Sharp Ellipsoid Embeddings and Toric Mutations

joint with Roger Casals (UC Davis)

$a \le b$

$E(a,b) \subseteq \mathbb{R}^4, \omega_{std}$; $E(a,b) = \left\{ \frac{x_1^2+y_1^2}{a} + \frac{x_2^2+y_2^2}{b} \le 1 \right\}$

When $E(a,b) \xhookrightarrow{symp} (X^4, \omega)$?

. Gromov's non-squeezing Thm (1985)

$E(1,b) \hookrightarrow D^2(R) \times \mathbb{R}^2 \iff 1 \le R$

. $\boxed{\text{Constructions}}$: (packings)

McDuff, Biran, Buse-Hind, Guth, Schlenk,
Frenkel-Müller, Hind-Lisi, Opshtein,
Ramos, Ramos-Sepe, Cristofaro-Gardiner-Hind-McDuff...

. $\boxed{\text{Obstructions}}$: Symplectic capacities: Gromov,
Ekland, Hofer-Zehnder, ..., Hutchings (ECH capacities)

⇒ Infinite staircases:

for $(X,\omega) = (B^4(1), \omega_{std})$

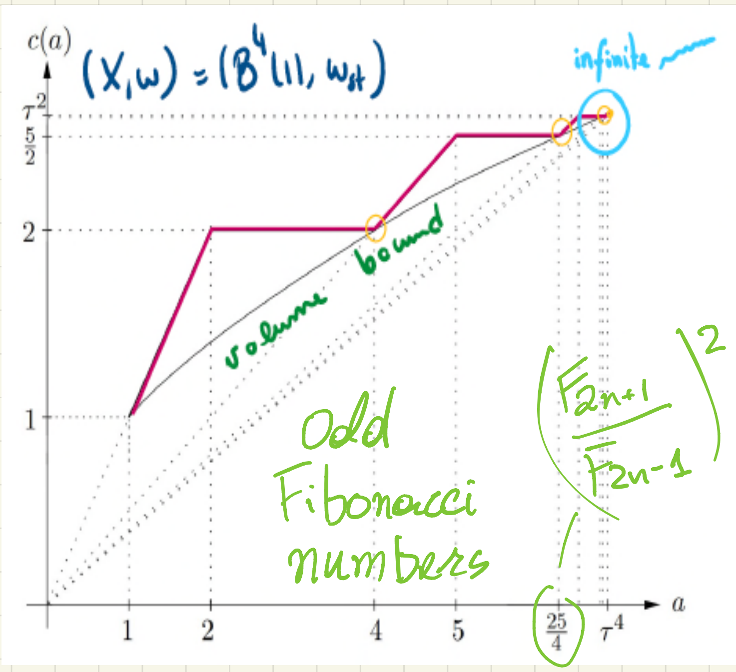

D. McDuff

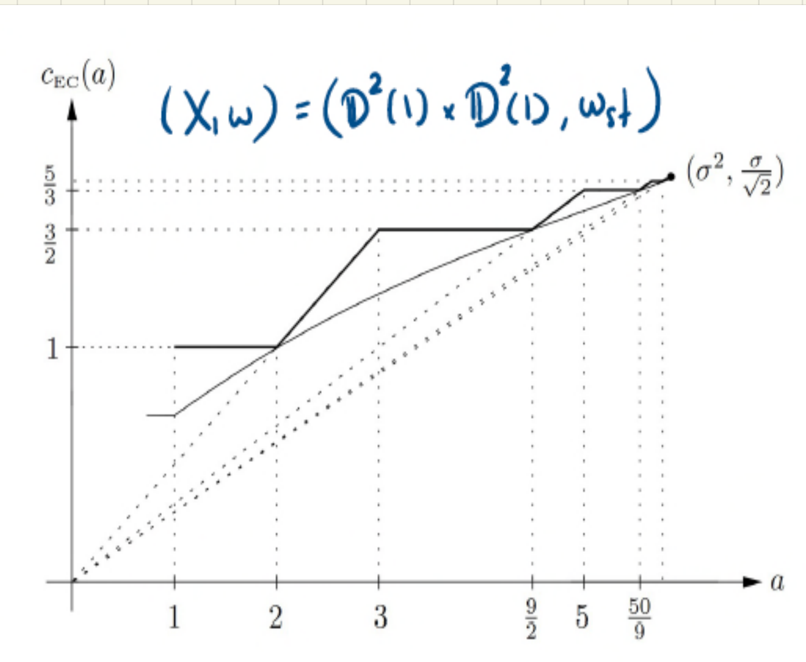

F. Schlenk

$(D^2(1) \times D^2(1), \omega_{std})$ and $E(2,3)$

Frenkel - Müller Cristofaro-Gardiner

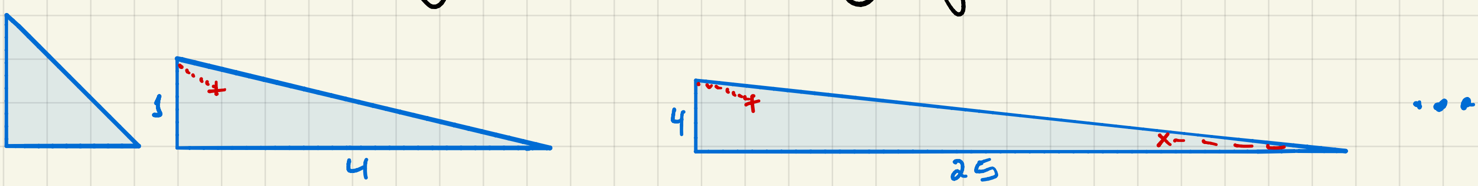
$$\boxed{C_X(a) = \inf\{\lambda \in \mathbb{R} : E(1,a) \hookrightarrow (X, \lambda\omega)\}}$$





Casals' idea:

(– 14) ∞ many monotone Lagragian tori in $\mathbb{C}P^2$:



Mutation known in algebraic geometry; Galkin - Usnich

Ahktar - Kasprzyk

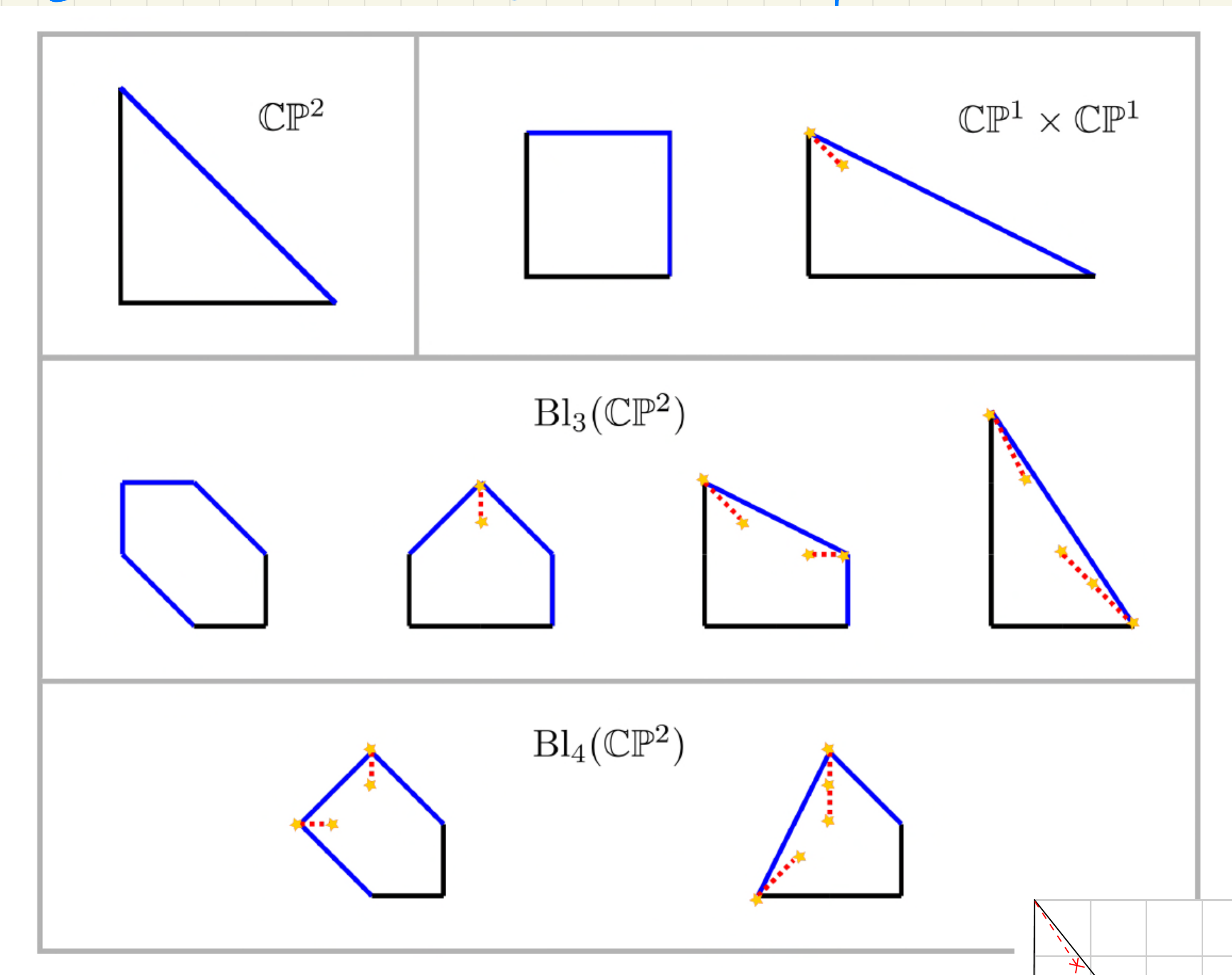
Polytopes $\rightsquigarrow \mathbb{C}P^2(p^2, q^2, r^2)$; (p,q,r) - Markov triple

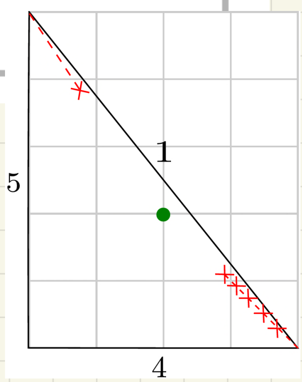
$$p^2 + q^2 + r^2 = 3pqr$$

$p = 1$: $\quad 1 + q^2 = (3q - r)\, r$ $\qquad$ mutation $\quad r \leftrightarrow 3q - r$

$$F_{2n+1} = F_{2n} + F_{2n-1} = 2F_{2n-1} + F_{2n-2} = 3F_{2n-1} - F_{2n-3}$$

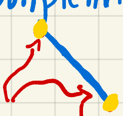
recover odd Fibonacci sequence!





(C.G-H-M-P)

Thm (Casals-V.) Sharp ellipsoid embeddings for 9 domains above | complement of



Lag spheres

"Infinite Staircases and Reflexive Polytopes"

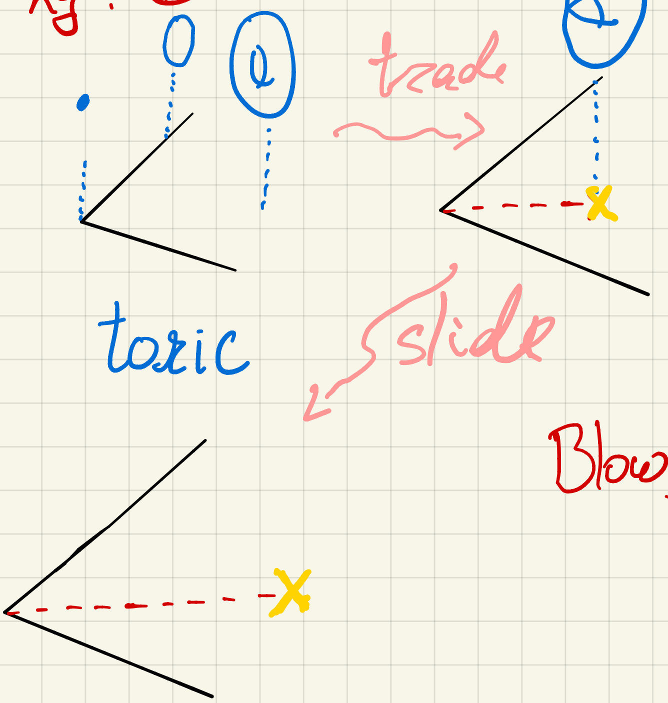
Our Proof:

* Almost toric mutations
* Symplectic tropical curves on ATFs.


M. Symington

Almost toric fibrations: $(X^4,\omega) \longrightarrow B^2$

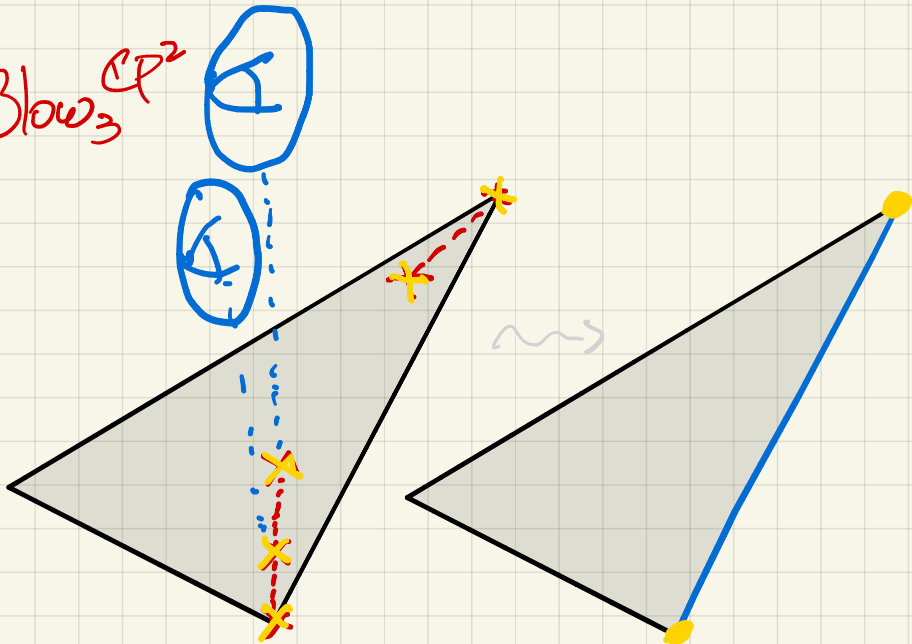
Eg: $\mathbb{C}^2$



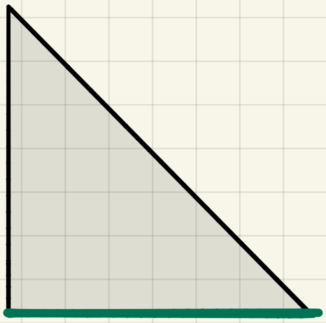
allows focus-focus singularity

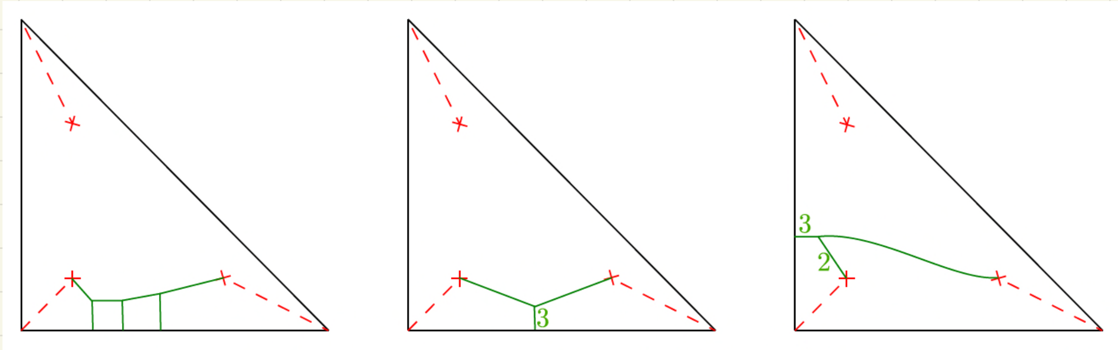
affine manifold

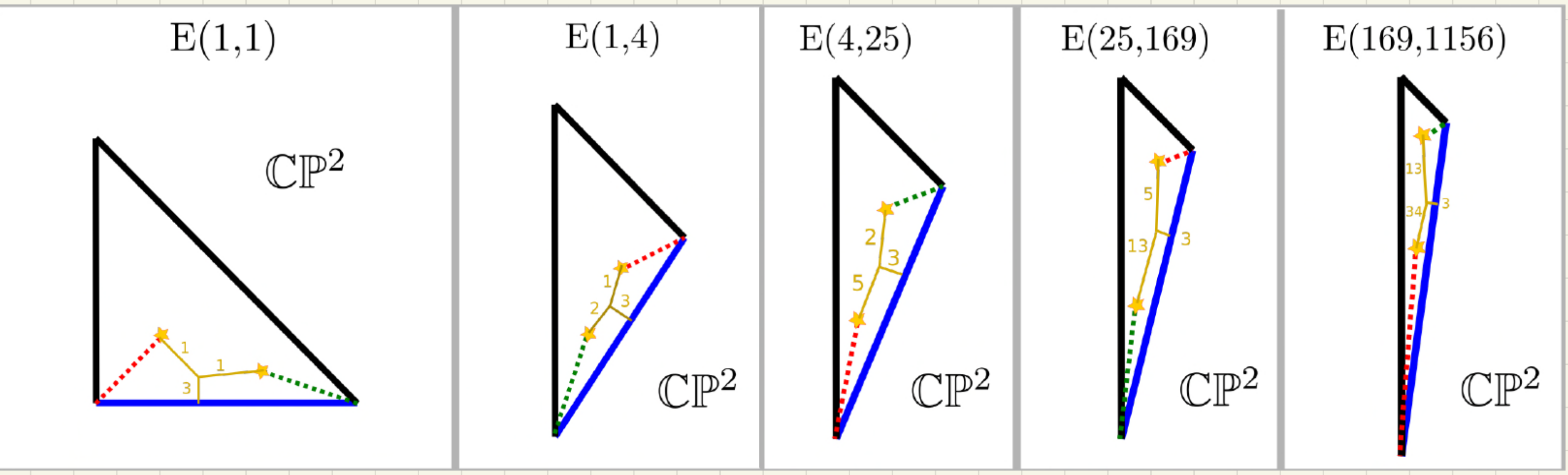
$\mathrm{Blow}_3 \mathbb{CP}^2$

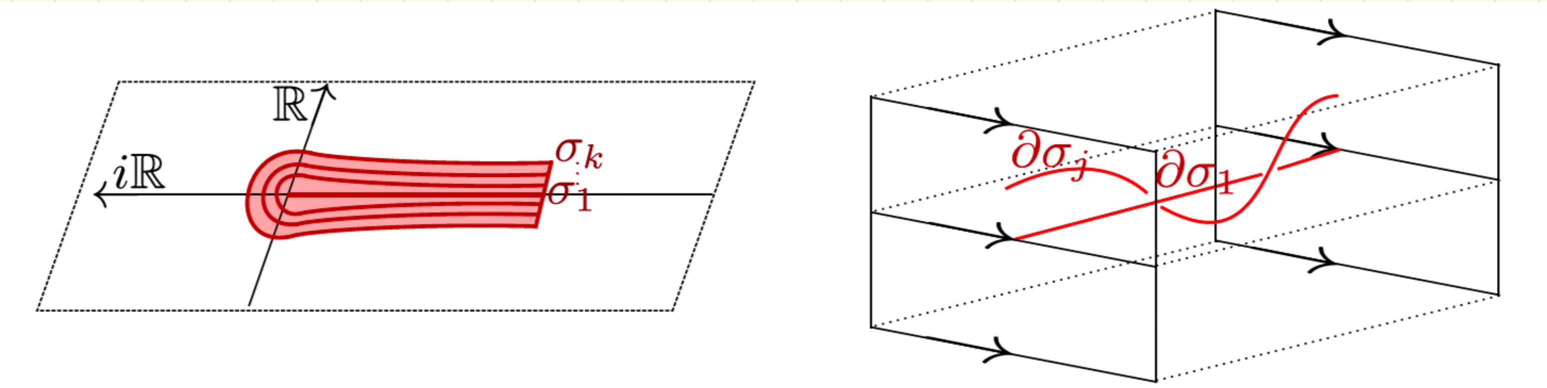


Symplectic tropical curves:

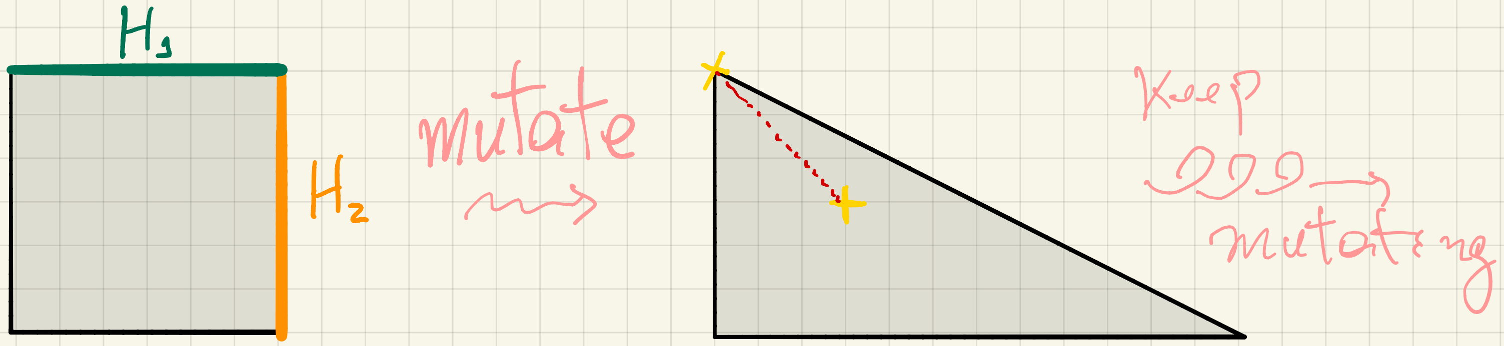




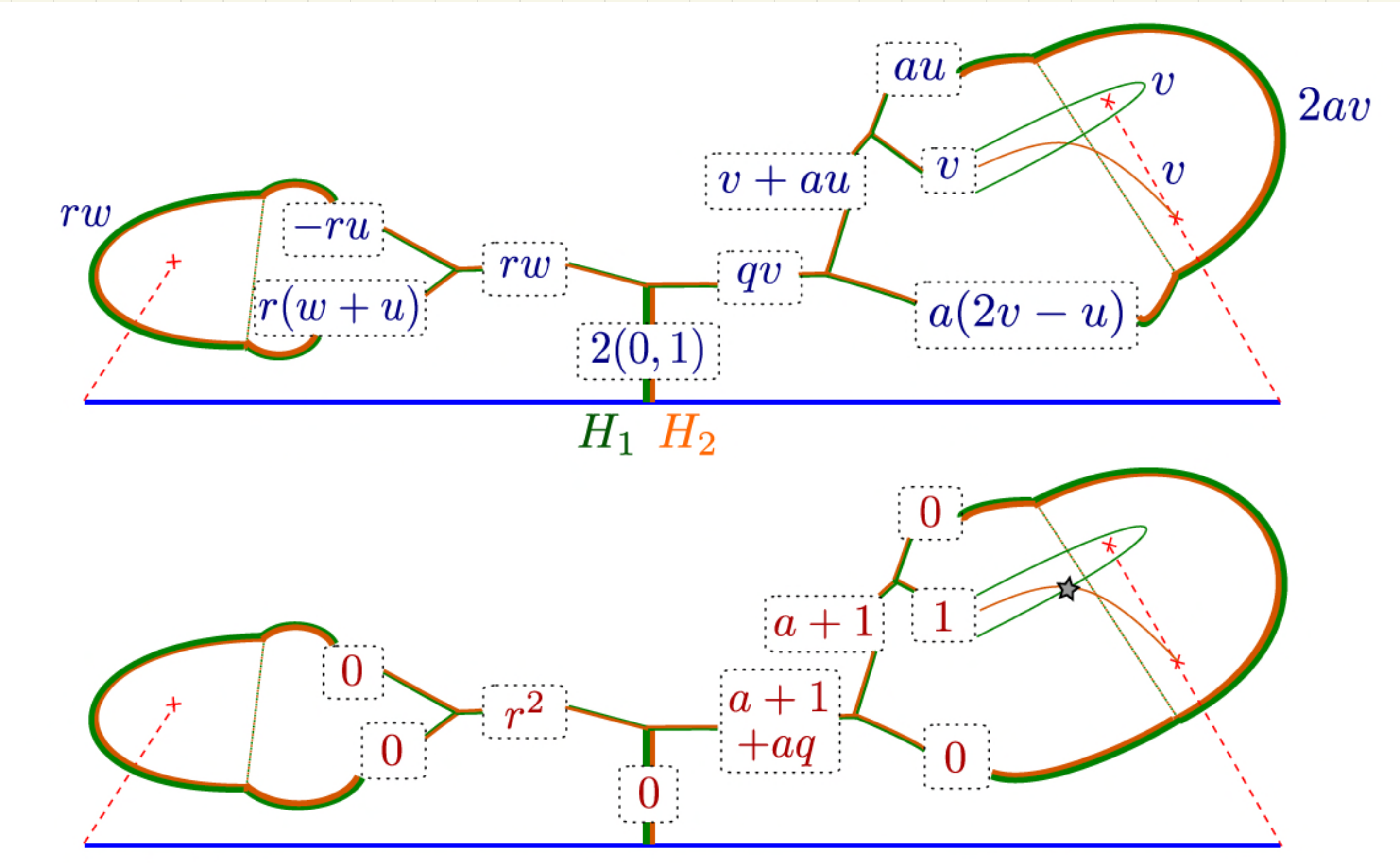




Theorem (Casals, _) In all triangular shape ATFs associated with $E(a,b) \hookrightarrow X = \begin{bmatrix} \mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1, \\ \mathrm{Blow}_3 \mathbb{CP}^2, \mathrm{Blow}_4 \mathbb{CP}^2 \end{bmatrix}$ $\exists$ confugurations of symplectic tropical curves associated with the divisors on the 9 initial diagrams, in $X \setminus E(a,b)$.



Sequence of ellipsoids associated with solutions of $1 + q^2 + 2r^2 = 4qr$; $p=1$. $\rightsquigarrow$ Frenkel-Müller's Pell Staircase.



Further directions:

* Neighbourhoods in T^*S^2, $T^*\mathbb{RP}^2$, "T^*Pinwheels" (and plumbins with S^2)
* Higher dimensions: Combinatorial mutations Akhtar-Coates-Galkin-Kasprzyk