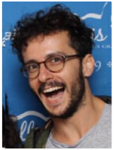
Actions of connected algebraic groups on rational 3-dimensional Mori fibrations

(joint work with Jérémy Blanc and Andrea Fanelli)



References :

- arXiv:1707.01462, "Automorphisms of $\mathbb{P}^1$-bundles over rational surfaces", 52 p.
- arXiv:1912.11364, "Connected algebraic groups acting on 3-dimensional Mori fibrations", 81 p.

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However, $\mathrm{Bir}(\mathbb{P}^n)$ contains algebraic subgroups. For instance, it contains all the $\mathrm{Aut}^0(X)$ with X a rational projective n-fold.
(In fact $\mathrm{Bir}(\mathbb{P}^n)$ contains $\varphi \mathrm{Aut}^0(X) \varphi^{-1}$ with $\varphi \colon X \dashrightarrow \mathbb{P}^n$.)

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Question: What are the (maximal) connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$ when $n \geq 2$?

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2. Compactify G-equivariantly X_1 (Sumihiro [1974]) to obtain a *rational projective n-fold* $\iota \colon X_1 \hookrightarrow X_2$ such that $\varphi_2 G \varphi_2^{-1} \subseteq \mathrm{Aut}^0(X_2)$ with $\varphi_2 = \iota \circ \varphi_1$.

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3. Resolve G-equivariantly the singularities of X_2 (Kollár [2007]) to obtain a *rational smooth projective n-fold* X_3 such that $\varphi_3 G \varphi_3^{-1} \subseteq \mathrm{Aut}^0(X_3)$ with $\varphi_3 \colon \mathbb{P}^n \dashrightarrow X_3$ a birational map.

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4. Apply a Minimal Model Program to X_3 to get a *Mori fibration* $\pi\colon X \to Y$ such that $\varphi G \varphi^{-1} \subseteq \mathrm{Aut}^0(X)$ for some birational map $\varphi\colon \mathbb{P}^n \dashrightarrow X$.

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Partial conclusion: The connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$ are those acting biregularly on rational Mori fiber spaces.

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Proposition (Case $n = 2$, Enriques [1893])

Any connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ *is conjugate to a subgroup of* $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1)$ *or* $\mathrm{Aut}(\mathbb{F}_k)$ *with* $k \geq 2$*. Moreover, these algebraic subgroups are maximal in* $\mathrm{Bir}(\mathbb{P}^2)$*, and so any connected algebraic subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$ *is contained into a maximal one.*

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- 8 *discrete families of* $\mathbb{P}^1$*-bundles and* $\mathbb{P}^2$*-bundles depending on* 1 *or* 2 *parameters* (*e.g.* $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{F}_k)$, $\mathrm{Aut}(\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(-k) \oplus \mathcal{O}_{\mathbb{P}^2}))$, $\mathrm{Aut}(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-k_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-k_2) \oplus \mathcal{O}_{\mathbb{P}^1}))$ *etc*)*; or*

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- 1 *continuous family of automorphism groups of smooth quadric fibrations over* $\mathbb{P}^1$.

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- If $\dim(Y) = 1$, then $X \to Y = \mathbb{P}^1$ is a *Mori del Pezzo fibration* over $\mathbb{P}^1$, i.e. a general fiber of π is a del Pezzo surface.
  (Recall that the smooth del Pezzo surfaces are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}^2$ blown-up at r points in general position with $1 \leq r \leq 8$.)

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- If $\dim(Y) = 2$, then $X \to Y$ is a *Mori* $\mathbb{P}^1$*-fibration* over a rational surface.

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If $\pi\colon X \to \mathbb{P}^1$ is a Mori del Pezzo fibration, then a general fiber of π cannot be $\mathbb{P}^2$ blown-up at one or two points.

Proof (by *reductio ad absurdum*).

Assume that a general fiber of π is $\mathbb{P}^2$ blown-up at one point.
Let $K = \mathbb{C}(\mathbb{P}^1)$ and let $X_{\overline{K}} \simeq BL_p(\mathbb{P}^2_{\overline{K}})$ be the geometric generic fiber of π. Then $\mathrm{Pic}(X_{\overline{K}}) = \mathbb{Z}\langle L, E\rangle \simeq \mathbb{Z}^2$, with L a generic line and E the exceptional divisor. Also $\mathrm{Gal}(\overline{K}/K)$ fixes E and the canonical class $-3L+E$, hence $\mathrm{Pic}(X_{\overline{K}})^{\mathrm{Gal}(\overline{K}/K)}$ is a sublattice of rank 2. Thus

$$1 = \rho(X/\mathbb{P}^1) = \mathrm{rg}(\mathrm{Pic}(X_K)) = \mathrm{rg}(\mathrm{Pic}(X_{\overline{K}})^{\mathrm{Gal}(\overline{K}/K)}) = \mathrm{rg}(\mathbb{Z}^2) = 2.$$

The proof for $\mathbb{P}^2$ blown-up at two points is similar. □

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If a general fiber of the del Pezzo fibration $\pi\colon X \to \mathbb{P}^1$ *is* $\mathbb{P}^2$ *blown-up at three points or more, then* $\mathrm{Aut}^0(X)$ *is an algebraic torus.*

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where $H \subseteq \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$ and $\mathrm{Aut}^0(X)_{\mathbb{P}^1}$ acts trivially on $\mathbb{P}^1$.

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Case of Mori del Pezzo fibrations: conclusion

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Proposition (well-known, Popov [2013])

All tori of dimension $d \in \{1, 2, 3\}$ are conjugate in $\mathrm{Bir}(\mathbb{P}^3)$. In particular, they are all conjugate to a strict subgroup of $\mathrm{Aut}(\mathbb{P}^3) = \mathrm{PGL}_4(\mathbb{C})$.

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Consequence: If $\pi\colon X \to \mathbb{P}^1$ is a Mori del Pezzo fibration whose a general fiber is neither $\mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$, then $\mathrm{Aut}^0(X)$ is conjugate to a strict subgroup of $\mathrm{Aut}(\mathbb{P}^3)$.

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- a general fiber of π is $\mathbb{P}^1 \times \mathbb{P}^1$, then we reduce to an element of the continuous family of quadric fibrations over $\mathbb{P}^1$ mentioned earlier.

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  - if the generic fiber of π is not $\mathbb{P}^1$, then $\mathrm{Aut}^0(X)$ is again an algebraic torus.
- When $\pi\colon X \to Y$ is a $\mathbb{P}^1$-bundle, we have a *descent lemma* to reduce to the case where Y is a minimal smooth rational surface, i.e. Y is $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{F}_k$ with $k \geq 2$.

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The main tool for this last step is the *equivariant Sarkisov program* for threefolds (whose validity follows from the work of Corti [1995], for the dimension 3, and of Floris [2020], for the dimension ≥ 3).

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- What are the maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$ when $n \geq 4$? Is there a pattern? Is any connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^n)$ always contained into a maximal one?

Thank you for your attention!