, Song-Weinkove, Chen, Dinezza-Lu)

Claim: We have

(*)

$$n\frac{[\theta]\cdot[\omega]^{n-1}}{[\omega]^n} - (n-1)\,\sigma([\theta],[\omega]) < \Delta(\omega).$$

When $>0$?

where $\sigma([\theta],[\omega]) := \inf\{\lambda\in\mathbb{R} : \lambda[\omega] - [\theta] > 0\}$.

Enough if $K_X$ nef $\Rightarrow$ (*) $>0$.

Show: The if close enough to $K_X$.

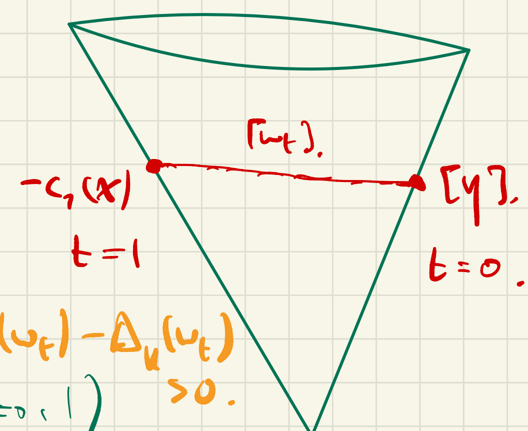
Work on coh. classes. Assume $-c_1(X) \in \partial \mathcal{C}_X$, $[\omega] \in \mathcal{C}_X$.

Pick $[\eta] \in \partial \mathcal{C}_X$ s.t.

$$[\omega_t] := (1-t)[\eta] + t(-c_1(X)) \in \mathcal{C}_X \quad \forall t \in (0,1).$$

Consider

$$\boxed{R_\varepsilon(t) := n \frac{(-c_1(X) + \varepsilon[\omega_t]) \cdot [\omega_t]^{n-1}}{[\omega_t]^n} - (n-1)\,\sigma(-c_1(X) + \varepsilon[\omega_t], [\omega_t])}$$




**Want:** $R_{\varepsilon_{\omega_t}}(\omega_t) > 0$ where $\varepsilon_{\omega_t} = \Delta(\omega_t) - \Delta_k(\omega_t) > 0$.

Then $R_0(1) = 1$,

$\exists\, t_0 \in (0,1)$ s.t. $R_0(t) > 0 \quad \forall t \in (t_0, 1)$

$$\boxed{R_\varepsilon(t) = R_0(t) + \varepsilon.}$$

Apply with $\varepsilon_{w_t} := \underline{\Delta(w_t) - \Delta_R(w_t) > 0.}$ $\|$

$$\Rightarrow \boxed{R_{\varepsilon_{w_t}}(t) \geq 0} \quad \forall t \in (t_0, 1). \quad \rightsquigarrow \quad \Delta(w_t) > 0 \ \forall t \in (t_0, 1).$$

$\Rightarrow$ M is proper over these classes

$$-c_1(X) + \varepsilon[\omega].$$

$\boxtimes$

What we proved ...

$\downarrow$

**Theorem** (_ '20) Suppose that $X$ is a compact Kähler manifold with $-c_1(X)$ nef. Then for any Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R})$, there is $\varepsilon_{X,[\omega]} > 0$, such that for all $\varepsilon \in (0, \varepsilon_{X,[\omega]})$, the Mabuchi K-energy is proper over the Kähler class $-c_1(X) + \varepsilon[\omega]$.

(In particular true for any minimal model. We can also consider $X$ non-projective with the same proof.)

This is stronger than showing $\exists$ cscK!

↳ Application:

## Theorem (Arezzo-Pacard '06)

Let $(X, \omega)$ be a constant scalar curvature compact Kähler manifold. Assume that there is no nonzero holomorphic vector field vanishing somewhere on $X$. Then, given finitely many points $p_1, \dots, p_m$ and positive numbers $a_1, \dots, a_m > 0$, there exists $\varepsilon_0 > 0$ such that the blowup of $X$ at $p_1 \dots p_m$ carries constant scalar curvature Kähler forms

$$\omega_\varepsilon \in \pi^*[\omega] - \varepsilon^2 \left(a_1 [E_1] + \dots + a_m [E_m]\right)$$

where $\varepsilon \in (0, \varepsilon_0)$.

It turns out: Properness + $\exists$ cscK metric

$\Rightarrow$ cannot exist a Hamiltonian v.f. on $X$.

$\Rightarrow$ Arezzo-Pacard applies:

**Cor:** (Arezzo-Pacard + Main thm)

Suppose $X$ is a compact Kähler mfld with $-c_1(X)$ nef. Then $Bl_{p_1 \dots p_m}(X)$ admits cscK metrics.

**Cor**: Suppose $X$ is a compact Kähler mfld with $-c_1(X)$ nef. Then $\mathrm{Aut}_0(X)$ is a complex torus.