Stable maps to Looijenga pairs

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To the experts on polytopes, I present some new polytopial manipulations.

Overview

(Y, D) Looijenga pair (= log Calabi-Yau surface of maximal boundary):

- Y (smooth) projective surface.
- $|-K_Y| \ni D = D_1 + \cdots + D_l$, $l > 1$.
- D_j smooth nef.

There are 19 deformation-families of such, ∞-many if we allow for orbifold singularities at the $D_i \cap D_j, i \neq j$. We focus on $l = 2$.

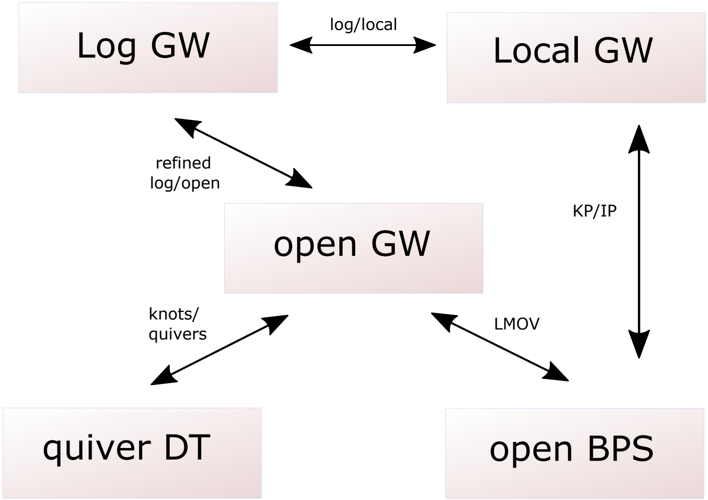
Example: $\mathbb{P}^2(1,4) = \mathbb{P}^2$ with $D_1 = H$ a line and $D_2 = 2H$ a conic.

Example: $\mathrm{dP}_3(0,2)$ = blow up of $\mathbb{P}^2$ in 3 points with $D_1 = H - E_3$ and $D_2 = 2H - E_1 - E_2$.

Theme

- 5 different enumerative theories built from (Y, D).
- They are all equivalent.
- They are all closed-form solvable.

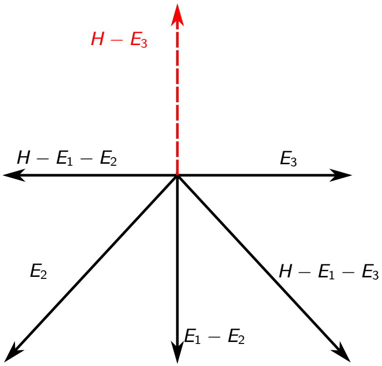
I describe some of these through examples.


Log GW
log/local
Local GW
refined
log/open
KP/IP
open GW
knots/
quivers
LMOV
quiver DT
open BPS

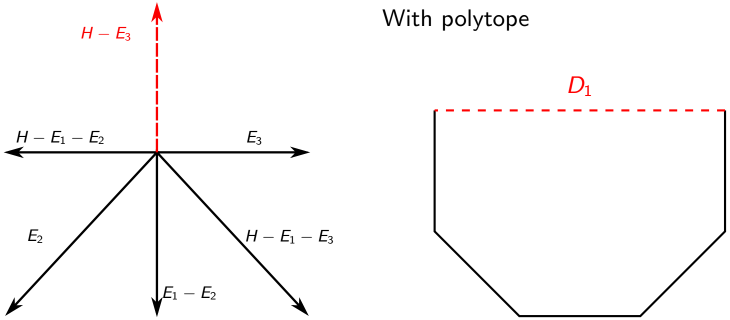
Geometric Mechanism log $\to$ open

By example of $\mathrm{dP}_3(0,2) = \left(\mathrm{Bl}_{3pts}\,\mathbb{P}^2\right)(D_1 = H - E_3, D_2 = 2H - E_1 - E_2)$.

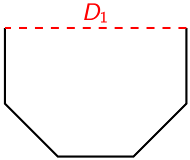
Fan of a deformation where D_1 is toric:



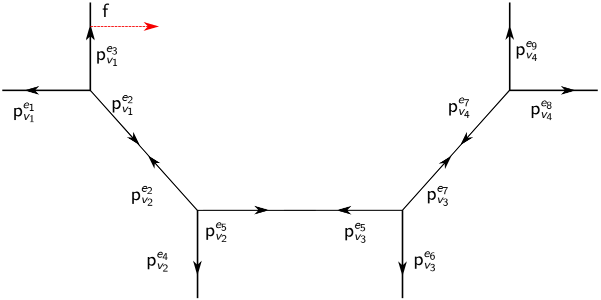
d



With polytope
$H - E_3$
$H - E_1 - E_2$
E_3
E_2
$H - E_1 - E_3$
$E_1 - E_2$
D_1

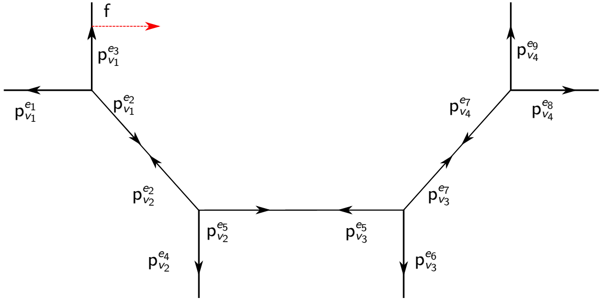


Replace edge D_1 by framing and balance the vertices:



Open geometry associated to Looijenga pair

$\rightsquigarrow$ toric CY graph = discriminant locus of the SYZ torus fibration of the toric Calabi–Yau 3-fold $\mathrm{dP}_3^{\mathrm{op}}(0,2) := \mathrm{Tot}\left(K_{\mathrm{dP}_3(0,2)\setminus D_1}\right)$.



The framing f determines an Aganagic–Vafa Lagrangian *A-brane*. The construction is eminently reversible.

Geometric manipulations

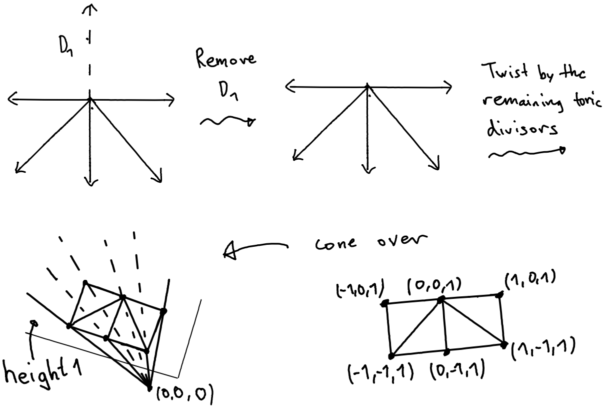
We designated D_1 to be open and D_2 to be local. Starting from the fan, we

1. removed the ray D_1 (and remembered where it was through the framing),
2. twisted by the remaining toric rays (whose sum is lin. equiv. to D_2).

$\rightsquigarrow$ toric CY3.

This construction works for 15 (out of 19) deformation families, and for ∞-many if we allow orbifold singularities.

Ignoring the framing, we can build the fan of the toric CY3 $\mathrm{dP}_3^{\mathrm{op}}(0,2)$ directly:



Associated Calabi–Yau fourfold

Variant: Declare both D_1 and D_2 to be local. Twist by the toric divisors summing to D_2 as before and twist by D_1 in the fourth dimension. $\rightsquigarrow$ toric CY4

$$\mathrm{dP}_3^{\mathrm{loc}}(0,2) := \mathrm{Tot}\left(\mathcal{O}_{\mathrm{dP}_3}(-D_1) \oplus \mathcal{O}_{\mathrm{dP}_3}(-D_2)\right).$$

In general, $Y^{\mathrm{loc}}(D_1, \ldots, D_l)$ need not be toric.

Remark

A priori, both D_1 and D_2 open another option. However: computational tools for open Gromov-Witten invariants (topological vertex, topological recursion) only available for toric CY3.

Summary so far

For each of the 19 $Y(D_1, \ldots, D_l)$ we can build $Y^{\mathrm{loc}}(D_1, \ldots, D_l)$, for 15 of them we can build $Y^{\mathrm{op}}(D_1, \ldots, D_l)$ and for the 10 with $l = 2$, we also have some associated quivers.

We come to the enumerative theories and first focus on $\mathbb{P}^2(1, 4) := \mathbb{P}^2(D_1, D_2)$ with $D_1 =$ line, $D_2 =$ conic.

$\mathbb{P}^2(1,4)$ with log CY boundary (line + conic)

The space of degree d rational curves in $\mathbb{P}^2$ is of dimension $3d-1$. One may formulate enumerative questions by asking a rational curve to

- pass through a point $\leftrightarrow$ codim 1,
- be maximally tangent to a line/conic $\leftrightarrow$ codim $d-1/2d-1$.

Let $R_d := \#\{$ degree d rational curves in $\mathbb{P}^2$ through 1 point and maximally tangent to both line and conic $\}$.

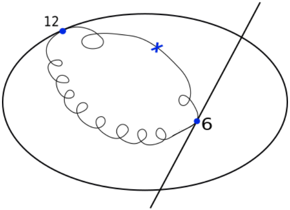
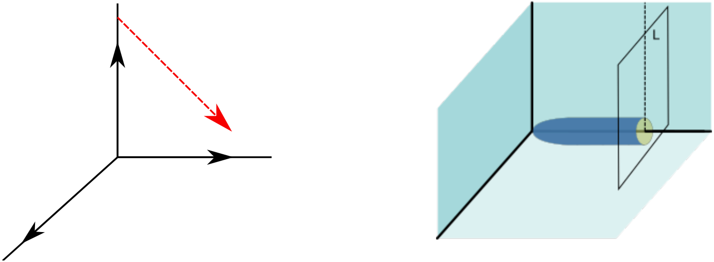


Figure: A degree 6 rational curve contributing to $R_6 = 924$.

Then $R_d = \binom{2d}{d}$.

The open geometry $(\mathbb{P}^2)^{\mathrm{op}}(1,4)$

The previous construction $\rightsquigarrow (\mathbb{P}^2)^{\mathrm{op}}(1,4) = (\mathbb{C}^3, L)$, where L is the A-brane determined by the framing.



Open GW invariants

Let O_d be the Katz-Liu count of disks in $\mathbb{C}^3$ with boundary on L, of winding number d and with framing 1 (defined by localization).

$$O_d = \frac{(-1)^d}{2d^2}\binom{2d}{d} = \frac{(-1)^d}{2d^2} R_d.$$

A local CY4 geometry

$$(\mathbb{P}^2)^{\mathrm{loc}}(1,4) := \mathrm{Tot}\left(\mathcal{O}(-1) \oplus \mathcal{O}(-2) \longrightarrow \mathbb{P}^2\right)$$

Local GW invariants

$$N_d := \#\{\text{ degree } d \text{ rational curves in } (\mathbb{P}^2)^{\mathrm{loc}}(1,4) \text{ through 1 point }\}.$$

Theorem (Klemm-Pandharipande '07)

$$N_d = \frac{(-1)^d}{2d^2}\binom{2d}{d} = \frac{(-1)^d}{2d^2} R_d$$

Definition/Conjecture (Klemm-Pandharipande '07)

$$N_d = \sum_{k|d} \frac{1}{k^2} KP_{d/k}$$

and $KP_d \in \mathbb{Z}$.

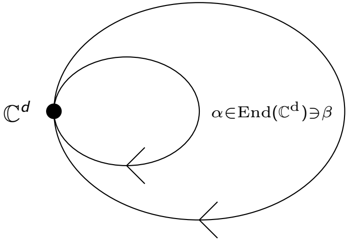
Quiver associated to $\mathbb{P}^2(1,4)$

The sequence $(-1)^d KP_d$ is OEIS sequence A131868:

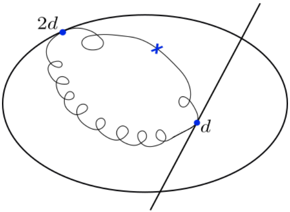
Konvalinka, Tewari, *Some natural extensions of the parking space*, arXiv:2003.04134.

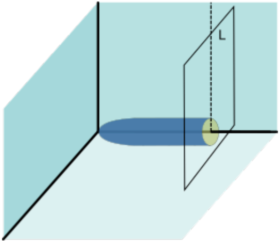
$$
\begin{aligned}
(-1)^d KP_d &= (-1)^d \sum_{k|d} \frac{\mu(k)}{k^2} \frac{(-1)^{d/k}}{2d^2/k^2} \binom{2d/k}{d/k} \\
&= \frac{(-1)^d}{d^2} \sum_{k|d} \mu(d/k) \frac{(-1)^k}{2} \binom{2k}{k} \\
&= \frac{(-1)^d}{d^2} \sum_{k|d} \mu(d/k) (-1)^k \binom{2k-1}{k-1} = \mathrm{DT}_d(Q),
\end{aligned}
$$

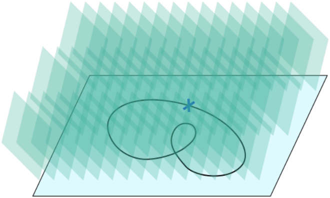
where Q is the 2-loop quiver (=oriented graph consisting of one vertex and two loops) and $\mathrm{DT}_d(Q)$ is its dth quiver DT invariant (Reineke '12).

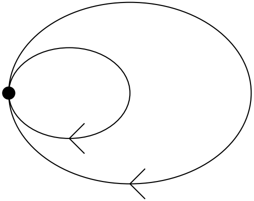


4 different geometries in different dimensions exhibit 5 sets of equivalent invariants:





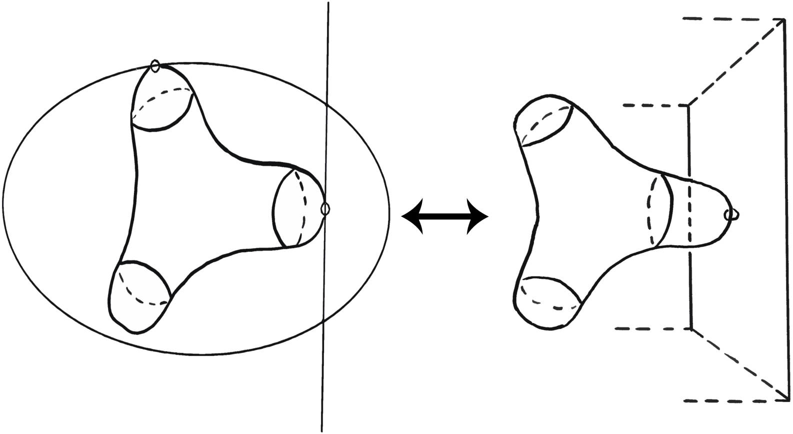




Log/local (N. Takahashi, Gathmann, vG-Graber-Ruddat, Bousseau-Brini-vG, Nabijou-Ranganathan, Tseng-You)

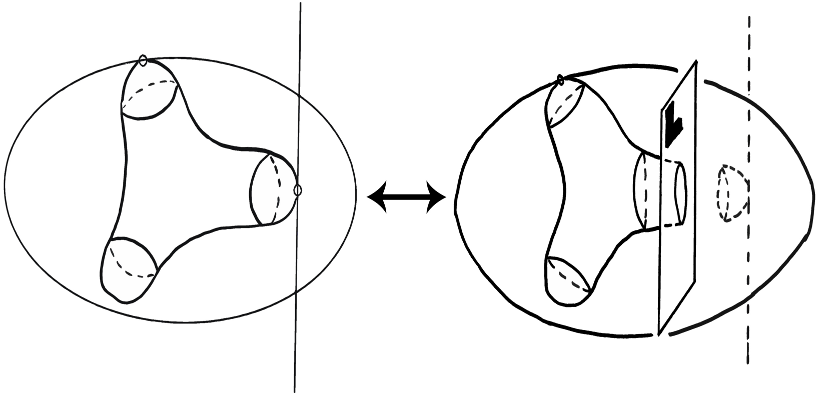
In large families of cases, equivalence of log & local invariants through:

Impose maximal tangency with	D_j	$\longleftrightarrow$	twist by $\mathcal{O}_Y(-D_j)$ and multiply by $(-1)^{d\cdot D_j-1}\, d\cdot D_j$.

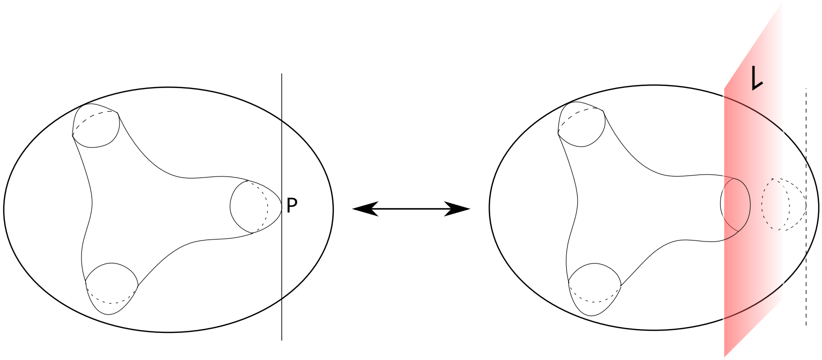


Log/open

Maximal tangency with D_j $d \cdot D_j$ choices of contact points	$\longleftrightarrow$	replace D_j by a Lagrangian L near D_j multiply by $(-1)^{d \cdot D_j - 1} d \cdot D_j$.



Curves in Looijenga surface $Y(D_1, D_2) \leftrightarrow$ disks in open CY3 $Y^{\text{op}}(D_1, D_2)$



Higher genus theorem for log/open

Under a positivity assumption (tameness), to each $Y(D_1, \dots, D_l)$ we associate an open geometry $Y^{\mathrm{op}}(D_1, \dots, D_l)$ and prove that

$$\mathsf{O}_{\iota^{-1}(d)}(Y^{\mathrm{op}}(D)) = \frac{1}{[1]_q^2} \prod_{i=1}^{l} \frac{(-1)^{d \cdot D_i + 1}[1]_q}{[d \cdot D_i]_q} \prod_{i=1}^{l-1} \frac{[d \cdot D_i]_q}{d \cdot D_i} \mathsf{R}_d^{\log}(Y(D)),$$

where $\mathsf{O}_{\iota^{-1}(d)}(Y^{\mathrm{op}}(D))$, resp. $\mathsf{R}_d^{\log}(Y(D))$, are the generating functions of open, resp. log, Gromov-Witten invariants,

and where $[n]_q := q^{\frac{n}{2}} - q^{-\frac{n}{2}}$ are the q-integers.

Scattering diagrams in the Gross-Siebert program

For today, a scattering diagram is a 2-dim complete fan Σ with focus-focus singularities $\times$ on the rays of Σ indicating blow ups $E(\times)$ on the smooth loci of the prime toric divisors corresponding to that rays.

It is a scattering diagram for $Y(D)$ if the associated variety with its boundary (= toric variety + blow ups $E(\times)$ at smooth loci corresponding to $\times$'s) can be transformed into $Y(D)$ by a sequence of toric blow ups and blow downs.

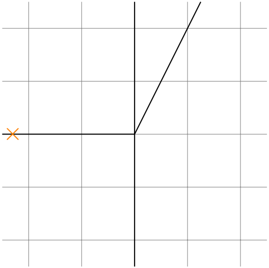


Figure: Scattering diagram for $\mathbb{P}^2(1, 4)$.

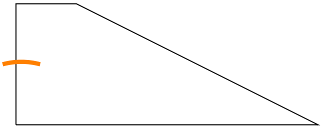
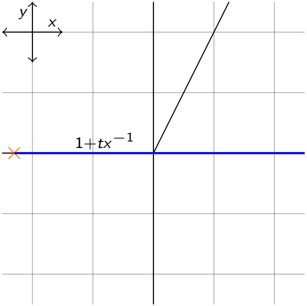


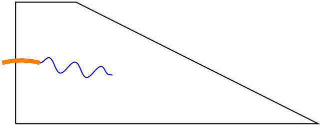
Figure: Polytope picture.

Building degree d curves in $\mathbb{P}^2$

A wall emanating out of $\times$ with wall-crossing function $1 + tx^{-1}$. $t = t^{[E(\times)]}$ keeps track of the intersection multiplicity with $E(\times)$ (on $\mathbb{P}^2$).



A (Maslov index 0) disk emanating out of the newly created singular fiber of the SYZ-fibration.



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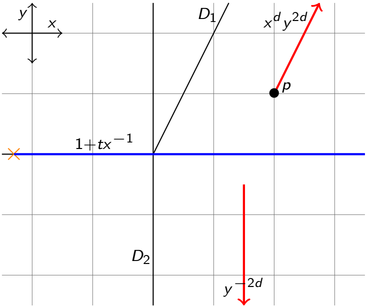

Two broken lines coming from the D_1, resp. D_2, directions of index d, resp. $2d$, captured by their attaching functions.

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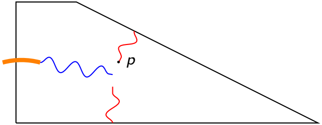


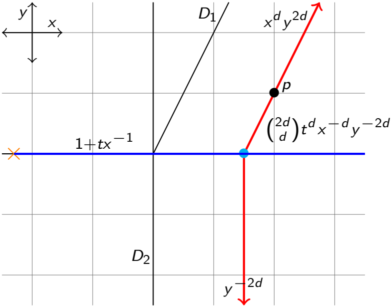
Two disks emanating out of the boundary of tangency d, resp. $2d$.

Point condition at p.

Building degree d curves in $\mathbb{P}^2$

A wall emanating out of $\times$ with wall-crossing function $1 + tx^{-1}$. $t = t^{[E(\times)]}$ keeps track of the intersection multiplicity with $E(\times)$ (on $\mathbb{P}^2$).

Two broken lines coming from the D_1, resp. D_2, directions of index d, resp. $2d$, captured by their attaching functions.



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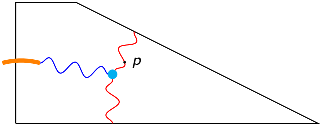


The bottom broken line is crossing the wall according to the Gross-Siebert wall-crossing automorphism, picking up a contribution from $\times$. A priori there are many choices, but only one that guarantees the line is straight at p. It is the only contribution producing the correct t^d corresponding to the intersection multiplicity d with $E(\times)$.

The algorithm produces a coefficient, in this case $\binom{2d}{d}$. It is the result of **a** multiplication of two broken lines with *asymptotic monomials* $z^{d\,[D_1]} = x^d y^{2d}$, resp. $z^{2d\,[D_2]} = y^{-2d}$.

More precisely, it is the identity component of the result of multiplying two broken lines.

Moreover, summing over broken lines gives the theta functions.

Theorem (Mandel '19, Keel-Yu '19, Gross-Siebert '19) Frobenius Conjecture (Gross-Hacking-Keelv1 '11)

Let $Y(D_1, D_2)$ be a log Calabi-Yau surface with scattering diagram Σ and let d be a curve class.

For any general p, denote by R_d the sum of the coefficients of all the possible results of multiplying broken lines with asymptotic monomials

$$z^{(d\cdot D_1)\,[D_1]} \text{ and } z^{(d\cdot D_2)\,[D_2]}.$$

Then

$$R_d = R_d(Y(D)).$$

And that's why

$$R_d(\mathbb{P}^2(1,4)) = \binom{2d}{d}.$$

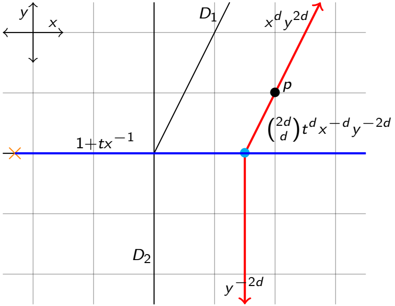
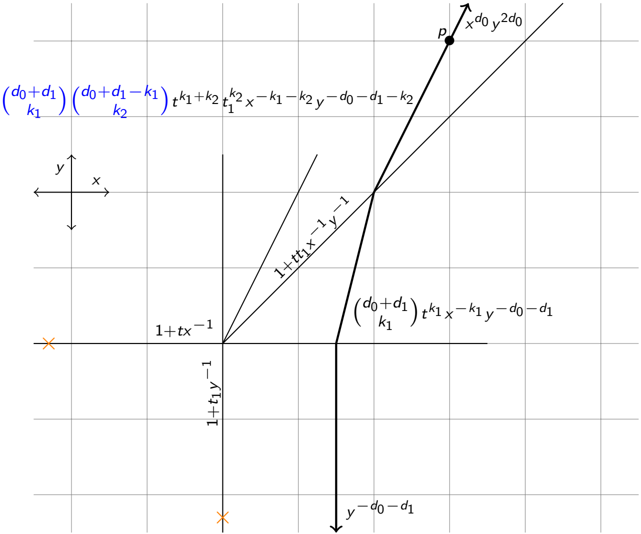


Figure: Scattering diagram for $\mathbb{P}^2(1,4)$ and two broken lines opposite at p.

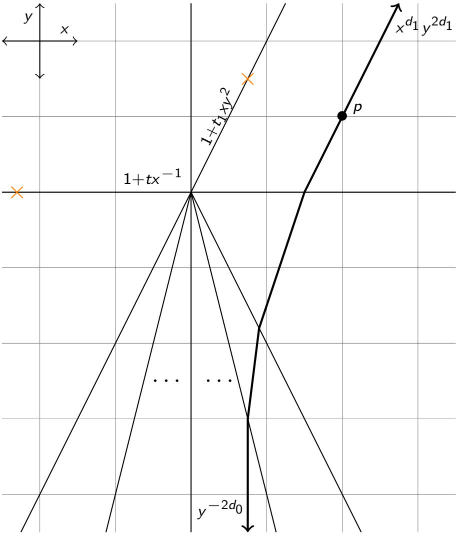
Scattering diagram for $\mathbb{F}_1(1,3)$

Two × interacting in a simple way.



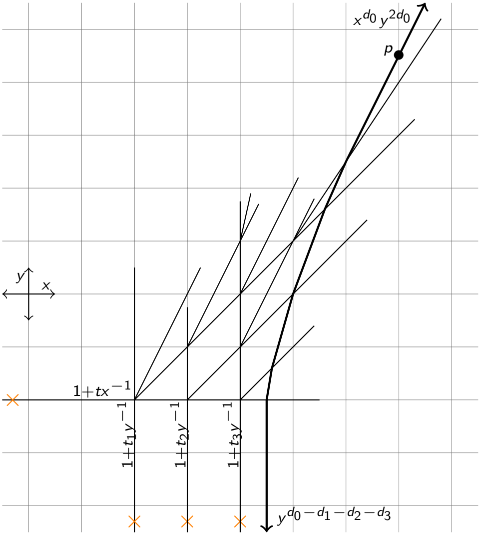
Scattering diagram for $\mathbb{F}_1(0,4)$

Two × creating infinite scattering.



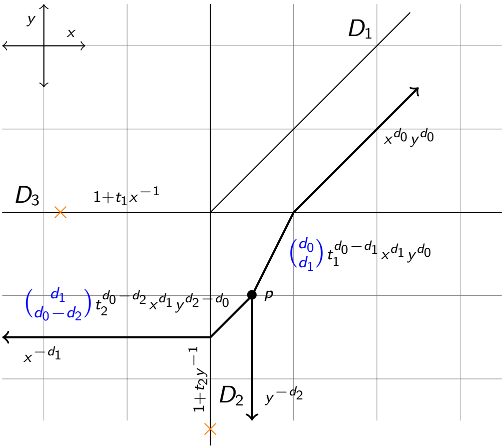
Scattering diagram for $\mathrm{dP}_3(1,1)$

Four × creating finite scattering.



2-pointed invariants I

$$R_d^{\psi}(\mathrm{dP}_2(1,0,0)) = \binom{d_0}{d_1}\binom{d_1}{d_0 - d_2}.$$



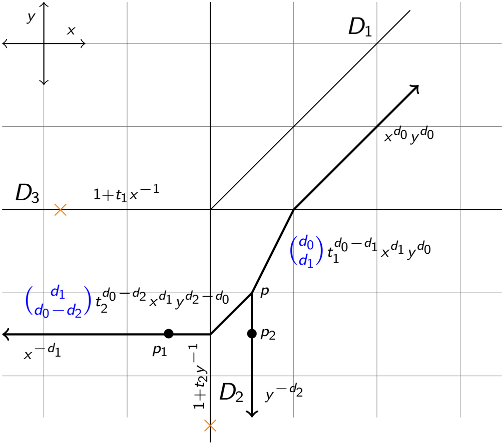
2-pointed invariants II

For the 2-point invariant, the tropical multiplicity at p is

$$\left| \det \begin{pmatrix} d_1 & 0 \\ d_0 & -d_2 \end{pmatrix} \right| = d_1 \, d_2$$

and hence

$$R_d(\mathrm{dP}_2(1,0,0) = d_1 \, d_2 \binom{d_0}{d_1} \binom{d_1}{d_0 - d_2}.$$



Refinement by example I

Recall/define

$$\mathbb{P}^2(1,4) = \quad \mathbb{P}^2 \text{ with boundary (line + conic)},$$

$$KP_d(\mathbb{P}^2(1,4)^{\mathrm{loc}}) = \sum_{k|d} \frac{\mu(k)}{k^2} N_{d/k}(\mathbb{P}^2(1,4)^{\mathrm{loc}}) = O_d^{\mathrm{BPS}}(\mathbb{C}^3, L).$$

Refine the log invariants by higher genus invariants

$$R_{g,d}(\mathbb{P}^2(1,4)) := \int_{[\overline{\mathrm{M}}^{\mathrm{log}}_{g,1}(Y(D),d)]^{\mathrm{vir}}} (-1)^g \lambda_g \,\mathrm{ev}^*([pt])$$

By Bousseau '18, the *quantized* scattering diagram computes $R_{g,d}(\mathbb{P}^2(1,4))$:

$$\mathrm{R}_d(\hbar) := \sum_{g \geq 0} R_{g,d}(\mathbb{P}^2(1,4))\, \hbar^{2g}.$$

Refinement by example II

After the change of variable $q = e^{i\hbar}$,

$$\mathsf{R}_d(q) = \begin{bmatrix} 2d \\ d \end{bmatrix}_q ,$$

which is the quantized binomial coefficient

$$\mathrm{Coeff}_{x^d}(1 + q^{-\frac{2d-1}{2}}x)(1 + q^{-\frac{2d-1}{2}+1}x)\dots(1 + q^{\frac{2d-1}{2}}x)\,.$$

Theorem (Higher genus log/open correspondence)

$$\mathsf{O}_{\iota^{-1}(d)}(Y^{\mathrm{op}}(D)) = \frac{1}{[1]_q^2} \prod_{i=1}^{l} \frac{(-1)^{d \cdot D_i + 1}[1]_q}{[d \cdot D_i]_q} \prod_{i=1}^{l-1} \frac{[d \cdot D_i]_q}{d \cdot D_i} \mathsf{R}_d^{\log}(Y(D))\,,$$

Refinement by example III

Multiple cover formula for open GW:

$$O_d(\mathbb{C}^3, L) = \sum_{k|d} \frac{1}{k^2} O^{\mathrm{BPS}}_{d/k}(\mathbb{C}^3, L).$$

Lifting to a refinement of $KP_d(\mathbb{P}^2(1,4))$

Following Ooguri-Vafa, there is a Laurent polynomial refinement $\Omega_d(q)$ of $O^{\mathrm{BPS}}_d(\mathbb{C}^3, L)$, i.e. such that

$$\Omega_d(q=1) = O^{\mathrm{BPS}}_d(\mathbb{C}^3, L) = KP_d(\mathbb{P}(1,4)^{\mathrm{loc}}).$$

Theorem (Higher genus open BPS integrality)

$$\Omega_d(q^{-1}) = \Omega_d(q) \in q^{-\binom{d-1}{2}}\, \mathbb{Z}[q].$$

E.g.

$$\Omega_1(q) = -1, \quad \Omega_2(q) = 1, \quad \Omega_3(q) = -\left(1 + \left(q^{1/2} - q^{-1/2}\right)^2\right),$$

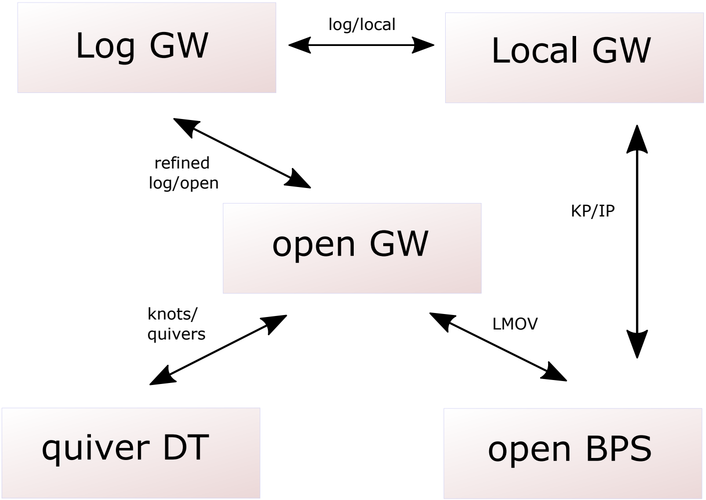
$$\Omega_4(q) = 2 + 6\left(q^{1/2} - q^{-1/2}\right)^2 + 5\left(q^{1/2} - q^{-1/2}\right)^4 + \left(q^{1/2} - q^{-1/2}\right)^6, \ldots$$

Summary of today

Theorem

For each tame $Y(D)$ and its associated toric CY3 (X, L),

- The higher genus log/open correspondence holds.
- The higher genus open BPS invariants are Laurent polynomials with integer coefficients
- and provide a refinement for the $KP_d(Y(D)^{\mathrm{loc}}) \leftarrow$ of interest to the enemerative geometry of CY4.


Log GW
log/local
Local GW
refined
log/open
KP/IP
open GW
knots/
quivers
LMOV
quiver DT
open BPS