Seshadri constants on toric surfaces

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Seshadri Constants

Definition

Let X be a projective surface, H a nef line bundle.

- The **Seshadri constant** of H at a smooth point $x \in X$ is

$$\varepsilon(X, H, x) := \inf \left\{ \frac{H \cdot C}{\operatorname{mult}_x(C)} \right\}.$$

- The **multiple Seshadri constant** of H at general $x_1, \ldots, x_n \in X$ is

$$\varepsilon(X, H, n) := \inf \left\{ \frac{H \cdot C}{\sum_i \operatorname{mult}_{x_i}(C)} \right\}.$$

Seshadri Constants

Proposition

If $\pi\colon \tilde{X} \to X$ is the blowing up at x and E exceptional divisor, then

$$\varepsilon(X,H,x) = \sup\{t \mid \pi^*H - tE \text{ is nef}\}.$$

Corollary

H nef, $x \in X$ smooth point.

- $\varepsilon(X,H,x) \leq \sqrt{H^2}$.
- $\varepsilon(X,H,n) \leq \sqrt{\frac{H^2}{n}}$.

Proof. $\pi^*H - \varepsilon E$ nef $\Rightarrow$ $(\pi^*H - \varepsilon E)^2 \geq 0$ $\Rightarrow$ $H^2 - \varepsilon^2 \geq 0$ □

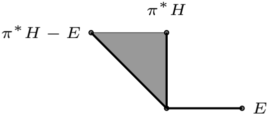
Seshadri constant

Remark

- *If* $\varepsilon < \sqrt{H^2} \;\Rightarrow\; \exists C$ **submaximal** *such that* $\varepsilon = \frac{H \cdot C}{\mathrm{mult}_x C} \in \mathbb{Q}$.
- *No* **irrational Seshadri** *are known.*
- *Knowing* $\mathrm{Nef}(\tilde{X})$ *we can compute the Seshadri constant.*

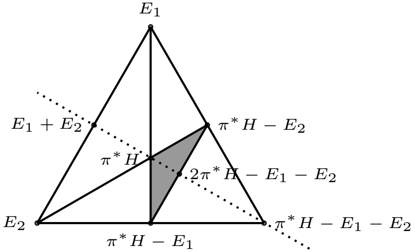
Example. If $X := \mathbb{P}^2$, $H = \mathcal{O}(1)$.

- $\mathrm{Eff}(\tilde{X}) = \langle \pi^* H - E, E \rangle \;\; \Rightarrow \;\; \mathrm{Nef}(\tilde{X}) = \langle \pi^* H - E, \pi^* H \rangle$.
- $\varepsilon(\mathbb{P}^2, H, x) = 1 = \sqrt{H^2}$.



Example. If $X := \mathbb{P}^2$, $\tilde{X} = \mathrm{Bl}_n(\mathbb{P}^2)$, $H = \mathcal{O}(1)$.

- If $n \leq 8$, $\mathrm{Eff}(\tilde{X})$ polyhedral, known $\Rightarrow$ $\mathrm{Nef}(\tilde{X})$ known.
- $\varepsilon(\mathbb{P}^2, H, n) = 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{5}, \frac{2}{5}, \frac{3}{8}, \frac{6}{17},$ for $n = 1, \ldots, 8$.
- Case $n = 2$:



Conjecture (Nagata)

For any $n \geq 9 \;\Rightarrow\; \varepsilon(\mathbb{P}^2, \mathcal{O}(1), n) = \frac{1}{\sqrt{n}}$.

Remark

True if $n = k^2$**, with** $k \in \mathbb{N}$.

- *Fix* C *with* $\deg(C) = k$ *and* $p_1, \ldots, p_{k^2} \in C$.
- $\tilde{C} = \pi^* kH - \sum E_i$ *is nef on the blowing up of* $\mathbb{P}^2$ *at the* p_i.
- *By semicontinuity it is also nef on the blowing up at* k^2 *general points.*
- $\pi^* H - \frac{1}{k} \sum E_i$ *nef* $\;\Rightarrow\; \varepsilon \geq \frac{1}{k}$.

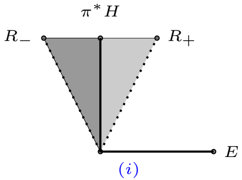
Example. If $X := \mathbb{P}(a,b,c)$, $\pi\colon \tilde{X} \to X$ blow-up at a general point $e \in X$.

- $\mathrm{Cl}(X) = \langle H \rangle$, with $H^2 = \frac{1}{abc} \;\Rightarrow\; \mathrm{Cl}(\tilde{X}) = \langle \pi^* H, E \rangle$.
- In general $\mathrm{Eff}(\tilde{X})$ is unknown.
- **Positive light cone** Q with rays $R_\pm = \pi^* H \pm \frac{1}{\sqrt{abc}} E$.
- By Riemann-Roch $\mathrm{Eff}(\tilde{X}) \supseteq Q$.

There are two possibilities.

(i) $\mathrm{Eff}(\tilde{X})$ bounded by the $\mathbb{R}$-divisor R_- $\Leftrightarrow$ $\varepsilon = \frac{1}{\sqrt{abc}}$.

(ii) $\mathrm{Eff}(\tilde{X})$ bounded by a negative class C $\Leftrightarrow$ $\varepsilon < \frac{1}{\sqrt{abc}}$.



Remark

- *For many gradings* (a, b, c) *it is known the existence of the negative curve* C *bounding the effective cone (e.g. [GK-16] and [Hau&al-18]).*
- *It is conjectured that for some gradings, i.e.* $(9, 10, 13)$*, there does not exist the negative curve.*

Remark ([CK-11])

If $\varepsilon(\mathbb{P}(a,b,c),H,e) \notin \mathbb{Q}$, then Nagata Conjecture holds for $n = abc$.

Proof.

- $f\colon \mathbb{P}^2 \to \mathbb{P}(a,b,c)$ defined by $(x,y,z) \mapsto (x^a, y^b, z^c)$.
- $\tilde{Y}$ = blowing-up of $\mathbb{P}^2$ at the $n := abc$ points of $f^{-1}(e)$.
- R_- is nef $\Rightarrow$ $f^*R_- = L - \frac{1}{\sqrt{n}}\sum_{i=1}^n E_i$ is nef on $\tilde{Y}$.
- By semicontinuity it is nef on the blowing-up in n general points. □

Toric surfaces

From lattice polygons

Definition

- $\Delta \subseteq \mathbb{Q}^2$ lattice polygon, $N := |\Delta \cap \mathbb{Z}^2|$,

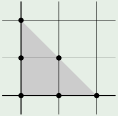
$$\begin{array}{rcl} \varphi\colon (\mathbb{C}^*)^2 & \to & \mathbb{P}^{N-1} \\ (s,t) & \mapsto & (s^a t^b : (a,b) \in \Delta \cap \mathbb{Z}^2). \end{array}$$

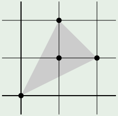
- $X_\Delta := \overline{\varphi((\mathbb{C}^*)^2)} \subseteq \mathbb{P}^{N-1}$ **projective toric surface** associated to Δ.
- $\Delta \cap \mathbb{Z}^2 \quad \Leftrightarrow \quad$ sections of $\mathcal{L}_\Delta := |H_\Delta|$, H_Δ ample.
- The image of $(1,1)$ is the **general point** $e \in \varphi((\mathbb{C}^*)^2) \subseteq X_\Delta$.
- $m \in \mathbb{Z}_{>0} \quad \Rightarrow \quad \mathcal{L}_\Delta(m) := \{C \in \mathcal{L}_\Delta \mid \mathrm{mult}_e(C) \geq m\} \subseteq \mathcal{L}_\Delta$.

Toric surfaces

From lattice polygons

Example

- $\Delta =$  $\Rightarrow \quad X_\Delta = \mathbb{P}^2$, $\mathcal{L}_\Delta = \mathcal{O}(2)$.

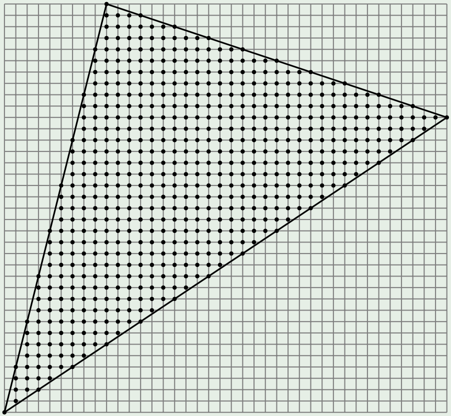
- $\Delta =$  $\Rightarrow \quad X_\Delta \subseteq \mathbb{P}^3$ singular.

Toric surfaces

From lattice polygons

Example

- The triangle Δ gives $X_\Delta = \mathbb{P}(9, 10, 13)$, $H_\Delta = 9 \cdot 10 \cdot 13 \cdot H$



Definition

$\Delta \subseteq \mathbb{Q}^2$ lattice polygon, $v \in \mathbb{Z}^2$.

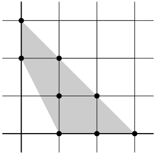
- **Lattice width of Δ with respect to v**

$$\mathrm{lw}_v(\Delta) := \max_{w \in \Delta}\{v \cdot w\} - \min_{w \in \Delta}\{v \cdot w\}.$$

- **Lattice width of Δ**

$$\mathrm{lw}(\Delta) := \min_{v \in \mathbb{Z}^2}\{\mathrm{lw}_v(\Delta)\}.$$

Example.



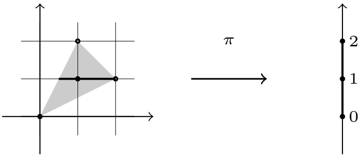
- $\mathrm{lw}_{(1,0)}(\Delta) = 3,\ \mathrm{lw}_{(1,1)}(\Delta) = 2 = \mathrm{lw}(\Delta).$

Seshadri on toric surfaces

Remark

- $x \in X_\Delta$ *fixed point or a point on a fixed curve* $\mathrm{Nef}(\tilde{X}) \;\Rightarrow\;$ *Seshadri constant ([Bau&al-09], [Ito-14]).*
- $e \in X_\Delta$ *general point* $\;\Rightarrow\;$ *upper bound* $\varepsilon \leq \mathrm{lw}(\Delta)$, *lower bound ([Ito-14]).*

Example.



$$\varepsilon(X_\Delta, H_\Delta, e) \geq \min\{2, 3/2\}.$$

Proposition ([LU-21])

$\Delta \subseteq \mathbb{Q}^2$ *lattice polygon*, (X, H) *toric pair*, $\varepsilon := \varepsilon(X, H, e)$. *Then:*

1. *If* $\mathrm{Vol}(\Delta) > \mathrm{lw}(\Delta)^2 \;\Rightarrow\; \varepsilon \in \mathbb{Q}$.
2. *If* $\exists m \in \mathbb{N}$ *such that* $\mathcal{L}_\Delta(m) \neq \emptyset$ *and* $\mathrm{Vol}(\Delta) \leq m^2$, *then:*
   - $\varepsilon \in \mathbb{Q}$;
   - $\varepsilon \leq \mathrm{Vol}(\Delta)/m$;
   - *if* $\mathcal{L}_\Delta(m)$ *contains an irreducible curve, then* $\varepsilon = \mathrm{Vol}(\Delta)/m$.

Proof.

1. $\varepsilon \leq \mathrm{lw}(\Delta) < \sqrt{\mathrm{Vol}(\Delta)} = \sqrt{H^2} \;\Rightarrow\; \exists$ submaximal curve.
2. $C = \pi^* H - mE \subseteq \tilde{X}$ effective, $C^2 = \mathrm{Vol}(\Delta) - m^2 \leq 0$.

$$\begin{array}{rcl} (\pi^* H - \varepsilon E) \cdot C & \geq & 0 \\ (\pi^* H - mE + (m - \varepsilon)E) \cdot C & \geq & 0 \\ C^2 + (m - \varepsilon)E \cdot C & \geq & 0 \\ C^2 + m^2 - \varepsilon m & \geq & 0 \end{array} \quad \Rightarrow \quad \varepsilon \leq \frac{m^2 + C^2}{m}.$$

□

Definition

$f \in \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ irreducible, Δ Newton polygon, m multiplicity at $(1,1)$. The strict transform $C \subseteq \tilde{X}$ of the closure of $V(f) \subseteq (\mathbb{C}^*)^2$ is the **intrinsic curve** defined by f, and it is:

- **intrinsic negative** (resp. **non-positive**) if $C^2 < 0$ (resp. ≤ 0);
- **intrinsic** $(-n)$**-curve** if $C^2 = -n < 0$ and $p_a(C) = 0$;
- **expected** if $|\Delta \cap \mathbb{Z}^2| > \binom{m+1}{2}$.

Remark

In the above setting:

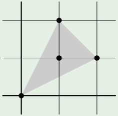
- $\overline{V(f)} \in \mathcal{L}_\Delta(m) \;\Rightarrow\; C^2 = \mathrm{Vol}(\Delta) - m^2.$
- $p_a(C) = \frac{1}{2}\left(\mathrm{Vol}(\Delta) - m^2 + m - |\partial\Delta \cap \mathbb{Z}^2|\right) + 1.$
- *Intrinsic* (-1)*-curve*

$$\mathrm{Vol}(\Delta) = m^2 - 1, \quad |\partial\Delta \cap \mathbb{Z}^2| = m + 1.$$

- C *expected* $\Rightarrow$ *we expect* $\mathcal{L}_\Delta(m) \neq \emptyset$.

Example

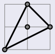
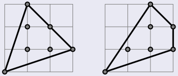
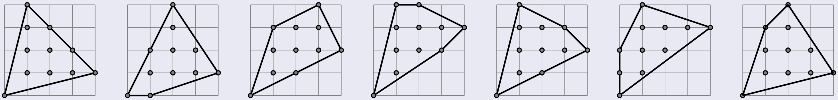
- $f := u^2v + uv^2 - 3uv + 1$, irreducible with $m = 2$.
- Newton polygon



- $\mathrm{Vol}(\Delta) = 3 = m^2 - 1$ and $|\partial\Delta \cap \mathbb{Z}^2| = 3 = m + 1$.
- f defines an **intrinsic** (-1)-**curve**.

Proposition ([LU-21])

Non-equivalent polygons for intrinsic non-positive curves, $m \leq 7$.

m	Δ
2	
3	
4	

Intrinsic curves

Expected

Proposition ([LU-21])

C intrinsic **expected** *non-positive with Newton Δ and multiplicity m. Then one of the following holds:*

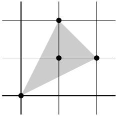
	$\mathrm{Vol}(\Delta)$	$\lvert\partial\Delta \cap \mathbb{Z}^2\rvert$	C^2	$p_a(C)$
$i)$	m^2	m	0	1
$ii)$	m^2	$m+2$	0	0
$iii)$	m^2-1	$m+1$	-1	0

Corollary ([LU-21])

If C is an intrinsic non-positive curve corresponding to a pair (Δ, m), and $\varepsilon := \varepsilon(X_\Delta, H_\Delta, e)$ is the Seshadri constant of the corresponding toric surface, then

$$\varepsilon = \frac{\mathrm{Vol}(\Delta)}{m}.$$

Example.



$\Rightarrow$ Seshadri constant $\varepsilon = \frac{\mathrm{Vol}(\Delta)}{m} = \frac{3}{2}$.

Proposition ([LU-21] Infinite families of non-positive intrinsic curves)

	vertices of Δ	$\mathrm{lw}(\Delta)$	C^2	$g(C)$
(i)	$\begin{bmatrix} 0 & m & 1 \\ 0 & 1 & m \end{bmatrix}$	$m \geq 2$	-1	0
(ii)	$\begin{bmatrix} 0 & m-3 & m & m-1 & m-2 \\ 0 & 0 & 1 & m & m-1 \end{bmatrix}$	$m \geq 4$	-1	0
(iii)	$\begin{bmatrix} 0 & 0 & 2 & m-4 & m-1 & m & m-1 \\ 0 & 1 & m & m & m-1 & m-2 & m-3 \end{bmatrix}$	$m = 2k \geq 8$	-2	0
(iv)	$\begin{bmatrix} 0 & m-2 & m & m-1 & m-2 \\ 0 & 0 & 1 & m & m-1 \end{bmatrix}$	$m \geq 4$	0	0

Proof. Given homogeneous $f_1, \dots, f_4 \in \mathbb{C}[s,t]_m$:

1. it is possible to describe the Newton polygon of $\overline{\psi(\mathbb{P}^1)}$ ([DS-10])

$$\begin{array}{rcl} \psi\colon \mathbb{P}^1 & \dashrightarrow & (\mathbb{C}^*)^2 \\ (s,t) & \mapsto & \left(\frac{f_1}{f_2}, \frac{f_3}{f_4}\right) \end{array}$$

2. if $(f_1, f_2) = (f_3, f_4) = 1$ and $f_1 + f_3 = f_2 + f_4$, then $\overline{\psi(\mathbb{P}^1)}$ has multiplicity at least m at $(1,1)$.

□

Example. Consider $g := s^{m-1} + ts^{m-2} + \cdots + t^{m-1}$ and

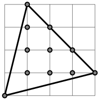
$$f_1 = -s^m, \quad f_2 = t \cdot g, \quad f_3 = t^m, \quad f_4 = -s \cdot g.$$

- The vanishing order of $\psi = \left(\frac{-s^m}{t \cdot g}, \frac{-t^m}{s \cdot g}\right)$ is $(0,0)$ unless

$$\begin{array}{rcl} \mathrm{ord}_{(0,1)} & = & (m,-1) \\ \mathrm{ord}_{(1,0)} & = & (-1,m) \\ \mathrm{ord}_{q_j} & = & (-1,-1) \end{array}$$

  where $q_1, \ldots, q_{m-1}$ are the roots of g.

- The rays of the normal fan of Δ are $(m, -1)$, $(-1, m)$, $(-1, -1)$.
- The integer lengths of the corresponding edges are given by the number of zeroes $1, 1, m-1$.
- Then Δ is



Remark

Let $X := \mathbb{P}(9,10,13)$, $\varepsilon := \varepsilon(X,H,e)$.

- $\varepsilon = 1/\sqrt{9\cdot 10\cdot 13} \;\Leftrightarrow\; d_n\pi^*H - m_nE$, *s.t.* $d_n/m_n \to \sqrt{9\cdot 10\cdot 13}$.
- *It is possible to compute a minimal generating set of the Cox ring of* $\tilde{X}$, *consisting of homogeneous elements of given bounded multiplicity at* e *(see [Hau&al-16]).*

Intrinsic curves

$\mathbb{P}(9, 10, 13)$

- For $m \leq 30$, we found the following 52 generators.
- There are many intrinsic (-1)-curves which are positive in $\tilde{X}$.

d	m	C^2	p_a
36	1	0	0
39	1	0	0
40	1	0	0
83	2	-1	0
109	3	-1	0
110	3	-1	0
113	3	-1	0
139	4	-1	0
140	4	-1	0
143	4	-1	0
208	6	-1	0
209	6	-1	0
210	6	-1	0
213	6	-2	0
243	7	0	1
309	9	-1	0
310	9	-1	0
312	9	-1	0

d	m	C^2	p_a
313	9	-1	0
378	11	-1	0
379	11	-1	0
380	11	-1	0
413	12	-1	0
481	14	-1	0
482	14	-1	0
483	14	-1	0
516	15	0	1
549	16	-1	0
550	16	-1	0
551	16	-1	0
585	17	-1	0
652	19	-1	0
653	19	-1	0
686	20	0	1
720	21	1	2

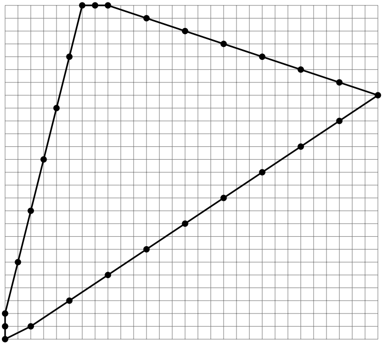
d	m	C^2	p_a
721	21	-1	0
755	22	-1	0
789	23	0	1
790	23	1	2
823	24	-1	0
824	24	-1	0
858	25	-1	0
891	26	-1	0
892	26	-1	0
893	26	0	1
893	26	3	3
926	27	-1	0
959	28	0	1
960	28	0	1
994	29	-1	0
1028	30	0	1
1029	30	-1	0

Intrinsic curves

$\mathbb{P}(9, 10, 13)$

Remark

Best approximation of $\sqrt{9 \cdot 10 \cdot 13} = 34.20526\ldots$ *given by an intrinsic* (-1)*-curve,* $891/26 = 34.26923\ldots$



Intrinsic curves

$\mathbb{P}(9, 10, 13)$

Question

Is it possible to construct an infinite family of intrinsic (-1)-curves appearing as positive curves in $\tilde{X}$, and whose slopes approach $\sqrt{9 \cdot 10 \cdot 13}$?

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