Residual categories of quadric surface bundles

arxiV 2203.01031

Def: T triangulated cat

$T = \langle \mathcal{A}_1, \cdots, \mathcal{A}_n \rangle$ is a <u>semiorthogonal decomposition</u> (by full triang subcat $\mathcal{A}_i$) if

Hom = 0

(1) $\mathrm{Hom}_T(\mathcal{A}_j, \mathcal{A}_i) = 0 \quad \forall j > i$

(2) $\forall t \in \mathrm{Obj}\, T \ \exists$ filtration $0 = t_n \to t_{n-1} \to \cdots \to t_0 = t$

s.t. $\mathrm{cone}(t_i \to t_{i-1}) \in \mathcal{A}_i$.

Convention: X scheme $D^b(X) := D^b(\mathrm{Coh}\, X)$
R alg $D^b(R) := D^b(\text{mod-}R)$
finitely gen right mod

For a flat family of Fano varieties $p: X \to S$
(can be singular)

with $\omega_{X/S}^{-1} = \mathcal{O}_{X/S}(n)$

$\exists$ SOD $D^b(X) = \langle \mathcal{R}_X, p^* D^b(S) \otimes \mathcal{O}_{X/S}(1), \cdots, p^* D^b(S) \otimes \mathcal{O}_{X/S}(n) \rangle$

where

$$\mathcal{R}_X = \{ t \in D^b(X) \mid \mathrm{Hom}_{D^b(X)}(p^* D^b(S) \otimes \mathcal{O}_{X/S}(i), t) = 0 \quad \forall\ 1 \le i \le n \}$$

Def: $\mathcal{R}_X$ is called the <u>residual cat</u> (or Kuznetsov component) of X.

I. Quadric hypersurfaces

k field $k=\bar{k}$ char $k=0$

$Q = Q^n$ quadric of dim n over k.

① (Kapranov) Q smooth

$$R_Q \simeq \begin{cases} \langle T_1, T_2 \rangle \simeq D^b(k\times k) & n \text{ even} \\ \langle T \rangle \simeq D^b(k) & n \text{ odd} \end{cases}$$

where T_1, T_2, T are spinor bundles on Q.

eg. $n=2$ $Q^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $\{T_1, T_2\} = \{\mathcal{O}(1,0), \mathcal{O}(0,1)\}$

② corank 1 = Q is a cone over smooth quadric (i.e. vertex is nodal)

$$R_Q \simeq \begin{cases} \langle \text{spinor sheaf} \rangle \simeq D^b\left(\frac{k[\varepsilon]}{\varepsilon^2}\right) & n \text{ even} \\ D^b_{\mathbb{Z}/2\mathbb{Z}}\left(\frac{k[\varepsilon]}{\varepsilon^2}\right) \simeq D^b(R) & n \text{ odd} \end{cases}$$

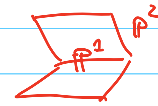
where R is a quaternion alg.

③ [Kuz] [ABB]

In general, $R_Q \simeq D^b(\mathrm{Cliff}_0)$

where Cliff_0 is the even Clifford alg of Q.

(Xie) $n=2$ corank 2 i.e $Q \cong \mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$



$R_Q \cong D^b(Y)$ where

$(Y, \mathcal{O}_Y)$ is a dg scheme

- $\mathcal{O}_Y$ concentrated in deg $-1, 0$
- underlying scheme $\pi_0 Y \cong \mathbb{P}^1$.

II Quadric surface bundles

1. Set up and main results

k field char $k \neq 2$

S = integral noetherian scheme over k

$\mathcal{E}$ vb, $\mathcal{L}$ lb on S

Def: $q: \mathcal{E} \to \mathcal{L}$ is a (line bundle valued) quadratic form

on S if q is $\mathcal{O}_S$-homo of deg 2 s.t.

$b_q: \mathcal{E} \times \mathcal{E} \to \mathcal{L}$ defined by $\quad \mathrm{Sym}^2 \mathcal{E} \to \mathcal{L}$

$b_q(v,w) = q(v+w) - q(v) - q(w)$ is symmetric bilinear.

$q: \mathcal{E} \to \mathcal{L}$ corr to

$$q \in \Gamma(\mathbb{P}_S(\mathcal{E}), \mathcal{O}_{\mathbb{P}_S(\mathcal{E})/S}(2) \otimes \pi^* \mathcal{L}) \cong \Gamma(S, \mathrm{Sym}^2(\underset{\text{dual}}{\mathcal{E}^\vee}) \otimes \mathcal{L})$$

$\pi: \mathbb{P}_S(\mathcal{E}) \to S$

Def: Assume $q \neq 0$. Let $Q = \{q=0\} \subset \mathbb{P}_S(\mathcal{E})$.

$p: Q \to S$ is called a <u>quadric bundle</u>.

Def. $q: \mathcal{E} \to \mathcal{L}$ is called <u>primitive</u> if $\forall\, s \in S$

$q_s := q \otimes_k k(s) \neq 0$

Then $p: Q \to S$ is flat $\Leftrightarrow$ $q: \mathcal{E} \to \mathcal{L}$ is primitive.

Denote by $S_\ell = \{ s \in S \mid \operatorname{corank} q_s \geq \ell \}$ $\ell \in \mathbb{N}$

$S = S_0 \supset S_1 \supset S_2 \cdots$

locus of singular fibers

Theorem $(\overset{\text{char}=0}{\text{Kuz}}, \overset{\text{any char}}{\text{ABB}})$ even Clifford alg of p.

$p: Q \to S$ flat $\Rightarrow$ $\mathcal{R}_Q \simeq D^b(S, \mathcal{C}\ell iff_0)$

bounded derived cat of coh sheaves on S with right $\mathcal{C}\ell iff_0$-mod structures.

In general, $\mathcal{R}_Q$ is noncommutative

Goal: When is $\mathcal{R}_Q$ geometric?

That is, $\mathcal{R}_Q \cong D^b(Z, \mathcal{A})$ where

- Z scheme over S
- $\mathcal{A}$ Azumaya alg on Z

Known (Kuz, ABB)

If $p: Q \to S$ has simple degeneration (each fiber has corank ≤ 1) and relative dim is even, then

$R_Q \cong D^b(\hat{S}, \hat{A})$ where $\hat{A}$ Azumaya

$\hat{S} \to S$ is the double cover ramified along S_1.

My expectation:

When relative dim of $p: Q \to S$ is even and $S_2 \subsetneqq S$, $S_3 = \phi$, R_Q is geometric.

Now we focus on $p: Q \to S$ flat quadric surf bundle

Main Results.

$p: Q \to S$ flat quadric surf bundle.

R_Q is geometric when

① $S_2 \subsetneqq S$ and p has a smooth section (consists of smooth points of fibers). In this case, twist is trivial.

② $k = \overline{k}$, char $k = 0$, Q smooth, S smooth surf

Remarks:

(1) In both cases $S_3 = \emptyset$, i.e., fibers have corank ≤ 2

(2) For any flat quadric surf bundle $p: Q \to S$ with $S_3 = \emptyset$, étale locally p has a smooth section.

$\Rightarrow$ It's possible to generalise ① to any $p: Q \to S$ with $S_2 \subsetneq S$ and $S_3 = \emptyset$.

(3) Proof of ② is geometric but can't be generalised.

Main Ideas: make use of

- hyperbolic reduction $\quad q = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & \bar{q} \end{pmatrix} \quad \begin{pmatrix} \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} & 0 \\ 0 & \bar{q} \end{pmatrix}$
- relative Hilbert scheme of lines

Let $p: Q \to S$ be a flat quadric surface bundle.

Theorem 1 (–)

$S_2 \subsetneq S$, $p: Q \to S$ has a smooth section

$\Rightarrow R_Q \simeq D^b(\bar{Q})$ where $\bar{Q}$ is the hyperbolic reduction wrt the smooth section.

Theorem 2 (–)

$k = \bar{k}$, char $k = 0$, Q smooth, S smooth surf

$\Rightarrow R_Q \simeq D^b(S^+, A^+)$ where

$S_2 \subset S$

$S^+ = Bl_{S_2}\tilde{S}$ = resolution of the double cover $\tilde{S}$ over S ramified along S_1 ($\tilde{S}$ is nodal along $S_2 \subset \tilde{S}$) and A^+ is Azumaya on S^+.

Moreover, $[A^+] \in Br(S^+)$ is trivial $\Leftrightarrow$ $p: Q \to S$ has a rational section.

Example: $Q = \{xy + tzw = 0\} \subset \overset{x\,y\,z\,w}{\mathbb{P}^3} \times \overset{t}{\mathbb{A}^1} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & t & 0 \end{pmatrix}$

$p: Q \to \mathbb{A}^1$ $Q_0 :=$ fiber over $0 \in \mathbb{A}^1$ has corank 2.

Smooth section $= \{y = z = w = 0\}$ (or $\{x = z = w = 0\}$)

Hyperbolic reduction $\bar{Q} = \{tzw = 0\} \subset \mathbb{P}^1 \times \mathbb{A}^1$ $\bar{Q} \to \mathbb{A}^1$ non-flat

Theorem 1 $\Rightarrow$ Residual category $\mathcal{R}_Q \simeq D^b(\bar{Q})$

Base change $\Rightarrow$ $\mathcal{R}_{Q_0} \simeq D^b(\bar{Q} \overset{\mathbb{L}}{\underset{\mathbb{A}^1}{\times}} \{0\})$ $Y := \bar{Q} \overset{\mathbb{L}}{\underset{\mathbb{A}^1}{\times}} \{0\}$

$(\{0\}, \mathcal{O}_{\{0\}}) \simeq (\mathbb{A}^1 \quad \mathcal{O}_{\mathbb{A}^1} \xrightarrow{t} \mathcal{O}_{\mathbb{A}^1})$

deg -1 0

$Y \simeq \bar{Q} \underset{\mathbb{A}^1}{\times} (\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1} \xrightarrow{t} \mathcal{O}_{\mathbb{A}^1}) \simeq (\bar{Q}, \mathcal{O}_{\bar{Q}} \xrightarrow{t} \mathcal{O}_{\bar{Q}})$

$\mathcal{H}^0(\mathcal{O}_{\bar{Q}} \xrightarrow{t} \mathcal{O}_{\bar{Q}}) \simeq \mathcal{O}_{\mathbb{P}^1}$ $\mathcal{H}^{-1}(\mathcal{O}_{\bar{Q}} \xrightarrow{t} \mathcal{O}_{\bar{Q}}) \neq 0$

2. Ideas for the proofs of Theorem 1

Two proofs: one easy, one harder

harder proof describes the embedding functor

$\mathcal{B}_Q \hookrightarrow D^b(Q)$ explicitly.

$q: \mathcal{E} \to \mathcal{L}$ $\quad$ $p: Q \to S$

Def: $W \subseteq \mathcal{E}$ subbundle

- W is <u>isotropic</u> if $q|_W = 0$ $\left(\Leftrightarrow \mathbb{P}_S(W) \subset Q \right)$
- W is <u>regular isotropic</u> if more over $\forall s \in S$

$\mathbb{P}_S(W) \cap Q_s \left(:= Q \times_S k(s) \right)$ is contained in the smooth locus of Q_s.

(smooth section $\leftrightarrow$ regular isotropic (b))

W regular isotropic

$$0 \to W^\perp \to \mathcal{E} \xrightarrow{b_q|_{W \times \mathcal{E}}} \mathcal{H}om(W, \mathcal{L}) \to 0$$

$$v \mapsto b_q(-, v)$$

$\downarrow$ W regular

$q|_W = 0 \Rightarrow W \subset W^\perp$

$b_q(W, W^\perp) = 0 \Rightarrow q|_{W^\perp}: W^\perp \to \mathcal{L}$ induces

a new quadratic form $\bar{q}: W^\perp/W \to \mathcal{L}$

$S = \operatorname{Spec} \bar{k}$

$\Rightarrow b_q$

$= \begin{pmatrix} \begin{matrix} 0 & I \\ I & 0 \end{matrix} & 0 \\ 0 & b_{\bar{q}} \end{pmatrix}$

Def: Denote $\bar{\mathcal{E}} = W^{\perp}/W$

$\bar{q} : \bar{\mathcal{E}} \to \mathcal{L}$ ($\bar{Q} = \{\bar{q} = 0\} \subseteq \mathbb{P}_S(\bar{\mathcal{E}})$) is the <u>hyperbolic reduction</u> of $q : \mathcal{E} \to \mathcal{L}$ ($Q = \{q = 0\} \subseteq \mathbb{P}_S(\mathcal{E})$) wrt regular isotropic W.

$$\begin{tikzcd}[column sep=small] & Q' \cong Bl_{\mathbb{P}_S(W)} Q \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbb{P}_S(W) \subset Q \arrow[rr, dashed] & & \mathbb{P}_S(\mathcal{E}/W) \supset \bar{Q} = \{\bar{q} = 0\} \end{tikzcd}$$

where g has fiber $\mathbb{P}^r$ over $\bar{Q}$ ($r = \operatorname{rank} W$)

$\mathbb{P}^{r-1}$ over its complement.

If $\operatorname{rank} W = 1$, $Q' \cong Bl_{\bar{Q}} \mathbb{P}_S(\mathcal{E}/W)$ i.e.,

$$\begin{tikzcd}[column sep=small] & Q' \cong Bl_{\mathbb{P}_S(W)} Q \cong Bl_{\bar{Q}} \mathbb{P}_S(\mathcal{E}/W) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbb{P}_S(W) \subset Q \arrow[rr, dashed] & & \mathbb{P}_S(\mathcal{E}/W) \supset \bar{Q} = \{\bar{q} = 0\} \end{tikzcd}$$

Proofs of Theorem 1:

$p: Q \to S$ flat quadric surface bundle

$S_2 \subsetneq S$, p has a smooth section $\mathbb{P}_S(W)$

Proof 1 (easy):

[Jiang 21] Blow-up formula

In the setting of Theorem 1 ($S_2 \subsetneq S$ + smooth section)

$\Rightarrow D^b(Q') = \langle D^b(Q), D^b(S) \otimes \mathcal{O}_E \rangle$ E exc locus of f.

$= \langle D^b(\bar{Q}), D^b(\mathbb{P}_S(\mathcal{E}/W)) \rangle$

Mutations $\Rightarrow R_Q \cong D^b(\bar{Q})$

Note $\bar{p}: \bar{Q} \to S$ is <u>not</u> flat!

- $\bar{p}^{-1}(S \setminus S_2) \to S \setminus S_2$ double cover ramified along $S_1 \setminus S_2$
- $\bar{q}|_{S_2} = 0 \Rightarrow \bar{p}^{-1}(S_2) = \mathbb{P}_{S_2}(\bar{\mathcal{E}}|_{S_2})$ is a $\mathbb{P}^1$-bundle.

III. Examples (Applications of Main Theorems)

Example 1 (Xie)

X quintic del Pezzo 3-folds (terminal Gorenstein Fano 3-folds of index 2 and degree 5)

X nodal and number of nodes ≤ 3

Let $a \in X$ be a node.

$X \subset \mathbb{P}^6$ embedded projective tangent space $T_a X \cong \mathbb{P}^4$

Consider linear projection $X \dashrightarrow \mathbb{P}^1$ from $T_a X$

$f : Y \cong Bl_{T_a X \cap X} X \to X$ resolution at a

exceptional locus $E \cong \mathbb{P}^1$

$g : Y \to \mathbb{P}^1$ flat quadric surface bundle with a smooth section E

X has 1 or 2 nodes $\Rightarrow$ $Y \to \mathbb{P}^1$ has fibers of corank ≤ 1

X has 3 nodes $\Rightarrow$ $Y \to \mathbb{P}^1$ has a fiber of corank 2

Theorem 1 $\Rightarrow$ residual cat $R_Y \cong D^b$(hyperbolic reduction)

Example 2 (Moschetti + Kuznetsov)

$X \subset \mathbb{P}^5$ smooth cubic 4-fold containing a plane $\Sigma = \mathbb{P}^2$

$$\begin{tikzcd} & Y = Bl_\Sigma X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \Sigma \subset X \arrow[rr, dashed, "\text{projection from } \Sigma"'] & & \mathbb{P}^2 \end{tikzcd}$$

g is a flat quadric surf bundle with possibly a finite number of corank 2 fibers.

$$R_X \simeq R_Y \underset{\text{Theorem2}}{\simeq} D^b(\text{smooth K3 surf}, \mathcal{A})$$

Example 3

$X = Q_1 \cap Q_2 \cap Q_3$ smooth c.i.

where $Q_i \subseteq \mathbb{P}^{2m+3}$ quadrics ($\Rightarrow \dim X = 2m$)

net of quadrics $\Rightarrow$

$p: Q \to \mathbb{P}^2$ flat quadric bundle of relative dim $2m+2$

Homological Projective Duality

$\Rightarrow$ Residual categories $R_X \simeq R_Q$

Prop: Assume $m \leq 5$.

$$R_X \simeq D^b(S^{2m}, \underset{\swarrow \text{ Azumaya}}{A^{2m}})$$

where $S^{2m} \to \mathbb{P}^2$ is the resolution of the double cover over $\mathbb{P}^2$ ramified along a nodal curve of deg $2m+4$.

Moreover, if $m \geq 3$, then X is rational

if $m=2$ and $0=[A^4] \in Br(S^4)$, then X is rational.

Idea:

$X = Q_1 \cap Q_2 \cap Q_3$ net of quadrics $\Rightarrow$

$p: Q \to \mathbb{P}^2$ flat quadric bundle of relative dim $2m+2$

For $m \leq 5$, $\exists$ $\Sigma_m = \mathbb{P}^{m-1} \subset X$

$\Rightarrow$ $\Sigma_m \times \mathbb{P}^2 \subset Q$ corr to regular isotropic subbundle

hyperbolic reduction

$\Rightarrow$ $\bar{p}: \bar{Q} \to \mathbb{P}^2$ flat quadric surf bundle

with $\bar{Q}$ smooth $\square$