Jacobi algebras on the two-loop quiver and applications

Michael Wemyss

Nottingham geometry seminar, 29th April 2021.

(joint with Gavin Brown)

Plan of Talk

1. Jacobi algebras, and the Main Problem.
2. Geometric Interlude: flops and div-to-curve contractions.
3. Results in 'Type A', and 'Type D'.
4. Geometric Consequences.

Algebraic Setup

Consider the free algebra $\mathbb{C}\langle x, y\rangle$. Elements are finite sums like

$$f = \lambda_1 + \lambda_2 x + \lambda_3 y + \lambda_4 x^2 + \lambda_5 xy + \lambda_6 yx + \lambda_7 y^2.$$

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...and the completed version $\mathbb{C}\langle\langle x, y\rangle\rangle$. Basically the same, except now allow infinite sums

$$f = \lambda_1 + \lambda_2 x + \lambda_3 y + \lambda_4 x^2 + \lambda_5 xy + \lambda_6 yx + \lambda_7 y^2 + \dots$$

Both these rings are *not* noetherian, and have exponential growth (GKdim ∞)

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Given any $f \in \mathbb{C}\langle\langle x, y \rangle\rangle$, the Jacobi algebra is

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$$\text{e.g.} \quad \mathfrak{Jac}(x^4 + xy^2) = \frac{\mathbb{C}\langle\langle x, y\rangle\rangle}{((4x^3 + y^2, xy + yx))}.$$

Main Algebraic Question

...classify all possible Jacobi algebras, up to isomorphism.

Problem

For every $n \geq 0$, produce a set of potentials $\mathcal{S}_n$ from which we can realise *every* Jacobi algebra of Gelfand–Kirillov (GK) dimension n, up to isomorphism.

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We insist that the elements of $\mathcal{S}_n$ should be a *normal form*, meaning that if $f, g \in \mathcal{S}_n$ with $f \neq g$, then the resulting Jacobi algebras are not isomorphic.

Notation: write $f \cong g$ to mean $\mathrm{Jac}(f) \cong \mathrm{Jac}(g)$.

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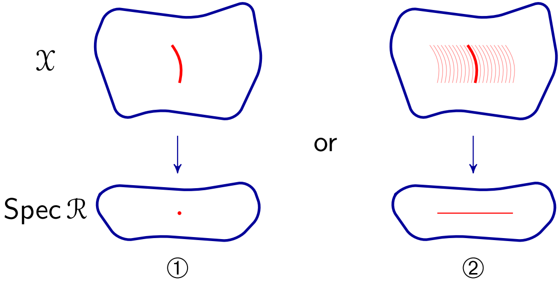
2. ...there are no moduli. Just very few countable families.
3. ...the classification is ADE.
4. ...this algebraic classification *is* (and implies) the classification of flops, and of crepant divisorial contractions to curves.

Back up: where to find Jacobi algebras?

Contraction algebras arise in the birational geometry.

Today: focus on crepant contractions of two types:



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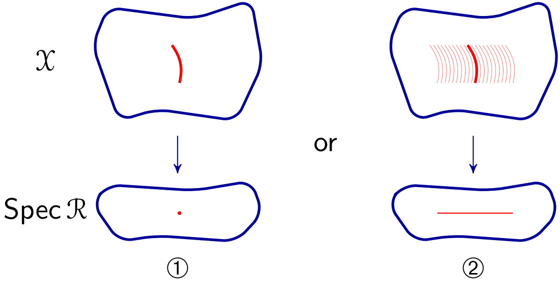

Assumptions: $\mathcal{X}$ is smooth, and only one curve above the origin.

To this data we associate the contraction algebra $\mathrm{A}_{\mathrm{con}}$ as follows...

Contraction Algebras

The contraction algebra $\mathrm{A}_{\mathrm{con}}$ is defined using (noncommutative) deformation theory of the reduced fibre above the origin.

Details are unimportant, the only facts we need today are:

1. Since only one curve, $\mathrm{A}_{\mathrm{con}}$ is a factor of $\mathbb{C}\langle\langle x, y\rangle\rangle$.
2. Since $\mathcal{X}$ is smooth, there exists f such that $\mathrm{A}_{\mathrm{con}} \cong \mathcal{J}\mathrm{ac}(f)$.

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Theorem (Donovan–W)

$$\text{Situation ① (flopping)} \iff \operatorname{GKdim} \mathrm{A}_{\mathrm{con}} = 0.$$
$$\text{Situation ② (div}\to\text{curve)} \iff \operatorname{GKdim} \mathrm{A}_{\mathrm{con}} = 1.$$

...motivates studying f such that $\operatorname{GKdim} \mathcal{J}\mathrm{ac}(f) \leq 1$.

Two Conjectures

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Realisation Conjecture (Brown–W)

Contraction algebras=Jacobi algebras. If $f \in \mathbb{C}\langle\langle x, y\rangle\rangle$ satisfies $\operatorname{GKdim} \mathcal{J}\mathrm{ac}(f) \leq 1$, then $\mathcal{J}\mathrm{ac}(f) \cong \mathrm{A}_{\mathrm{con}}$ for either a flopping contraction (GK zero), or div$\to$curve contraction (GK 1).

...so, blind to any geometry, off we go to classify all $\mathcal{J}\mathrm{ac}(f)$!

We will classify *first*, using only algebra, then at the end relate this to geometry.

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Rules

Since scalars differentiate to zero, and linear terms differentiate to units, to classify f, we can assume f contains only quadratic terms and higher. Write this as $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 2}$.

‘Type A’

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Warm-Up Result

Suppose $f \in \mathbb{C}\langle\langle x, y\rangle\rangle_{\geq 2}$ with $f_2 \neq 0$. Then either

$$f \cong \begin{cases} x^2 \\ x^2 + y^n & \text{for some } n \geq 2. \end{cases}$$

In all cases, $\operatorname{GKdim} \mathfrak{Jac}(f) \leq 1$, $\mathfrak{Jac}(f)$ is commutative, as either

$$\mathfrak{Jac}(f) \cong \mathbb{C}[\![y]\!] \quad \text{or} \quad \mathbb{C}[\![y]\!]/y^{n-1}.$$

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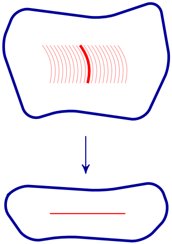
Notes:

- $f_2 \neq 0$ in fact equivalent to $\mathfrak{Jac}(f)$ being commutative.
- Generic behaviour is $\mathfrak{Jac}(f) \cong \mathbb{C}$.

Compare Reid's Pagoda

For Type A contractions, either:



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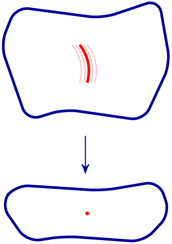

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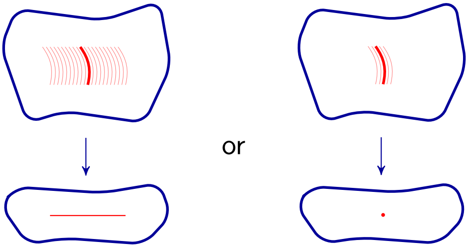

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Having 2 or 3 factors $\rightarrow$ 'Type D'.
Having only 1 factor $\rightarrow$ the exceptional, or 'Type E' case.

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Theorem (Brown–W)

Consider $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$ with $f_3 \neq 0$ such that f_3^{ab} has two or three distinct factors. Then either

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These are normal forms. All satisfy GKdim $\mathfrak{Jac}(f) \leq 1$.

Note: there are no moduli!

All are geometric!

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[Donovan–W] div→curve

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Corollary

The Realisation Conjecture is true, except possibly the only remaining case $f = x^3 +$ higher.

Is Type D now finished?

Theorem* (Brown–W)

Suppose that $f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ is *any* smooth type D flop, or div$\to$curve contraction, one curve above the origin. Then

$$\mathrm{A}_{\mathrm{con}} \cong \mathcal{J}\mathrm{ac}(f)$$

for some f on the previous slide.

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...the conjectures suggest, but don't yet prove, that *these are all* Type D flops, and div$\to$curve, extending Reid from 80s. Even if you don't believe conjectures, there are still geometric corollaries!

GV invariants

To every flop is an associated tuple of numbers $(n_1, \dots, n_6)$ called the Gopakumar–Vafa (GV) invariants.

..basically deform your flopping curve C into a disjoint union of $(-1, -1)$ curves, and count those. It is a bit more refined than this: n_j equals the number of such curves with curve class $j[C]$.

GV invariants

To every flop is an associated tuple of numbers $(n_1, \dots, n_6)$ called the Gopakumar–Vafa (GV) invariants.

..basically deform your flopping curve C into a disjoint union of $(-1,-1)$ curves, and count those. It is a bit more refined than this: n_j equals the number of such curves with curve class $j[C]$.

- Type A (Pagoda flops) have GV invariants $(n, 0, 0, 0, 0, 0)$. The data of n is enough to distinguish elements in this family. All possible n arise.
- Type D flops have GV invariants $(a, b, 0, 0, 0, 0)$ for some $a, b \in \mathbb{N}$. Different flops can have the same GV invariants.

  Question. What possible (a, b) can arise?

Gaps in GV

Corollary

For Type D flops, the only possible GV invariants (a, b) are:

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For Type D flops, the only possible GV invariants (a, b) are:

x^3+x^4	x^3+x^6	x^3+x^8	x^3+x^{10}	x^3+x^{12}	x^3+x^{14}
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)					
	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)
	(7,2)				
		(8,3)	(8,4)	(8,5)	(8,6)
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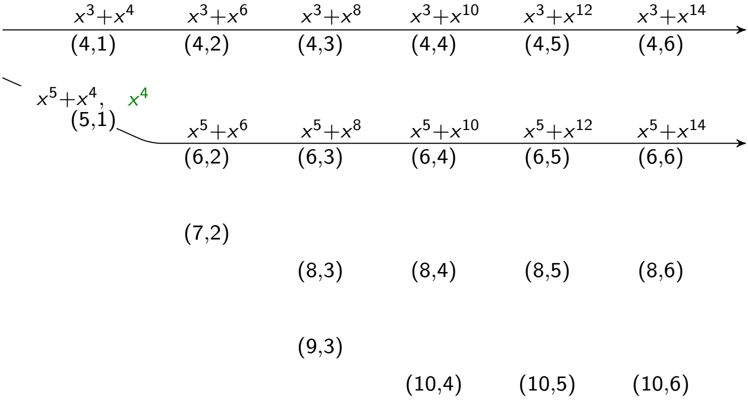

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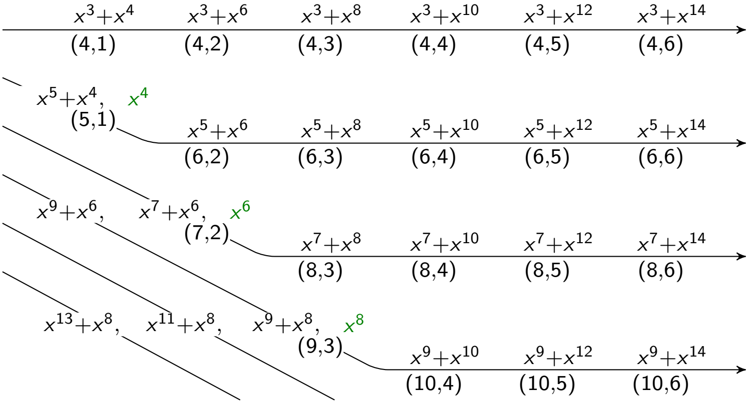

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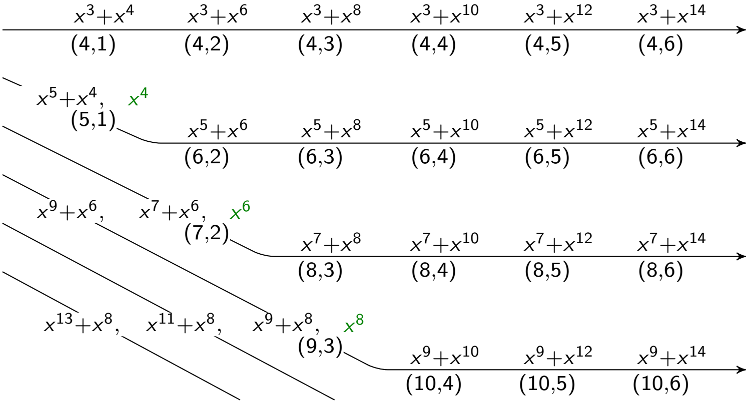

The obstruction to e.g. $(5, 2)$ existing is noncommutative.

Towards Type E

The final case $f = x^3 +$ higher is work in progress.

We have already found the first infinite family of type E flops, plus some div$\to$curve contractions.

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...looks like a *full* analytic classification of single-curve flops, and at the same time div$\to$curve contractions, may indeed be possible. Here is the beginning:

A	$x^2 + y^t$	$t \in \mathbb{N} \cup \{\infty\}$
D	$xy^2 + \varepsilon x^{2n} + \varepsilon x^{2m-1}$	$n, m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$
E	$x^3 + ?$	