# MINKOWSKI POLYNOMIALS AND MUTATIONS

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Abstract. Given a Laurent polynomial $f$, one can form the period of $f$: this is a function of one complex variable that plays an important role in mirror symmetry for Fano manifolds. Mutations are a particular class of birational transformations acting on Laurent polynomials in two variables; they preserve the period and are closely connected with cluster algebras. We propose a higher-dimensional analog of mutation acting on Laurent polynomials $f$ in $n$ variables. In particular we give a combinatorial description of mutation acting on the Newton polytope $P$ of $f$, and use this to establish many basic facts about mutations. Mutations can be understood combinatorially in terms of Minkowski rearrangements of slices of $P$, or in terms of piecewise-linear transformations acting on the dual polytope $P^*$ (much like cluster transformations). Mutations map Fano polytopes to Fano polytopes, preserve the Ehrhart series of the dual polytope, and preserve the period of $f$. Finally we use our results to show that Minkowski polynomials, which are a family of Laurent polynomials that give mirror partners to many three-dimensional Fano manifolds, are connected by a sequence of mutations if and only if they have the same period.

## 1. Introduction

Given a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, one can form the *period* of $f$:

$$\pi_f(t) = \left(\frac{1}{2\pi \mathtt{i}}\right)^n \int_{|x_1|=\cdots=|x_n|=1} \frac{1}{1 - t f(x_1, \ldots, x_n)} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \qquad t \in \mathbb{C},\ |t| \ll \infty \tag{1.1}$$

The period of $f$ gives a solution to a GKZ hypergeometric differential system associated to the Newton polytope of $f$ (see [9, §3]). Periods of Laurent polynomials and the associated differential systems are interesting from the point of view of mirror symmetry, because certain Laurent polynomials arise as mirror partners to $n$-dimensional Fano manifolds [4–6, 8, 11, 17]. In its most basic form (which will suffice for what follows) the statement that a Laurent polynomial $f$ is a mirror partner for a Fano manifold $X$ means that the Taylor expansion of the period of $f$:

$$\pi_f(t) = \sum_{k \geq 0} c_k t^k$$

coincides with a certain generating function for Gromov–Witten invariants of $X$ called the *quantum period* of $X$ [9, §4]. The Taylor coefficient $c_k$ here is the coefficient of the unit monomial in the Laurent polynomial $f^k$. We refer to the sequence $(c_k)_{k \geq 0}$ as the *period sequence* for $f$.

We expect that if the Laurent polynomial $f$ is a mirror partner to a Fano manifold $X$, then there is a geometric relationship between $f$ and $X$ as follows (cf. [24]). Let $N_f$ be the lattice generated by the exponents of monomials of $f$. Consider the Newton polytope $\mathrm{Newt}(f)$ of $f$, and assume that the origin lies in its strict interior. Let $X_f$ denote the toric variety defined by the spanning fan of $\mathrm{Newt}(f)$ in $N_f \otimes \mathbb{Q}$; in general $X_f$ will be singular. We expect that $X_f$ admits a smoothing with general fiber $X$. Note that our assumption that $\mathrm{Newt}(f)$ contains the origin is not restrictive: if the origin is outside $\mathrm{Newt}(f)$ then the period of $f$ is constant, and hence cannot be the quantum period of a Fano manifold; if the origin is contained in a proper face of $\mathrm{Newt}(f)$ then we can reduce to a lower-dimensional situation. Note also that the lattice $N_f$ may be a proper sublattice of $\mathbb{Z}^n$: see Example 3.16 and [2]. The picture described here implies that one might expect many Laurent polynomial mirrors for a given Fano manifold, as a smooth Fano manifold can degenerate to many different singular toric varieties.

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The motivating case for this paper is that of three-dimensional *Minkowski polynomials*. These are a family of Laurent polynomials in three variables, defined in §4 below, which provide mirror partners to many of the three-dimensional Fano manifolds. The correspondence between three-dimensional Minkowski polynomials and Fano manifolds is not one-to-one, in part because many Minkowski polynomials give rise to the same period sequence. There are several thousand Minkowski polynomials $f\colon (\mathbb{C}^\times)^3 \to \mathbb{C}$, up to change of co-ordinates on $(\mathbb{C}^\times)^3$, but between them these give only 165 distinct period sequences[1]. In what follows we give a conceptual explanation for this phenomenon. We define birational transformations, called *mutations*, that preserve periods and show that any two Minkowski polynomials with the same period are related by a sequence of mutations. Our birational transformations are higher-dimensional generalisations of the mutations defined by Galkin–Usnich [14]. Combining our results in §5 with a theorem of Ilten [18] shows that whenever $f$ and $g$ are Minkowski polynomials with the same period sequence, the toric varieties $X_f$ and $X_g$ occur as fibers of a flat family over a curve. This is consistent with our conjectural picture, which implies that whenever Laurent polynomials $f$ and $g$ are mirror partners for the same Fano manifold $X$, the toric varieties $X_f$ and $X_g$ are deformation equivalent.

The paper is organised as follows. We define mutations algebraically in §2 and combinatorially in §3. Algebraic mutations operate on Laurent polynomials, whereas combinatorial mutations operate on polytopes. An algebraic mutation of a Laurent polynomial $f$ induces a combinatorial mutation of its Newton polytope $\mathrm{Newt}(f)$; the converse statement is discussed in Remark 3.5 on page 6. We establish various basic properties of combinatorial mutations: they send Fano polytopes to Fano polytopes (Proposition 3.11); there are, up to isomorphism, only finitely many mutations of a given polytope (Proposition 3.13); and mutation-equivalent polytopes have the same Hilbert series (Proposition 3.15). We define Minkowski polynomials in §4, and in §5 show by means of a computer search that all Minkowski polynomials with the same period sequence are connected by a sequence of mutations. Period sequences for Minkowski polynomials are listed in Appendix A, and mutations connecting the Minkowski polynomials with the same period sequence are listed in Appendix B.

## 2. Mutations

In this section we define mutations. These are a class of birational transformations $\varphi\colon (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n$ with the property that if two Laurent polynomials $f$ and $g$ are related by a mutation $\varphi$, so that $g = \varphi^\star f$, then the periods of $f$ and $g$ coincide. We begin with two examples.

**Example 2.1.** Consider a Laurent polynomial:

$$f = A(x,y)z^{-1} + B(x,y) + C(x,y)z$$

where $A, B, C$ are Laurent polynomials in $x$ and $y$. The pullback of $f$ along the birational transformation $(\mathbb{C}^\times)^3 \dashrightarrow (\mathbb{C}^\times)^3$ given by:

$$(x,y,z) \mapsto \big(x, y, A(x,y)z\big) \tag{2.1}$$

is:

$$g = z^{-1} + B(x,y) + A(x,y)C(x,y)z$$

We say that the Laurent polynomials $f$ and $g$ are related by the mutation (2.1).

**Example 2.2.** Consider a Laurent polynomial:

$$f = \sum_{i=k}^{l} C_i(x,y)z^i$$

with $k < 0$ and $l > 0$ where each $C_i$, $i \in \{k, k+1, \ldots, l\}$, is a Laurent polynomial in $x$ and $y$. Let $A$ be a Laurent polynomial in $x$ and $y$ such that $C_i$ is divisible by $A^{-i}$ for $i \in \{k, k+1, \ldots, -1\}$. The pullback of $f$ along the birational transformation $(\mathbb{C}^\times)^3 \dashrightarrow (\mathbb{C}^\times)^3$ given by:

$$(x,y,z) \mapsto \big(x, y, A(x,y)z\big) \tag{2.2}$$

[1] 98 of these periods are the quantum periods for the three-dimensional Fano manifolds with very ample anticanonical bundle. The remaining 67 periods are not the quantum period for any three-dimensional Fano manifold, although they may correspond to Fano orbifolds. See [9, §7].

is:

$$g = \sum_{i=k}^{l} A(x,y)^i C_i(x,y) z^i$$

We say that the Laurent polynomials $f$ and $g$ are related by the mutation (2.2).

**Remark 2.3.** Note that the pullback of a Laurent polynomial along a birational transformation of the form (2.1) or (2.2) will not, in general, be a Laurent polynomial: the condition $A^{-i} \big| C_i$, $i \in \{k, k+1, \ldots, -1\}$, is essential.

**Definition 2.4.** A $\mathrm{GL}_3(\mathbb{Z})$-*equivalence* is an isomorphism $(\mathbb{C}^\times)^3 \to (\mathbb{C}^\times)^3$ of the form:

$$(x,y,z) \mapsto (x^a y^b z^c, x^d y^e z^f, x^g y^h z^i) \qquad \text{where } M := \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z})$$

For brevity, we write this isomorphism as $\mathsf{x} \mapsto \mathsf{x}^M$.

**Definition 2.5.** A *mutation* is a birational transformation $(\mathbb{C}^\times)^3 \dashrightarrow (\mathbb{C}^\times)^3$ given by a composition of:

(i) a $\mathrm{GL}_3(\mathbb{Z})$-equivalence;
(ii) a birational transformation of the form (2.2); and
(iii) another $\mathrm{GL}_3(\mathbb{Z})$-equivalence.

If $f, g$ are Laurent polynomials and $\varphi$ is a mutation such that $\varphi^\star f = g$ then we say that $f$ and $g$ are *related by the mutation* $\varphi$.

**Remark 2.6.** One can also define mutations of Laurent polynomials in $n$ variables, using the obvious generalisations of Example 2.2, Definition 2.4, and Definition 2.5.

**Example 2.7.** Consider the Laurent polynomial:

$$f = xyz + x + y + z + \frac{1}{x} + \frac{1}{xyz}$$

The Newton polytope $P$ of $f$ has two pairs of parallel facets, and we place one pair of them at heights 1 and $-1$ by applying the $\mathrm{GL}_3(\mathbb{Z})$-equivalence $\mathsf{x} \mapsto \mathsf{x}^M$ with:

$$M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

This transforms $f$ into the Laurent polynomial:

$$\frac{1}{z}\left(y + \frac{y}{x} + \frac{1}{x}\right) + z\left(1 + x + \frac{x}{y}\right)$$

We now apply the birational transformation (2.1) with $A(x,y) = y + \frac{y}{x} + \frac{1}{x}$, followed by the $\mathrm{GL}_3(\mathbb{Z})$-equivalence $\mathsf{x} \mapsto \mathsf{x}^{M^{-1}}$, obtaining:

$$g = xy^2z^2 + xyz + 2yz^2 + 2z + \frac{1}{z} + \frac{1}{y} + \frac{z^2}{x} + \frac{z}{xy}$$

This shows that the Laurent polynomials $f$ and $g$ are related by the mutation $\varphi$, where:

$$\varphi(x,y,z) = \left(\frac{xy^2z + yz + 1}{y}, \frac{xy^2}{xy^2z + yz + 1}, \frac{z(xy^2z + yz + 1)}{xy}\right)$$

**Lemma 2.8.** *If the Laurent polynomials $f$ and $g$ are related by a mutation $\varphi$, then the periods of $f$ and $g$ coincide.*

*Proof.* Let $\varphi\colon (\mathbb{C}^\times)^3 \dashrightarrow (\mathbb{C}^\times)^3$ be the birational transformation:

$$(x,y,z) \mapsto (x, y, A(x,y)z)$$

from (2.2). Since $\mathrm{GL}_3(\mathbb{Z})$-equivalence preserves periods, it suffices to show that if $g = \varphi^\star f$ then the periods of $f$ and $g$ coincide. Let:

$$Z = \left\{(x,y,z) \in (\mathbb{C}^\times)^3 : A(x,y) = 0\right\}$$

and let $U = (\mathbb{C}^\times)^3 \setminus Z$. Note that the restriction $\varphi|_U \colon U \to (\mathbb{C}^\times)^3$ is a morphism, and that:

$$(\varphi|_U)^\star\Big(\frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\Big) = \frac{dx}{x}\frac{dy}{y}\frac{dz}{z} \tag{2.3}$$

Let:

$$C_{a,b,c} = \big\{(x,y,z) \in (\mathbb{C}^\times)^3 : |x| = a,\ |y| = b,\ |z| = c\big\}$$

so that the period of $f$ is:

$$\Big(\frac{1}{2\pi\mathtt{i}}\Big)^n \int_{C_{1,1,1}} \frac{1}{1 - tf(x,y,z)} \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}$$

The amoeba of $Z$ is a proper subset of $\mathbb{R}^3$ [15, chapter 6, Corollary 1.8], so there exists $(a,b,c)$ such that $C_{a,b,c} \subset U$. The cycles $C_{a,b,c}$ and $C_{a',b',c'}$ are homologous in $(\mathbb{C}^\times)^3$ for any non-zero $a$, $b$, $c$, $a'$, $b'$, $c'$. Thus:

$$\begin{aligned}
\pi_g(t) &= \Big(\frac{1}{2\pi\mathtt{i}}\Big)^n \int_{C_{1,1,1}} \frac{1}{1 - tg(x,y,z)} \frac{dx}{x}\frac{dy}{y}\frac{dz}{z} \\
&= \Big(\frac{1}{2\pi\mathtt{i}}\Big)^n \int_{C_{a,b,c}} \frac{1}{1 - tg(x,y,z)} \frac{dx}{x}\frac{dy}{y}\frac{dz}{z} \\
&= \Big(\frac{1}{2\pi\mathtt{i}}\Big)^n \int_{\varphi(C_{a,b,c})} \frac{1}{1 - tf(x,y,z)} \frac{dx}{x}\frac{dy}{y}\frac{dz}{z} && \text{by the change of variable formula.}
\end{aligned}$$

Now the homology class $\big[\varphi(C_{a,b,c})\big] \in H_3\big((\mathbb{C}^\times)^3, \mathbb{Z}\big)$ is equal to $k\big[C_{1,1,1}\big]$ for some integer $k$, since $H_3\big((\mathbb{C}^\times)^3, \mathbb{Z}\big)$ is freely generated by $\big[C_{1,1,1}\big]$, and from (2.3) and the change of variable formula we see that $k = 1$. It follows that:

$$\pi_g(t) = \Big(\frac{1}{2\pi\mathtt{i}}\Big)^n \int_{C_{1,1,1}} \frac{1}{1 - tf(x,y,z)} \frac{dx}{x}\frac{dy}{y}\frac{dz}{z} = \pi_f(t)$$

□

**Example 2.9.** Consider the two Laurent polynomials:

$$\begin{aligned}
f_1 &= x + \frac{2x}{y} + \frac{x}{y^2} + y + z + \frac{1}{z} + \frac{z}{y} + \frac{4}{y} + \frac{1}{yz} + \frac{yz}{x} + \frac{2y}{x} + \frac{y}{xz} + \frac{2z}{x} + \frac{5}{x} + \frac{2}{xz} + \frac{yz}{x^2} + \frac{2y}{x^2} + \frac{y}{x^2z} \\
f_2 &= x + \frac{2x}{y} + \frac{x}{y^2} + y + z + \frac{1}{z} + \frac{z}{y} + \frac{3}{y} + \frac{1}{yz} + \frac{yz}{x} + \frac{3y}{x} + \frac{y}{xz} + \frac{2z}{x} + \frac{4}{x} + \frac{2}{xz} + \frac{yz}{x^2} + \frac{2y}{x^2} + \frac{y}{x^2z}
\end{aligned}$$

Since $f_1$ and $f_2$ have the same period sequence:

$$\pi_{f_1}(t) = \pi_{f_2}(t) = 1 + 28t^2 + 216t^3 + 3516t^4 + 49680t^5 + \cdots$$

and since $\mathrm{Newt}(f_1) = \mathrm{Newt}(f_2)$, it is tempting to assume that there is some $\mathrm{GL}_3(\mathbb{Z})$-equivalence that preserves the Newton polytope and sends $f_1$ to $f_2$. This is not the case. However, there does exist a birational map sending $f_1$ to $f_2$. This is a composition of mutations:

$$f_1 \xrightarrow{\varphi} f \xrightarrow{\psi} f_2$$

factoring through

$$f = xz^2 + 2xz + x + yz + y + 3z + \frac{2}{z} + \frac{z}{y} + \frac{1}{y} + \frac{y}{x} + \frac{y}{xz} + \frac{2}{x} + \frac{2}{xz} + \frac{1}{xz^2} + \frac{1}{xy} + \frac{1}{xyz}.$$

The maps $\varphi$, $\psi$ and their inverses are given by:

$$\begin{aligned}
\varphi &\colon (x,y,z) \mapsto \left(\frac{z(xyz + (y+1)^2)}{y}, \frac{xyz + (y+1)^2}{xy}, y\right), \\
\psi &\colon (x,y,z) \mapsto \left(\frac{(x+yz+y)(xz+yz+y)}{y^2z(x+y)}, \frac{1}{z}, \frac{y}{x}\right), \\
\varphi^{-1} &\colon (x,y,z) \mapsto \left(\frac{xz + y(z+1)^2}{y^2z}, z, \frac{xyz}{xz + y(z+1)^2}\right), \\
\psi^{-1} &\colon (x,y,z) \mapsto \left(\frac{(yz+z+1)(yz+y+z)}{xyz^2(z+1)}, \frac{(yz+z+1)(yz+y+z)}{xyz(z+1)}, \frac{1}{y}\right).
\end{aligned}$$

Set $P := \mathrm{Newt}(f_1) = \mathrm{Newt}(f_2)$ and $Q := \mathrm{Newt}(f)$. The polytopes $P$ and $Q$ are reflexive, but since $\mathrm{Vol}(P) = 32$ and $\mathrm{Vol}(Q) = 28$, $P$ and $Q$ are not isomorphic. However, as predicted by Proposition 3.15 below, the Ehrhart series $\mathrm{Ehr}_{P^*}(t)$ and $\mathrm{Ehr}_{Q^*}(t)$ are equal: in other words, the Hilbert series $\mathrm{Hilb}_{X_{f_i}}(-K_{X_{f_i}})$ and $\mathrm{Hilb}_{X_f}(-K_{X_f})$ agree.

## 3. Combinatorial mutations

The mutations of a Laurent polynomial $f$ defined in §2 induce transformations of the Newton polytope of $f$. In this section we give a combinatorial and coordinate-free definition of these transformations, which we call *combinatorial mutations*, in terms of the Newton polytope alone. We then establish some basic properties of combinatorial mutations. Let us begin by fixing our notation. Let $N$ be an $n$-dimensional lattice and let $P \subset N_{\mathbb{Q}} := N \otimes \mathbb{Q}$ be a convex lattice polytope such that:

(i) $P$ is of maximum dimension, $\dim P = n$;
(ii) the origin lies in the strict interior of $P$, $\mathbf{0} \in P^\circ$;
(iii) the vertices $\mathcal{V}(P) \subset N$ of $P$ are primitive lattice points.

We call such a polytope *Fano*.

Given any lattice polytope $P \subset N_{\mathbb{Q}}$, the *dual* polyhedron $P^* \subset M_{\mathbb{Q}}$, where $M := \mathrm{Hom}(N, \mathbb{Z})$, is defined by:

$$P^* := \{u \in M_{\mathbb{Q}} \mid u(v) \geq -1 \text{ for all } v \in P\}.$$

Condition ii ensures that, when $P$ is Fano, $P^*$ is a polytope. When $P^*$ is a *lattice* polytope, we say that $P$ is *reflexive*. Low-dimensional reflexive polytopes have been classified [20, 21]: up to the action of $\mathrm{GL}_n(\mathbb{Z})$ there are 16 reflexive polytopes in dimension two; $4,319$ in dimension three; and $473,800,776$ in dimension four. A Fano polytope $P \subset N_{\mathbb{Q}}$ is called *canonical* if $P^\circ \cap N = \{\mathbf{0}\}$. In two dimensions, the reflexive polytopes and canonical polytopes coincide. In general every reflexive polytope is canonical, although the converse is not true: there are $674,688$ canonical polytopes in dimension three [19].

**Definition 3.1.** Let $w \in M$ be a primitive lattice vector, and let $P \subset N_{\mathbb{Q}}$ be a lattice polytope. Set:

$$h_{\min} := \min\{w(v) \mid v \in P\} \qquad\qquad h_{\max} := \max\{w(v) \mid v \in P\}$$

We define the *width* of $P$ with respect to $w$ to be the positive integer:

$$\mathrm{width}_w(P) := h_{\max} - h_{\min}$$

If $\mathrm{width}_w(P) = l$ then we refer to $w$ as a *width $l$ vector* for $P$. We say that a lattice point $v \in N$ (resp. a subset $F \subset N_{\mathbb{Q}}$) is at *height $m$* with respect to $w$ if $w(v) = m$ (resp. if $w(F) = \{m\}$).

If $\mathbf{0} \in P^\circ$ then $h_{\min} < 0$ and $h_{\max} > 0$, hence $w$ must have width at least two. If $P$ is a reflexive polytope then for any $w \in \mathcal{V}(P^*)$ there exists a facet $F \in \mathcal{F}(P)$ at height $-1$; this is a well-known characterisation of reflexive polytopes [3]. For each height $h \in \mathbb{Z}$, $w$ defines a hyperplane $H_{w,h} := \{x \in N_{\mathbb{Q}} \mid w(x) = h\}$. Let

$$w_h(P) := \mathrm{conv}(H_{w,h} \cap P \cap N).$$

By definition, $w_{h_{\min}}(P) = H_{w,h_{\min}} \cap P$ and $w_{h_{\max}}(P) = H_{w,h_{\max}} \cap P$ are faces of $P$, hence $\mathcal{V}(w_{h_{\min}}(P)) \subseteq \mathcal{V}(P)$ and $\mathcal{V}(w_{h_{\max}}(P)) \subseteq \mathcal{V}(P)$. Furthermore, the face $w_{h_{\min}}(P)$ is a facet of $P$ if and only if $w = \overline{u}$ for some vertex $u \in \mathcal{V}(P^*)$, where $\overline{u}$ denotes the unique primitive lattice point on the ray from $\mathbf{0}$ through $u$. Similarly, $w_{h_{\max}}(P)$ is a facet if and only if $-w = \overline{u}$ for some $u \in \mathcal{V}(P^*)$.

**Definition 3.2.** Recall that the Minkowski sum of two polytopes $Q, R \subset N_{\mathbb{Q}}$, is

$$Q + R := \{q + r \mid q \in Q, r \in R\}.$$

Henceforth we adopt the convention that $Q + \varnothing := \varnothing$ for any polytope $Q$.

**Definition 3.3.** Suppose that there exists a lattice polytope $F \subset N_{\mathbb{Q}}$ with $w(F) = 0$, such that for every height $h_{\min} \leq h < 0$ there exists a possibly-empty lattice polytope $G_h \subset N_{\mathbb{Q}}$ satisfying $H_{w,h} \cap \mathcal{V}(P) \subseteq G_h + (-h)F \subseteq w_h(P)$. We call such an $F$ a *factor* for $P$ with respect to $w$. We define the *combinatorial mutation* given by width vector $w$, factor $F$, and polytopes $\{G_h\}$ to be the convex lattice polytope

$$\mathrm{mut}_w(P, F; \{G_h\}) := \mathrm{conv}\left( \bigcup_{h=h_{\min}}^{-1} G_h \cup \bigcup_{h=0}^{h_{\max}} (w_h(P) + hF) \right) \subset N_{\mathbb{Q}}.$$

Notice that one need only consider factors up to translation, since for any $v \in N$ such that $w(v) = 0$ we have $\mathrm{mut}_w(P, v + F; \{G_h + hv\}) \cong \mathrm{mut}_w(P, F; \{G_h\})$. In particular, if $w_{h_{\min}}(P)$ is zero-dimensional then combinatorial mutations with width vector $w$ leave $P$ unchanged.

**Example 3.4.** Consider the situation of Example 2.2, so that $f$ is the Laurent polynomial:

$$f = \sum_{i=k}^{l} C_i(x,y) z^i$$

with $k < 0$ and $l > 0$, $A$ is a Laurent polynomial in $x$ and $y$ such that:

$$A^{-i} \big| C_i \qquad\qquad \text{for } i \in \{k, k+1, \ldots, -1\}$$

$\varphi$ is the birational transformation $(x,y,z) \mapsto \big(x, y, A(x,y)z\big)$, and:

$$g = \sum_{i=k}^{l} A(x,y)^i C_i(x,y) z^i$$

The Laurent polynomials $f$ and $g$ are related by the algebraic mutation $\varphi$. This algebraic mutation induces a combinatorial mutation of $P = \mathrm{Newt}(f)$ with $w = (0,0,1)$, $F = \mathrm{Newt}(A)$, $h_{\min} = k$, $h_{\max} = l$,

$$G_h = \mathrm{Newt}\left(\frac{C_h}{A^{-h}}\right) \qquad\qquad \text{where}^{2}\ h_{\min} \le h \le -1$$

and $\mathrm{mut}_w(P, F; \{G_h\}) = \mathrm{Newt}(g)$.

**Remark 3.5.** Given a Laurent polynomial $f_1$ with Newton polytope $P$, there may exist combinatorial mutations of $P$ that do not arise from any algebraic mutation of $f_1$. Given a combinatorial mutation of $P$, however, there exists a Laurent polynomial $f_2$ with $\mathrm{Newt}(f_2) = P$ such that the combinatorial mutation arises from an algebraic mutation of $f_2$. See Example 3.9 below.

**Lemma 3.6.** *Let $Q := \mathrm{mut}_w(P, F; \{G_h\})$ be a combinatorial mutation of $P \subset N_{\mathbb{Q}}$. Then*

$$\mathrm{mut}_{-w}(Q, F; \{w_h(P)\})$$

*is a combinatorial mutation of $Q$ equal to $P$.*

*Proof.* Let $P' := \mathrm{mut}_{-w}(Q, F; \{w_h(P)\})$. Clearly this is well defined. Let $v \in \mathcal{V}(P)$ be a vertex of $P$ with height $h$. If $h \ge 0$ then $P' \supseteq w_h(P) \supseteq H_{w,h} \cap \mathcal{V}(P)$, so $v \in P'$. If $h < 0$ then $P' \supseteq G_h + (-h)F \supseteq H_{w,h} \cap \mathcal{V}(P)$. Hence $P \subseteq P'$. Conversely let $v \in \mathcal{V}(P')$ and set $h := w(v)$. If $h \ge 0$ then $v \in w_h(P)$, so $v \in P$. If $h < 0$ then $v \in G_h + (-h)F \subseteq w_h(P)$, so again $v \in P$. Hence $P = P'$. □

**Lemma 3.7.** *Let $Q := \mathrm{mut}_w(P, F; \{G_h\})$ be a combinatorial mutation of $P \subset N_{\mathbb{Q}}$. Then*

$$\mathcal{V}(Q) \subseteq \{v_P + w(v_P)v_F \mid v_P \in w_h(P) \cap N, v_F \in \mathcal{V}(F)\},$$
$$\textit{and } \mathcal{V}(P) \subseteq \{v_Q - w(v_Q)v_F \mid v_Q \in w_h(Q) \cap N, v_F \in \mathcal{V}(F)\}.$$

*Proof.* Pick any vertex $v \in \mathcal{V}(Q)$, and set $h := w(v)$. First consider the case when $h \ge 0$. Notice that $v \in \mathcal{V}(w_h(P) + hF)$, since otherwise $v$ lies in the convex hull of two (not necessarily lattice) points $v_1, v_2 \in Q$, and as such could not be a vertex. In general the vertices of a Minkowski sum are contained in the sum of the vertices of the summands, hence there exist lattice points $v_P \in \mathcal{V}(w_h(P))$ and $v_F \in \mathcal{V}(F)$ such that $v = v_P + hv_F$. If $h < 0$ then $v \in \mathcal{V}(G_h)$. In particular, $v + (-h)v_F \subseteq w_h(P)$ for all $v_F \in F$, so there exist lattice points $v_P \in w_h(P)$ and $v_F \in \mathcal{V}(F)$ such that $v = v_P + hv_F$. Hence we have the first equation in the statement.

The second equation follows from the first by considering the inverse combinatorial mutation

$$\mathrm{mut}_{-w}(Q, F; \{w_h(P)\}).$$

□

**Proposition 3.8.** *Let $P \subset N_{\mathbb{Q}}$ be a convex lattice polytope with width vector $w$ and factor $F$. Then $\mathrm{mut}_w(P, F; \{G_h\}) = \mathrm{mut}_w(P, F; \{G'_h\})$ for any two combinatorial mutations of $P$.*

[2] The Newton polytope of the zero polynomial is $\varnothing$.

*Proof.* Set $Q := \mathrm{mut}_w(P,F;\{G_h\})$ and $Q' := \mathrm{mut}_w(P,F;\{G'_h\})$, and suppose that $Q \neq Q'$. Then (possibly after exchanging $Q$ and $Q'$) there exists some vertex $q \in \mathcal{V}(Q')$ such that $q \notin Q$, $w(q) < 0$. In particular, there exists a supporting hyperplane $H_{u,l}$ of $Q$ separating $Q$ from $q$; i.e. $u(x) \leq l$ for all $x \in Q$, and $u(q) > l$.

For any $v \in \mathcal{V}(P)$ there exists $v_Q \in w_{w(v)}(Q) \cap N$, $v_F \in \mathcal{V}(F)$ such that $v = v_P + w(v_Q)v_F$. Hence

$$u(v) = u(v_Q) - w(v_Q)u(v_F) \leq \begin{cases} l - w(v_Q)u_{\min}, & \text{if } w(v_Q) \geq 0, \\ l - w(v_Q)u_{\max}, & \text{if } w(v_Q) < 0, \end{cases}$$

where $u_{\min} := \min\{u(v_F) \mid v_F \in \mathcal{V}(F)\}$, and $u_{\max} := \max\{u(v_F) \mid v_F \in \mathcal{V}(F)\}$.

Since $\mathrm{mut}_{-w}(Q', F; \{w_h(P)\}) = P$, so $q - w(q)F \subseteq P$. By definition there exists some $v_F \in \mathcal{V}(F)$ such that $u(v_F) = u_{\max}$, hence

$$u(q - w(q)v_F) = u(q) - w(q)u_{\max} > l - w(q)u_{\max},$$

a contradiction. Hence $Q = Q'$. □

In light of Proposition 3.8, we simply write $\mathrm{mut}_w(P,F)$ for a mutation $\mathrm{mut}_w(P,F;\{G_h\})$ of $P$.

**Example 3.9.** Consider the Laurent polynomial:

$$f = \frac{z^2}{y} + 2z^2 + yz^2 + \frac{2xz^2}{y} + 2xz^2 + \frac{x^2z^2}{y} + \frac{1}{x^2} + \frac{x^4}{y^2} + y^2 + \frac{1}{z}$$

The corresponding Newton polytope $P$ has vertices $\{(0,1,2), (0,-1,2), (2,-1,2), (0,2,0), (4,-2,0), (-2,0,0), (0,0,-1)\}$. Note that the sublattice generated by the non-zero coefficients has index one.

Set $w = (0,0,-1) \in M$. The height $-2$ slice $w_{-2}(P)$ Minkowski-factorizes into two empty triangles, whereas the height $-1$ slice $w_{-1}(P) = \mathrm{conv}\{(0,1,1),(-1,0,1),(1,-1,1),(2,-1,1)\}$ is indecomposable. Since there are no vertices of $P$ at height $-1$, Definition 3.3 allows us to take $G_{-1} = \varnothing$. This gives a combinatorial mutation to:

$$Q = \mathrm{conv}\{(0,-1,2),(4,-2,0),(0,2,0),(-2,0,0),(1,0,-1),(0,1,-1),(0,0,-1)\}$$

This corresponds to the algebraic mutation $\varphi : (x,y,z) \mapsto (x, y, z/(x+y+1))$ sending $f$ to:

$$\varphi^* f = \frac{z^2}{y} + \frac{x^4}{y^2} + y^2 + \frac{1}{x^2} + \frac{x}{z} + \frac{y}{z} + \frac{1}{z}.$$

Set $g = f + yz$. Then $\mathrm{Newt}(g) = P$ but, since $g$ has a non-zero coefficient on the slice $w_{-1}(P)$, the combinatorial mutation described above does not arise from any algebraic mutation of $g$.

**Example 3.10.** The weighted projective spaces $\mathbb{P}(1,1,1,3)$ and $\mathbb{P}(1,1,4,6)$ are known to have the largest degree $-K^3 = 72$ amongst all canonical *toric* Fano threefolds [19, Theorem 3.6] and amongst all *Gorenstein* canonical Fano threefolds [23]. We shall show that they are connected by a width three combinatorial mutation.

Let $P := \mathrm{conv}\{(1,0,0),(0,1,0),(0,0,1),(-1,-1,-3)\} \subset N_{\mathbb{Q}}$ be the simplex associated with $\mathbb{P}(1,1,1,3)$. The primitive vector $(-1,2,0) \in M$ is a width three vector on $P$, with $w_{-1}(P)$ equal to the edge $\mathrm{conv}\{(-1,-1,-3),(1,0,0)\}$, and $w_2(P)$ given by the vertex $(0,1,0)$. Let $F := \mathrm{conv}\{(0,0,0),(2,1,3)\}$, and consider the mutation $\mathrm{mut}_w(P,F)$. This has vertices:

$$\{(-1,-1,-3),(0,0,1),(0,1,0),(4,3,6)\}$$

and is the simplex associated with $\mathbb{P}(1,1,4,6)$.

**Proposition 3.11.** *Let $P \subset N_{\mathbb{Q}}$ be a lattice polytope. The combinatorial mutation $\mathrm{mut}_w(P,F)$ is a Fano polytope if and only if $P$ is a Fano polytope.*

*Proof.* Begin by assuming that $P$ is Fano, and set $Q := \mathrm{mut}_w(P,F)$. Let $v \in \mathcal{V}(Q)$, and define $h := w(v)$ to be the height of $v$ with respect to $w$. If $h \geq 0$ then $v = v_P + hv_F$ for some $v_P \in \mathcal{V}(P)$, $v_F \in \mathcal{V}(F)$. But $v_P$ is primitive by assumption, hence $v$ is primitive.

If $h < 0$ then $v \in \mathcal{V}(G_h)$. Without loss of generality we are free to take $G_h$ equal to the smallest polytope such that $G_h + (-h)F \supseteq H_{w,h} \cap \mathcal{V}(P)$. Suppose that $v$ is not primitive. Then for any $v_F \in \mathcal{V}(F)$, $v + (-h)v_F$ is not primitive, hence $v + (-h)v_F$ is not a point of $H_{w,h} \cap \mathcal{V}(P)$. But this implies that we can take $G'_h := \mathrm{conv}(G_h \cap N \setminus \{v\}) \subsetneq G_h$ with $G'_h + (-h)F \supseteq H_{w,h} \cap \mathcal{V}(P)$, contradicting our choice of $G_h$. Hence $v$ is primitive.

Finally, the "if and only if" follows by considering the inverse combinatorial mutation. □

**Corollary 3.12.** *Let $P \subset N_{\mathbb{Q}}$ be a lattice polytope, and let $w$ be a width two vector. The combinatorial mutation $\mathrm{mut}_w(P,F)$ is a canonical polytope if and only if $P$ is a canonical polytope.*

*Proof.* Begin by assuming that $P$ is a canonical polytope, and set $Q := \mathrm{mut}_w(P,F)$. Since $w$ is of width two, we need only show that $H_{w,0} \cap \mathcal{V}(P) = H_{w,0} \cap \mathcal{V}(Q)$; it follows that $Q$ is canonical. But $H_{w,0} \cap \mathcal{V}(Q) \subseteq H_{w,0} \cap \mathcal{V}(P) \subseteq H_{w,0} \cap \mathcal{V}(Q)$, where the second inclusion follows by considering the inverse combinatorial mutation. Once more, the "if and only if" follows by exchanging the roles of $P$ and $Q$ via the inverse combinatorial mutation. □

**Proposition 3.13.** *Let $P \subset N_{\mathbb{Q}}$ be a lattice polytope, $\mathbf{0} \in P^\circ$. Up to isomorphism, there are only finitely many combinatorial mutations $\mathrm{mut}_w(P,F)$.*

*Proof.* For fixed width vector $w$ there are clearly only finitely many factors $F$, and so only finitely many combinatorial mutations $\mathrm{mut}_w(P,F)$. Assume that $\mathbf{0} \in P^\circ$ and consider the spanning fan $\Delta$ in $M_{\mathbb{Q}}$ for the dual polytope $P^*$. Since $\mathbf{0} \in (P^*)^\circ$, any width vector $w$ lies in a cone $\sigma \in \Delta$. If we insist that $\dim \sigma$ is as small as possible, then $\sigma$ is uniquely determined. If $\dim \sigma = \dim P$ then $w_{h_{\min}}(P)$ is zero-dimensional, hence $\mathrm{mut}_w(P,F) \cong P$. So we may insist that $\sigma \in \Delta^{(n-1)}$ is of codimension at least one.

By definition of duality, $w \in \partial(-h_{\min} P^*)$. Consider the corresponding face $F := w_{h_{\min}}(P)$. Let $l_F \in \mathbb{Z}_{>0}$ be the largest integer such that there exist lattice polytopes $A, B \subset N_{\mathbb{Q}}$, $A \neq \varnothing$, with $F = l_F A + B$. Then, for a factor to exist, $l \leq l_F$. Define $l_P := \max\{l_F \mid F \text{ is a face of } P\}$. Then $w$ is a primitive vector in $\Delta^{(n-1)} \cap l_P P^* \cap M$, and the right hand side is a finite set that depends only on $P$. □

**Remark 3.14.** When $\dim P = 2$, $\mathbf{0} \in P^\circ$, we see that $\mathrm{mut}_w(P,F)$ is a non-trivial combinatorial mutation only if $w \in \{\overline{u} \mid u \in \mathcal{V}(P^*)\}$. In particular, there are at most $|\mathcal{V}(P)|$ choices for $w$, and at most $|\partial P \cap N|$ distinct non-trivial combinatorial mutations.

Let $Q \subset M_{\mathbb{Q}}$ be a rational polytope, and let $r \in \mathbb{Z}_{>0}$ be the smallest positive integer such that $rQ$ is a lattice polytope. In general there exists a quasi-polynomial, called the *Ehrhart quasi-polynomial*, $L_Q : \mathbb{Z} \to \mathbb{Z}$ of degree $\dim Q$ such that $L_Q(m) = |mQ \cap M|$. The corresponding generating function $\mathrm{Ehr}_Q(t) := \sum_{m \geq 0} L_Q(m) t^m$ is called the *Ehrhart series* of $Q$, and can be written as a rational function [25]

$$\mathrm{Ehr}_{Q^*}(t) = \frac{\delta_0 + \delta_1 t + \ldots + \delta_{r(n+1)-1} t^{r(n+1)-1}}{(1-t^r)^{n+1}}$$

with non-negative coefficients. We call $(\delta_0, \delta_1, \ldots, \delta_{r(n+1)-1})$ the *$\delta$-vector* of $Q$. In particular, the $\delta$-vector is palindromic if and only if $Q^* \subset N_{\mathbb{Q}}$ is a lattice polytope [12].

**Proposition 3.15.** *Let $P \subset N_{\mathbb{Q}}$ be a lattice polytope, $\mathbf{0} \in P^\circ$, and let $Q := \mathrm{mut}_w(P,F)$ be a combinatorial mutation of $P$. Then $\mathrm{Ehr}_{P^*}(t) = \mathrm{Ehr}_{Q^*}(t)$.*

*Proof.* Since $P^*$ is the dual of a lattice polytope, for any point $u \in M$ there exists a non-negative integer $k \in \mathbb{Z}_{\geq 0}$ such that $u \in \partial(kP^*)$. By definition of duality, $H_{u,-k} := \{u(v) = -k \mid v \in N_{\mathbb{Q}}\}$ is a supporting hyperplane for $P$. We begin by showing that the map $u \mapsto u - u_{\min} w$, where $u_{\min} := \min\{u(v_F) \mid v_F \in \mathcal{V}(F)\}$, gives a supporting hyperplane $H_{u-u_{\min}w,-k}$ for $Q$, hence $u - u_{\min} w \in \partial(kQ^*)$.

Let $v \in \mathcal{V}(Q)$. If $w(v) \geq 0$, we can write $v = v_P + w(v_P) v_F$ for some $v_P \in \mathcal{V}(P)$, $v_F \in \mathcal{V}(F)$. In particular,

$$(u - u_{\min} w)(v) = u(v_P) + w(v_P)(u(v_F) - u_{\min}) \geq -k.$$

Now suppose that $w(v) < 0$. For any $v_F \in \mathcal{V}(F)$ we have that $v - w(v) u_F \in P$, hence $u(v - w(v) v_F) \geq -k$. In particular, $u(v) \geq -k + u_{\min} w(v)$, so

$$(u - u_{\min} w)(v) = u(v) - u_{\min} w(v) \geq -k + u_{\min} w(v) - u_{\min} w(v) = -k.$$

Since $H_{u,-k}$ is a supporting hyperplane for $P$, there exists some $v_P \in \mathcal{V}(P)$ such that $u(v_P) = -k$. If $w(v_P) \geq 0$ then for any $v_F \in \mathcal{V}(F)$, $v_P + w(v_P) v_F \in Q$. Picking $v_F$ such that $u(v_F) = u_{\min}$, we obtain $(u - u_{\min} w)(v_P + w(v_P) v_F) = -k$. If $w(v_P) < 0$ then there exists some $v_Q \in \mathcal{V}(Q)$, $v_F \in \mathcal{V}(F)$ such that $v_P = v_Q - w(v_P) v_F$. Suppose that $u(v_F) > u_{\min}$, and let $v'_F \in \mathcal{V}(F)$ be such that $u(v'_F) = u_{\min}$. Then

$u(v_Q - w(v_P)v'_F) < u(v_Q - w(v_P)v_F) = -k$. But $v_Q - w(v_P)v'_F \in P$, which is a contradiction. Hence $u(v_F) = u_{\min}$, and so $(u - u_{\min}w)(v_Q) = -k$. Thus $H_{u-u_{\min}w,-k}$ is a supporting hyperplane for $Q$.

Finally, suppose that $u, u' \in \partial(kP^*)$ are lattice points such that $u - u_{\min}w = u' - u'_{\min}w$. Since for any $v_F \in \mathcal{V}(F)$, $(u - u_{\min}w)(v_F) = u(v_F)$, we see that $u_{\min} = u'_{\min}$ and so $u = u'$. Hence $|\partial(kP^*)| \leq |\partial(kQ^*)|$. By considering the inverse combinatorial mutation $\mathrm{mut}_{-w}(Q,F) = P$, we have equality. The result follows. □

**Example 3.16.** Consider the Laurent polynomials

$$f_1 = xyz^2u + x + y + z + \frac{1}{yz} + \frac{1}{x^2yz^2u}, \qquad f_2 = xyz^2u^3 + x + y + z + \frac{1}{yz} + \frac{1}{x^2yz^2u^3}.$$

The Newton polytopes $P_i := \mathrm{Newt}(f_i)$ are both four-dimensional reflexive polytopes in $N_{\mathbb{Q}}$, and the $\delta$-vectors of the dual polytopes are, respectively,

$$\delta_1 = (1, 95, 294, 95, 1), \qquad \delta_2 = (1, 29, 102, 29, 1).$$

Proposition 3.15 thus implies that there is no sequence of mutations connecting $f_1$ and $f_2$. However

$$\pi_{f_1}(t) = \pi_{f_2}(t) = 1 + 12t^3 + 900t^6 + 94080t^9 + 11988900t^{12} + \ldots$$

which we recognise as the period sequence for $\mathbb{P}^2 \times \mathbb{P}^2$. As in §1, let $N_{f_i}$ be the sublattice of $N$ generated by the exponents of monomials of $f_i$, and let $X_{f_i}$ be the toric variety defined by the spanning fan of $P_i$ in $N_{f_i} \otimes \mathbb{Q}$. The lattice $N_{f_1}$ is equal to $N$, and $X_{f_1}$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. The lattice $N_{f_2}$ is an index-three sublattice of $N$, but the restriction of $P_2$ to $N_{f_2}$ is isomorphic to $P_1$ and hence $X_{f_2}$ is also isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. Consider now the toric variety $\widetilde{X}_2$ defined by the spanning fan of $P_2$ in $N_{\mathbb{Q}}$. This is a three-fold cover of $X_2$, and is not deformation-equivalent to $\mathbb{P}^2 \times \mathbb{P}^2$: $\widetilde{X}_2$ has Hilbert $\delta$-vector $\delta_2$, whereas $\mathbb{P}^2 \times \mathbb{P}^2$ has Hilbert $\delta$-vector $\delta_1$. As this example illustrates, sublattices are invisible to period sequences.

Let $Q := \mathrm{mut}_w(P,F)$ be a combinatorial mutation of $P$, and let $r_P$ denote the smallest dilation $k$ of $P^*$ such that $kP^*$ is integral. When $P$ is a Fano polytope, we call $r_P$ the *Gorenstein index*. The minimum period common to the cyclic coefficients of $L_{P^*}$ divides $r_P$; when the period does not equal $r_P$ we have a phenomena known as *quasi-period collapse* [7, 16]. In general $r_P \neq r_Q$ but, by Proposition 3.15, $P^*$ and $Q^*$ are Ehrhart equivalent. Hence we have examples of quasi-period collapse. At its most extreme, when $P$ is reflexive $r_P = 1$ and so the period of $L_{Q^*}$ is one. Combinatorial mutations give a systematic way of producing families of rational polytopes that exhibit this behaviour.

**Corollary 3.17.** *A three-dimensional canonical polytope $P \subset N_{\mathbb{Q}}$ is combinatorial mutation equivalent to a reflexive polytope if and only if $P$ is reflexive.*

*Proof.* By inspecting the classification of canonical polytopes [19] we see that the period of $L_{P^*}$ equals $r_P$ for each $P$. □

The following example demonstrates that Corollary 3.17 does not hold in higher dimensions.

**Example 3.18.** For any even dimension $n = 2k \geq 4$, define $P_k$ to be the polytope with $n + 2$ vertices:

$$P_k := \mathrm{conv}\{(2,2,1,1,\ldots,1), (2,1,2,1,\ldots,1), \\ (0,1,\ldots,0), \ldots, (0,0,\ldots,1), (-1,-1,\ldots,-1)\}.$$

This is a canonical polytope, but is not reflexive: the facet defined by all but the final vertex is supported by the primitive vector $(n-1, -2, \ldots, -2) \in M$ at height $-2$. Let $w := (1, -1, -1, 0, \ldots, 0) \in M$ be a width two vector on $P_k$. The face $w_{-1}(P_k)$ is two-dimensional; this is the empty square, and Minkowski factors into two line segments. The corresponding combinatorial mutation gives the standard polytope for $\mathbb{P}^n$. By Proposition 3.15,

$$\mathrm{Ehr}_{P_k{}^*}(t) = \mathrm{Hilb}_{\mathbb{P}^n}(-K_{\mathbb{P}^n}).$$

Since $r_{P_k} = 2$, we have quasi-period collapse.

The map $u \mapsto u - u_{\min} w$ on $M$ described in the proof of Proposition 3.15 is a piecewise-linear transformation analogous to a cluster transformation. Let $\Delta_F$ be the normal fan of $F$ in $M_{\mathbb{Q}}$. Notice that the normal fan is well-defined up to translation of $F$. Furthermore, the cones of $\Delta_F$ are not strictly convex: they each contain the subspace $\mathbb{Z}w$. Let $\sigma$ be a maximum-dimensional cone of $\Delta_F$. Then there exists $M_\sigma \in \mathrm{GL}_n(\mathbb{Z})$ such that, for any $u \in -\sigma$, the map $u \mapsto u - u_{\min} w$ is equal to the map $u \mapsto uM_\sigma$.

We make this explicit. Suppose without loss of generality that $w = (0, \ldots, 0, 1) \in M$. Then a maximum-dimensional cone $\sigma \in \Delta_F$ corresponds to some vertex $v_\sigma = (v_{\sigma,1}, \ldots, v_{\sigma,n-1}, 0) \in \mathcal{V}(F)$, and:

$$M_\sigma := \begin{pmatrix} 1 & 0 & \ldots & 0 & -v_{\sigma,1} \\ 0 & 1 & \ldots & 0 & -v_{\sigma,2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -v_{\sigma,n-1} \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

**Example 3.19.** Let $w := (0,0,1) \in M$ and let $F := \mathrm{conv}\{(0,0,0),(1,0,0),(0,1,0)\} \subset N_{\mathbb{Q}}$. We have:

$$M_\sigma = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \sigma = \mathrm{cone}\{(-1,0,0),(0,-1,0),(0,0,\pm 1)\} \\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \sigma = \mathrm{cone}\{(1,1,0),(0,-1,0),(0,0,\pm 1)\} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \sigma = \mathrm{cone}\{(-1,0,0),(1,1,0),(0,0,\pm 1)\} \end{cases}$$

## 4. Three-dimensional Minkowski polynomials

Three-dimensional Minkowski polynomials are a family of Laurent polynomials in three variables with Newton polytopes given by three-dimensional reflexive polytopes. We consider Laurent polynomials $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$, and write $\mathsf{x}^v := x^a y^b z^c$ where $v = (a,b,c) \in \mathbb{Z}^3$.

**Definition 4.1.** Let $Q$ be a convex lattice polytope in $\mathbb{Q}^3$.

(i) We say that $Q$ is a *length-one line segment* if there exist distinct points $v, w \in \mathbb{Z}^3$ such that the vertices of $Q$ are $v$, $w$ (in some order) and $Q \cap \mathbb{Z}^3 = \{v, w\}$. In this case, we set:

$$f_Q = \mathsf{x}^v + \mathsf{x}^w$$

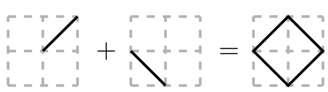
(ii) We say that $Q$ is an $A_n$ *triangle* if there exist distinct points $u, v, w \in \mathbb{Z}^3$ such that the vertices of $Q$ are $u$, $v$, $w$ (in some order) and $Q \cap \mathbb{Z}^3 = \{u, v_0, v_1, \ldots, v_n\}$, where $v_0 = v$, $v_n = w$, and $v_0, v_1, \ldots, v_n$ are consecutive lattice points on the line segment from $v$ to $w$. In this case, we set:

$$f_Q = \mathsf{x}^u + \sum_{k=0}^{n} \binom{n}{k} \mathsf{x}^{v_k}$$

Observe that, in each case, the Newton polytope of $f_Q$ is $Q$.

**Definition 4.2.** A *lattice Minkowski decomposition* of a lattice polytope $Q \subset \mathbb{Q}^m$ is a decomposition $Q = Q_1 + \cdots + Q_r$ of $Q$ as a Minkowski sum of lattice polytopes $Q_i$, $i \in \{1, 2, \ldots, r\}$, such that the affine lattice generated by $Q \cap \mathbb{Z}^m$ is equal to the sum of the affine lattices generated by $Q_i \cap \mathbb{Z}^m$, $i \in \{1, 2, \ldots, r\}$.

**Example 4.3.** The Minkowski decomposition:



is not a *lattice* Minkowski decomposition, because the affine lattices in the summands generate an index-two affine sublattice of $\mathbb{Z}^2$.

**Definition 4.4.** Let $Q$ be a lattice polytope in $\mathbb{Q}^m$. We say that lattice Minkowski decompositions:

$$Q = Q_1 + \cdots + Q_r, \qquad\qquad Q = Q'_1 + \cdots + Q'_s$$

of $Q$ are *equivalent* if $r = s$ and if there exist $v_1, \ldots, v_r \in \mathbb{Z}^m$ and a permutation $\sigma$ of $\{1, 2, \ldots, r\}$ such that $Q'_i = Q_{\sigma(i)} + v_i$.

**Definition 4.5.** Let $Q$ be a lattice polytope in $\mathbb{Q}^3$. A lattice Minkowski decomposition $Q = Q_1 + \cdots + Q_r$ of $Q$ is called *admissible* if each $Q_i$ is either a length-one line segment or an $A_n$ triangle. Given an admissible Minkowski decomposition $Q = Q_1 + \cdots + Q_r$ of $Q$, we set:

$$f_{Q:Q_1,\ldots,Q_r} = \prod_{i=1}^{r} f_{Q_i}$$

Observe that the Newton polytope of $f_{Q:Q_1,\ldots,Q_r}$ is $Q$. Observe also that if:

$$Q = Q_1 + \cdots + Q_r, \qquad\qquad Q = Q'_1 + \cdots + Q'_s$$

are equivalent lattice Minkowski decompositions of $Q$, then $f_{Q:Q_1,\ldots,Q_r} = f_{Q:Q'_1,\ldots,Q'_s}$.

**Definition 4.6.** Let $P$ be a three-dimensional reflexive polytope in $\mathbb{Q}^3$. Let $f$ be a Laurent polynomial in three variables. We say that $f$ is a *three-dimensional Minkowski polynomial with Newton polytope $P$* if:

(i) $\mathrm{Newt}(f) = P$, so that:

$$f = \sum_{v \in P \cap \mathbb{Z}^3} a_v \mathsf{x}^v$$

for some coefficients $a_v$.

(ii) $a_{\mathbf{0}} = 0$.

(iii) For each facet $Q \in \mathcal{F}(P)$ of $P$, we have:

$$\sum_{v \in Q \cap \mathbb{Z}^3} a_v \mathsf{x}^v = f_{Q:Q_1,\ldots,Q_r}$$

for some admissible lattice Minkowski decomposition $Q = Q_1 + \cdots + Q_r$ of $Q$.

Given a three-dimensional reflexive polytope $P$, there may be:

(i) no Minkowski polynomial with Newton polytope $P$, if some facet of $P$ has no admissible lattice Minkowski decomposition;

(ii) a unique Minkowski polynomial with Newton polytope $P$, if there is a unique equivalence class of admissible lattice Minkowski decomposition for each facet of $P$;

(iii) many Minkowski polynomials with Newton polytope $P$, if there is some facet of $P$ with several inequivalent admissible lattice Minkowski decompositions.

Up to isomorphism, there are 4319 three-dimensional reflexive polytopes [20]; 1294 of them support no Minkowski polynomial, and the remaining 3025 together support 3747 distinct[3] Minkowski polynomials.

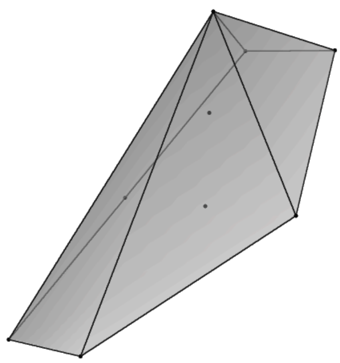


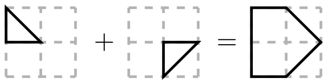
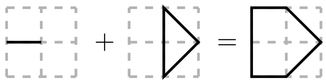
FIGURE 1. The polytope $P$ from Example 4.7.

[3] "Distinct" here means "distinct up to $GL_3(\mathbb{Z})$-equivalence": see Definition 2.4.

**Example 4.7.** Consider the three-dimensional reflexive polytope $P$ with vertices:

$$\{(-1,-1,-3),(1,0,0),(0,1,0),(0,0,1),(0,-1,-2),(-1,1,-1)\}$$

As facets, $P$ has four $A_1$ triangles, one $A_2$ triangle, and a pentagon. The pentagonal facet of $P$ has two inequivalent lattice Minkowski decompositions:

and thus there are two distinct Minkowski polynomials supported on $P$:

$$f_1 = x + y + z + \frac{1}{yz^2} + \frac{y}{xz} + \frac{2}{xz^2} + \frac{1}{xyz^3} + \frac{2}{z}$$
$$f_2 = x + y + z + \frac{1}{yz^2} + \frac{y}{xz} + \frac{2}{xz^2} + \frac{1}{xyz^3} + \frac{3}{z}$$

**Remark 4.8.** Recall from §1 that we denote by $N_f$ the lattice generated by the exponents of monomials of a Laurent polynomial $f$. For a three-dimensional reflexive polytope $P$, the lattice generated by $P \cap \mathbb{Z}^3$ is equal to $\mathbb{Z}^3$; thus whenever $f$ is a three-dimensional Minkowski polynomial we have $N_f = \mathbb{Z}^3$. In dimension four there are precisely 16 reflexive polytopes $P$ such that the lattice generated by $P \cap \mathbb{Z}^4$ is a proper sublattice of $\mathbb{Z}^4$ [2].

**Remark 4.9.** If $f$ is a Minkowski polynomial and $g$ is a Laurent polynomial related to $f$ via a mutation $\varphi$ then in general $g$ will not be a Minkowski polynomial; indeed $\mathrm{Newt}(g)$ will in general not be reflexive or even canonical (cf. Corollary 3.17).

## 5. Mutations between Minkowski polynomials

Consider now the set of all three-dimensional Minkowski polynomials, and partition this into smaller sets, which we call *buckets*, such that $f$ and $g$ are in the same bucket if and only if the first eight terms of the period sequences for $f$ and $g$ agree. There are 165 buckets. Using Proposition 3.13 and a computer search, we have determined all mutations of elements of each bucket. For all except two buckets any two Laurent polynomials $f$, $g$ in the bucket are connected by a sequence of mutations that involves only Minkowski polynomials (and hence involves only Laurent polynomials from that bucket). In fact:

(i) for 156 of the buckets, any two Laurent polynomials $f$, $g$ in the bucket are connected by a sequence of *width two* mutations that involves only Minkowski polynomials in that bucket.
(ii) for 7 of the buckets, any two Laurent polynomials $f$, $g$ in the bucket are connected by a sequence of *width two and width three* mutations that involves only Minkowski polynomials in that bucket.

For the remaining two buckets[4], any two Laurent polynomials $f$, $g$ in the bucket are connected by a sequence of mutations, but it is not possible to insist that all of the Laurent polynomials involved are Minkowski polynomials. (It is, however, possible to insist that all of the Laurent polynomials have reflexive Newton polytope.)

The sequence of mutations connecting two Minkowski polynomials $f$ and $g$ is far from unique, but representative such mutations are shown in Appendix B. The mutations shown there suffice to connect any two three-dimensional Minkowski polynomials in the same bucket.

**Corollary 5.1.** *Three-dimensional Minkowski polynomials $f$ and $g$ have the same period sequence if and only if they have the same first eight terms of the period sequence.*

*Proof.* Any two Minkowski polynomials in the same bucket are connected by a sequence of mutations. ☐

**Proposition 5.2.** *If $f$ and $g$ are three-dimensional Minkowski polynomials with the same period sequence, then there exist a flat family $\pi : \mathcal{X} \to \Sigma$ over a possibly-reducible rational curve $\Sigma$ and two distinct points $0, \infty \in \Sigma$ such that $\pi^{-1}(0)$ is isomorphic to the toric variety $X_f$ and $\pi^{-1}(\infty)$ is isomorphic to the toric variety $X_g$.*

*Proof.* Combine the results in Appendix B with [18, Theorem 1.3]. ☐

[4] Numbering as in Appendix A, these are the buckets with IDs 148 and 161.

In particular this gives a geometric proof that if $f$ and $g$ are three-dimensional Minkowski polynomials with the same period sequence, then $X_f$ and $X_g$ have the same Hilbert series. One might hope that substantially more is true: for example that given a bucket of Minkowski polynomials $\{f_i : i \in I\}$ corresponding to a Fano manifold $X$, the toric varieties $X_{f_i}$, $i \in I$, all lie in the same component of the Hilbert scheme, and that $X$ lies in this component too. It is possible that this hope is too naïve, as the geometry of Hilbert schemes is in many ways quite pathological, but in any case, as things currently stand, results of this form seem to be out of reach.

**Remark 5.3.** One might regard our results as further evidence that the mirror to a Fano manifold $X$ should be some sort of cluster variety $X^\vee$ together with a function (the superpotential) on $X^\vee$. This is familiar from the work of Auroux [1]. Given a special Lagrangian cycle and a complex structure on a Fano manifold $X$, mirror symmetry associates to this choice a Laurent polynomial. The Laurent polynomial depends on our choice: there is a wall-and-chamber structure on the space of parameters, and Laurent polynomials from neighbouring chambers are expected to be related by an elementary birational transformation. Reformulating this: the mirror to $X$ should be a pair $(X^\vee, W)$ where $X^\vee$ is obtained by gluing tori (one for each chamber) along birational transformations (coming from wall-crossing), and the superpotential $W$ is a global function on $X^\vee$. When $X$ is two-dimensional, the elementary birational transformations that occur here are closely related to those in the theory of cluster algebras [13]. In our three-dimensional situation, we need to allow a still more general notion of cluster algebra.

More precisely: the fundamental objects in the theory of cluster algebras are *seeds* and their mutations. A seed is a collection of combinatorial and algebraic data: the so-called exchange matrix and a basis for a field $k = \mathbb{Q}(x_1, ..., x_n)$ of rational functions of $n$ variables, called a cluster. Seeds can be transformed into new seeds through a process called *mutation*. The *Laurent phenomenon* theorem of [13] says that any cluster variable is a Laurent polynomial when expressed in terms of any other cluster variable in any other mutation equivalent seed. This produces a large subalgebra in $k$, called the *upper bound*, consisting of those elements that are Laurent polynomials when expressed in terms of cluster variables of any seed. The "Laurent phenomenon" arising from mirror symmetry for Fano manifolds (i.e. the presence of a Laurent polynomial that remains a Laurent polynomial under some collection of elementary birational transformations) does not fit into the framework of cluster algebras. In cluster algebras the number of ways to mutate a seed naturally is always less than or equal to the transcendence degree of $k$, whereas the Laurent polynomials that are mirror dual to a del Pezzo surface $S$ have $\chi(S) = \rho(S) + 2$ ways to mutate them [10, 14] and the transcendence degree of $k$ here is 2. Furthermore, in contrast to the case of cluster algebras, the transcendence degree of the upper bound in this setting equals just one; in fact, the upper bound is just the ring of polynomials generated by $W$. In dimension three there are additional complications: in the Fano setting the exchange polynomials need not be binomials (cf. [22]), and we do not see how to define the 2-form that occurs in the cluster theory. In this paper we do not consider the problem of defining seeds in dimension three, nor do we single out the relevant notion of mutation of seeds. These are two important questions for further study.

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