Relative quantum cohomology under birational transformations

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Outline

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Gromov–Witten theory

Gromov–Witten theory is a curve couting theory. However, instead of studying curves, we study maps from curves to the target variety.

Moduli Space of Stable Maps

Let X be a smooth projective variety.
The moduli space $\overline{M}_{g,n}(X,d)$ of stable maps of degree d from genus g nodal curves with n-markings to X consists of

$$(C, \{p_i\}_{i=1}^n) \xrightarrow{f} X,$$

where

- C is a projective, connected, nodal curve of genus g;
- $p_1, \dots, p_n$ are distinct nonsingular points of C;
- $f_*[C] = d \in H_2(X)$;
- stable: automorphisms of the map is finite

Definition

Given cohomological classes $\gamma_i \in H^*(X)$, one can define the Gromov–Witten invariant

$$\left\langle \prod_{i=1}^{n} \tau(\gamma_i) \right\rangle_{g,n,d}^{X} := \int_{[\overline{M}_{g,n}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^{n} (\mathrm{ev}_i^* \gamma_i),$$

Quantum Cohomology

The quantum cohomology ring $QH^*(X)$ is a deformation of the usual cohomology ring using Gromov–Witten invariants.

Quantum Product

Given $\alpha, \beta \in H^*(X)$, the quantum product is defined using three-point Gromov–Witten invariants.

$$\alpha \circ \beta = \sum_{d \in H_2^{\text{eff}}(X)} \sum_k Q^d \langle \alpha, \beta, \phi_k \rangle_{0,3,d}^X \phi^k$$

where $\{\phi_k\}$ and $\{\phi^k\}$ are dual basis of $H^*(X)$.

Enumerative mirror symmetry

- The A-model data is a generating function of genus zero Gromov–Witten invariants called the J-function $J_X(\tau, z)$.
- The B-model data is period integrals called the I-function $I_X(y, z)$

Mirror theorem (Givental 1996, Lian–Liu–Yau 1997, ...)

$$J_X(\tau(y), z) = I_X(y, z)$$

where $\tau(y)$ is called the mirror map.

I-function

For quintic threefold the I-function is

$$I_X(y) = \sum_{d\geq 0} y^{H+d} \frac{\prod_{a=1}^{5d}(5H+a)}{\prod_{a=1}^{d}(H+a)^5}$$
$$= \sum_{d\geq 0} \frac{(5d)!}{(d!)^5} y^d H^0 + O(H).$$

Remark

Quantum cohomology can be reconstructed from the I-function.

Gromov–Witten theory under birational transformations

- Gromov–Witten invariants are deformation invariants, but not birational invariants.

Natural Question

How Gromov–Witten theory varies under birational transformations?

The crepant transformation conjecture (by Yongbin Ruan)

Given a birational transformation $\phi : X_+ \dashrightarrow X_-$. Suppose there is a $\tilde{X}$ with projective birational morphisms $f_\pm : \tilde{X} \to X_\pm$ such that the following diagram commute

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{f_+}\swarrow & & \searrow^{f_-} \\ X_+ & \overset{\phi}{\dashrightarrow} & X_- \end{array}$$

and

$$f_+^* K_{X_+} = f_-^* K_{X_-}.$$

Then the quantum cohomology

$$\mathrm{QH}(X_+) = \mathrm{QH}(X_-)$$

under analytic continuations.

The crepant transformation conjecture

- Ruan (2006)
- Coates–Ruan (2013)
- Lee–Lin–Wang (2010, 2016, 2019,...)
- Gonzalez–Woodward (2012)
- Coates–Iritani–Jiang (2018)
- ...

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Coates–Iritani–Jiang (2018)

For toric complete intersections, the I-functions of X_+ and X_- are related by analytic continuation.

The discrepant transformation

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If $f_+^* K_{X_+} \neq f_-^* K_{X_-}$, then ϕ is discrepant.

- Iritani (2020)
- Acosta–Shoemaker (2018, 2020)

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Question

The relation in the discrepant case is more complicated and more difficult to obtain. Is there an easier approach to this question?

Turning into log crepant

We consider divisors $D_+ \subset X_+$ and $D_- \subset X_-$ such that

$$f_+^*(K_{X_+} + D_+) = f_-^*(K_{X_-} + D_-).$$

Then compare the relative quantum cohomology of the pairs (X_+, D_+) and (X_-, D_-).

Outline

Set-up

Relative Gromov–Witten theory is the enumerative theory of counting curves with tangency condition along a divisor (a codimension one subvariety).

- X: a smooth projective variety.
- D: a smooth divisor of X.
- For $d \in H_2(X, \mathbb{Q})$, we consider a partition $\vec{k} = (k_1, \ldots, k_m)$ of $\int_d [D]$. That is,
$$\sum_{i=1}^{m} k_i = \int_d [D], \quad k_i > 0$$
- $\overline{M}_{g,\vec{k},n,d}(X, D)$: the moduli space of $(m+n)$-pointed, genus g, degree $d \in H_2(X, \mathbb{Q})$, relative stable maps to (X, D) such that the relative conditions are given by the partition $\vec{k}$.

Evaluation Maps

There are two types of evaluation maps.

$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X,D) \to D, \quad \text{for } 1 \leq i \leq m;$$
$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X,D) \to X, \quad \text{for } m+1 \leq i \leq m+n.$$

The first m markings are relative markings with contact order k_i, the last n markings are interior markings.

Data

- $\delta_i \in H^*(D,\mathbb{Q})$, for $1 \leq i \leq m$.
- $\gamma_{m+i} \in H^*(X,\mathbb{Q})$, for $1 \leq i \leq n$.

Definition

Definition

Relative Gromov–Witten invariants of (X, D) *are defined as*

$$\left\langle \prod_{i=1}^{m} \tau(\delta_i) \middle| \prod_{i=1}^{n} \tau(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{(X,D)} := \tag{1}$$

$$\int_{[\overline{M}_{g,\vec{k},n,d}(X,D)]^{vir}} \prod_{i=1}^{m} \mathrm{ev}_i^*(\delta_i) \prod_{i=1}^{n} \mathrm{ev}_{m+i}^*(\gamma_{m+i}).$$

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Relative quantum cohomology

Relative quantum cohomology is defined by [Fan–Wu-Y, 2018].

Relative mirror symmetry

Theorem (Fan–Tseng–Y, 2018)

The I-function for the pair (X, D) is

$$I_{(X,D)}(Q,t,z) = \sum_{d \in \overline{\mathrm{NE}}(X)} J_{X,d}(t,z) Q^d \left(\prod_{0 < a \leq D \cdot d - 1} (D + az) \right) [\mathbf{1}]_{-D \cdot d}.$$

Relative and orbifold Gromov–Witten invariants

Root Stack

$X_{D,r}$: r-th root stack of X along the divisor D, where r is a positive integer. Geometrically, $X_{D,r}$ is smooth away from D and has generic stabilizer μ_r along D.

Remark

These relative invariants of (X, D) coincide with orbifold invariants of the root stack $X_{D,r}$ when $r \gg 1$.

$$\langle \cdots \rangle^{(X,D)} = \langle \cdots \rangle^{X_{D,r}}$$

[Abramovich–Cadman–Wise, 2017] and [Fan–Wu–Y, 2018]

Invariants of simple normal crossing pairs

- X: smooth projective variety;
- $D = D_1 + \cdots + D_n$: simple normal crossings divisor;
- $\vec{r} = (r_1, \ldots, r_n) \in (\mathbb{Z}_{>0})^n$;
- $X_{D,\vec{r}}$: multi-root stack of X along D.

Theorem (Tseng-Y, 2020)

For $r_1, ..., r_n$ sufficiently large,
genus 0: $\langle\rangle^{X_{D,\vec{r}}}$ is independent of $r_1, ..., r_n$.

A Gromov–Witten invariants of snc pairs via orbifold

$X_{D,\infty}$: infinite root stack.

Definition (Tseng-Y, 2020)

The genus zero formal Gromov-Witten invariants of $X_{D,\infty}$ are defined as

$$\langle \cdots \rangle^{X_{D,\infty}} := \langle \cdots \rangle^{X_{D,\vec{r}}}$$

for sufficiently large $\vec{r}$.

Remark

There is a quantum cohomology ring for $X_{D,\infty}$. We also call relative quantum cohomology of the snc pair (X, D).

Log vs Orbifold

Remark

The orbifold invariants of $X_{D,\infty}$ are different from log invariants in general. Log invariants are invariant under birational transformations, but orbifold invariants are not.

Log vs Orbifold

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The orbifold invariants of $X_{D,\infty}$ are different from log invariants in general. Log invariants are invariant under birational transformations, but orbifold invariants are not.

Question

Are there still some kinds of birational invariance (not on the level of single invariants) in orbifold Gromov–Witten theory of $X_{D,\infty}$?

Outline

Log crepant transformations

Given a birational transformation $\phi : X_+ \dashrightarrow X_-$. Suppose there is a $\tilde{X}$ with projective birational morphisms $f_\pm : \tilde{X} \to X_\pm$ such that the following diagram commute

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{f_+}\swarrow & & \searrow^{f_-} \\ X_+ & \overset{\phi}{\dashrightarrow} & X_- \end{array}$$

We consider divisors $D_+ \subset X_+$ and $D_- \subset X_-$ such that

$$f_+^*(K_{X_+} + D_+) = f_-^*(K_{X_-} + D_-).$$

Log crepant transformation

Question

What is the relation between Gromov–Witten theories of the pairs (X_+, D_+) and (X_-, D_-)?

Log crepant transformation

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What is the relation between Gromov–Witten theories of the pairs (X_+, D_+) and (X_-, D_-)?

In the spirit of the crepant transformation conjecture, there should be some form of birational invariance for Gromov–Witten theory of (X, D).

Example: root constructions

- X: smooth projective variety
- $D \subset X$: smooth divisor

Let $X_{D,r}$ be the r-th root stack of X along D. Then the natural map

$$X_{D,r} \to X$$

is a discrepant transformation.

Relation between absolute invariants

Tseng–Y, 2016: $GW(X_{D,r})$ is determined by $GW(X)$ and $GW(D)$ and the restriction map $H^*(X) \to H^*(D)$.

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Relation between relative invariants

Abramovich–Fantechi, 2016: $GW(X_{D,r}, D_r) = GW(X, D)$.

The relation between relative invariants are simpler than the relation between absolute invariants.

Example: blow-ups

Let X_+ be a blow-up of X_- along a complete intersection center $D_{-,1} \cap \cdots \cap D_{-,n}$.

- We can choose **snc divisors**

$$D_- = D_{-,1} + \cdots + D_{-,n}, \tag{2}$$

  and

$$D_+ = D_{+,1} + \cdots + D_{+,n} + E, \tag{3}$$

  where $D_{+,i}$ are strict transform of $D_{-,i}$ and E is the exceptional divisor.

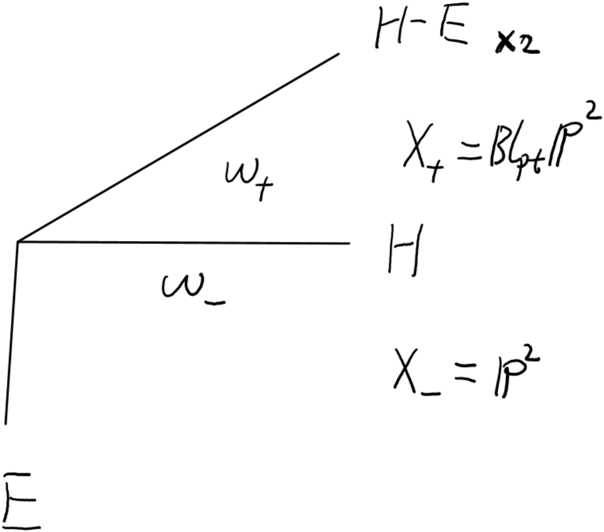
- We can also choose D_+ and D_- be **smooth divisors** that are linear equivalent to $D_{+,1} + \cdots + D_{+,n} + E$ and $D_{-,1} + \cdots + D_{-,n}$.

Toric wall-crossing

- Toric Deligne–Mumford stacks are defined as GIT quotients with a choice of stability condition.
- There is a wall and chamber structure in the secondary fan.
- A single wall-crossing gives a birational transformation between toric Deligne–Mumford stacks.
- For example, a toric blow-up along a complete intersection of toric divisors is given by a discrepant toric wall-crossing.

Toric wall-crossing

The secondary fan (GKZ fan)



Case 1) snc divisors

Theorem (Y, 2022)

Given

- *a birational transformation* $\phi : X_+ \dashrightarrow X_-$ *between toric Deligne–Mumford stacks (or toric complete intersections) is given by a single toric wall-crossing.*
- D_+ *and* D_- *are simple normal-crossings divisors such that toric divisors containing the loci of indeterminacy are the irreducible components.*

Then their relative I-functions are directly identified (without analytic continuation).

Case 2) smooth divisors

- We need to assume the divisors are nef. Then we apply one of the following results.
- 1) Via local-relative correspondence.
- 2) Via relative-orbifold correspondence and the hyperplane construction of root stacks.

We will explain 1).

Local-orbifold corrrespondence

Theorem (van Garrel–Graber–Ruddat, 2019: smooth divisors)

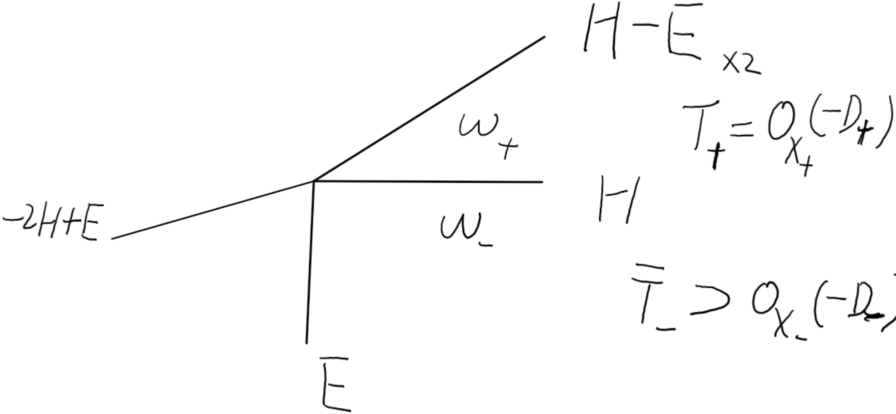
In genus zero with maximal contact, $GW(\mathcal{O}_X(-D))_0 = GW(X, D)_0$.

Theorem (Battistella–Nabijou–Tseng–Y, 2021: snc divisors)

In genus zero with maximal contact, $GW(\bigoplus_{i=1}^n \mathcal{O}_X(-D_i))_0 = GW(X_{D,\infty})_0$.

Relating local invariants

By the local-orbifold correspondence, we just need to compare local invariants of $\mathcal{O}_{X_+}(-D_+)$ and $\mathcal{O}_{X_-}(-D_-)$.



Relating local invariants

Then the result essentially follows from Mi-Shoemaker which states

Theorem (Mi-Shoemaker, 2020)

The narrow quantum D-modules of $\mathcal{O}_{X_+}(-D_+)$ *and* $\mathcal{O}_{X_-}(-D_-)$ *are related by analytic continuation and specialization of a variable.*

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Remark

Beyond maximal contacts, we need to use the relative-orbifold correspondence. We do not plan to talk about it here.

Outline

Connection to extremal transitions

Extremal transition

Two smooth projective varieties X and $\tilde{X}$ are said to be related by extremal transition if they are related by a birational contraction and a smoothing.

Reid's fantasy

Any pair of smooth Calabi–Yau threefolds may be connected via a sequence of flops and extremal transitions.

Cubic extremal transitions

Example

Cubic extremal transitions

- S : a cubic surface embedded in a smooth Calabi–Yau threefold $\tilde{Y}$ such that the rational curves in S generate an extremal ray in the sense of Mori.
- Consider a birational contraction $\tilde{Y} \to X_0$ where S is contracted to a point with local equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

- Deform the local equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = t(t \neq 0)$$

to obtain another Calabi–Yau threefold Y.

Cubic extremal transitions

Example

Cubic extremal transitions

- Y : quintic threefold in $\mathbb{P}^4$
- $\tilde{X} := \mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^4$.
- $\tilde{Y} \subset \tilde{X}$ is a hypersurface defined by the divisor $\tilde{D} = 5H - 3E$.

Connection to extremal transitions

- Li–Ruan, 2001: Gromov–Witten invariants of conifold transition via the degeneration formula.
- Lee–Lin–Wang, 2018: A+B theory in conifold transitions for Calabi-Yau threefolds.
- Rongxiao Mi, 2017: Gromov–Witten theory of cubic extremal transition via mirror symmetry.
- Mi–Shoemaker, 2020: Extremal transitions via quantum Serre duality.

Connection to extremal transitions

The set-up of Mi-Shoemaker

Let $\tilde{X} \to X$ be a toric blow-up. By Mi–Shoemaker, there are hypersurfaces $\tilde{D} \subset \tilde{X}$ and $D \subset X$ such that $\tilde{D}$ and D are related by extremal transitions. The pairs $(\tilde{X}, \tilde{D})$ and (X, D) are log-K-equivalent!

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Theorem (Mi-Shoemaker, 2020)

The ambient quantum D-modules of $\tilde{D}$ and D are related by analytic continuation and specialization of a variable.

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Theorem (Mi-Shoemaker, 2020)

The ambient quantum D-modules of $\tilde{D}$ and D are related by analytic continuation and specialization of a variable.

Proof via quantum Serre duality.

The ambient quantum D-modules of hypersurfaces D is equivalent to the narrow quantum D-module of $\mathcal{O}_X(-D)$. □

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Rank reduction in extremal transitions

- There is a rank discrepancy between the ambient quantum D-modules of $\tilde{Y}$ and Y: the rank of the ambient quantum D-module of $\tilde{Y}$ is 6 and the rank of the ambient quantum D-module of Y is 4.
- There are two extra solutions of the Picard-Fuchs equation coming from the analytic continuation of the ambient quantum D-module of $\tilde{Y}$.

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Theorem (Mi, 2017)

For cubic extremal transitions, the rank reduction is partially explained as the FJRW theory of the cubic singularity: The two extra solutions, after specialization, recover the regularized FJRW theory of the cubic singularity.

Rank reduction in extremal transitions

Question

Why FJRW theory? Why not Gromov–Witten theory directly?

Rank reduction in extremal transitions

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Recall:

Theorem (Mi, 2017)

After analytic continuation

$$I_{\tilde{Y}}(y_1, y_2 = 0) = I_Y(y_1)$$

Rank reduction in extremal transitions

Question

Why FJRW theory? Why not Gromov–Witten theory directly?

Recall:

Theorem (Mi, 2017)

After analytic continuation

$$I_{\tilde{Y}}(y_1, y_2 = 0) = I_Y(y_1)$$

Theorem (Y, 2022)

The rank reduction is partially explained as the local Gromov–Witten theory of the total space K_S of the canonical bundle of the cubic surface S:

$$\iota^* I_{\tilde{Y}}(y_1 = 0, y_2) = I_{K_S}, \quad \text{where } \iota : S \hookrightarrow \tilde{Y}.$$

Rank reduction in extremal transitions

In general, suppose toric hypersurfaces $\tilde{D} \subset \tilde{X}$ and $D \subset X$ are related by extremal transitions. let S be the subvariety of $\tilde{D}$ that is contracted under the transition.

Theorem (Y, 2022)

The rank reduction is partially explained as the local Gromov–Witten theory of the total space N_S of the normal bundle of S.

Rank reduction in extremal transitions

In general, suppose toric hypersurfaces $\tilde{D} \subset \tilde{X}$ and $D \subset X$ are related by extremal transitions. let S be the subvariety of $\tilde{D}$ that is contracted under the transition.

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Question

Why FJRW theory appears in [Mi, 2017]?

Outline

1. Gromov–Witten theory under birational transformations
   - Absolute Gromov–Witten theory
   - The crepant/discrepant transformation conjecture
2. Relative Gromov–Witten theory
   - Definition
   - Simple normal crossings pairs
3. Relative Gromov–Witten theory under birational transformations
   - Set-up
   - Toric wall-crossings
4. Connection to extremal transitions
5. Connection to FJRW theory

Connection to FJRW theory

- Gromov–Witten theory of Calabi–Yau hypersurface X in a weighted projective space $\mathbb{P}[w_1, \ldots, w_N]$.
- FJRW theory of a quasi-homogeneous polynomial W of degree $d = \sum_{i=1}^N w_i$ and a group $G = \langle J_W \rangle$, where

$$J_W := (\exp(2\pi i w_1/d), \ldots, \exp(2\pi i w_N/d)) \in (\mathbb{C}^\times)^N.$$

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LG/CY correspondence

Gromov–Witten theory of $X = \{W = 0\}$ matches with FJRW theory of (W, G) via analytic continuation.

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LG/CY correspondence

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- Chiodo–Ruan 2010.
- Chiodo–Iritani–Ruan 2014.
- Lee–Priddis–Shoemaker 2016.
- Clader–Ross 2018.
- Y. Zhao 2021.

LG/Fano correspondence

Beyond Calabi–Yau condition

P. Acosta (2014): LG/(Fano, general type) correspondence.

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Recall

The rank reduction in cubic extremal transition can be partially explained as either the FJRW theory of cubic singularity or the local/relative Gromov–Witten theory of the cubic surface. How are they related?

Mirror symmetry for Fano variety

$$\text{Fano } X \quad \overset{\text{Mirror}}{\rightleftarrows} \quad \text{Landau–Ginzburg model } (X^\vee, W)$$

For Fano/LG mirror symmetry, it is expected that the generic fiber of W: $W^{-1}(t)$ is mirror to the smooth anticanonical divisor of X. Therefore, it is more natural to consider mirror symmetry for a log Calabi–Yau pair (X, D).

Log Calabi–Yau mirror symmetry

For a smooth log Calabi–Yau pair (X, D)

$$\text{Log Calabi–Yau } (X, D) \quad \underset{}{\overset{\text{Mirror}}{\rightleftarrows}} \quad \text{LG model } (X^\vee, W)$$

$$\text{Noncompact Calabi–Yau } X \setminus D \quad \overset{\text{Mirror}}{\rightleftarrows} \quad X^\vee \text{ (without } W)$$

$$\text{Smooth anticanonical divisor } D \quad \overset{\text{Mirror}}{\rightleftarrows} \quad W^{-1}(t) \text{ for generic } t \in \mathbb{C}$$

Log Calabi–Yau pairs

Given a Fano hypersurface $X \subset \mathbb{P}[w_1, \ldots, w_N]$ with its smooth anticanonical divisor D. Is there a LG/log CY correspondence?

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The I-function for the relative Gromov–Witten theory of (X, D) can be analytically continued to the regularized I-function of the FJRW theory.

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Question

Is it a direct enumerative meaning of the regularized FJRW I-function?

The End

Thank you!