

# ON K-MODULI OF QUARTIC THREEFOLDS

HAMID ABBAN, IVAN CHELTSOV, ALEXANDER KASPRZYK, YUCHEN LIU, AND ANDREA PETRACCI

**ABSTRACT.** The family of smooth Fano 3-folds with Picard rank 1 and anticanonical volume 4 consists of quartic 3-folds and of double covers of the 3-dimensional quadric branched along an octic surface. They can all be parametrised as complete intersections of a quadric and a quartic in the weighted projective space  $\mathbb{P}(1, 1, 1, 1, 1, 2)$ , denoted by  $X_{2,4} \subset \mathbb{P}(1^5, 2)$ ; all such smooth complete intersections are K-stable. With the aim of investigating the compactification of the moduli space of quartic 3-folds given by K-stability, we exhibit three phenomena: (i) there exist K-polystable complete intersection  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$  Fano 3-folds which deform to quartic 3-folds and are neither quartic 3-folds nor double covers of quadric 3-folds – in other words, the closure of the locus parametrising complete intersections  $X_{2,4} \subset \mathbb{P}(1^5, 2)$  in the K-moduli contains elements that are not of this type; (ii) any quasi-smooth  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$  is K-polystable; (iii) the closure in the K-moduli space of the locus parametrising complete intersections  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$  which are not complete intersections  $X_{2,4} \subset \mathbb{P}(1^5, 2)$  contains only points which correspond to complete intersections  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ .

## 1. INTRODUCTION

A *Fano variety* is a normal projective variety over  $\mathbb{C}$  for which the anticanonical divisor is  $\mathbb{Q}$ -Cartier and ample. There has been spectacular recent progress on constructing moduli spaces of Fano varieties using K-stability [6, 13–15, 36, 40, 47, 56, 58]. For each positive integer  $n$  and for every positive rational number  $v$  there exists an Artin stack  $\mathcal{M}_{n,v}^{\text{Kss}}$ , called the *K-moduli stack*, which is of finite type over  $\mathbb{C}$  and which parametrises K-semistable  $n$ -dimensional Fano varieties with anticanonical volume  $v$ . Moreover, this stack admits a good moduli space  $M_{n,v}^{\text{Kps}}$ , called the *K-moduli space*, which is projective over  $\mathbb{C}$ . The closed points of  $M_{n,v}^{\text{Kps}}$  are in one-to-one correspondence with K-polystable  $n$ -dimensional Fano varieties with anticanonical volume  $v$ . We refer to [57] for a survey on these topics.

The next natural step is to investigate the geometry of K-moduli spaces, providing explicit descriptions when possible. K-moduli of smooth(able) 2-dimensional Fano varieties have been studied by Mabuchi–Mukai [49] and by Odaka–Spotti–Sun [51]. More generally, one of the most well-known classes of Fano varieties are hypersurfaces of degree  $d$  in  $n$ -dimensional projective space  $\mathbb{P}^n$ , with  $d \leq n$ ; all smooth Fano hypersurfaces are conjectured to be K-polystable [57, Part 3]. Substantial progress has made towards clarifying this conjecture [2, Theorem B], however very little is currently known about their K-moduli. One readily available compact moduli space for Fano hypersurfaces is the GIT moduli space, and Liu–Xu [46] showed that, for cubic 3-folds, the K-moduli space coincides with the GIT moduli space. An analogous result for cubic 4-folds was recently proved by Liu [43]. There are additional results in 2 dimensions, and in other settings [8–10, 45]. A particular observation is the following conjecture, which is verified up to dimension 4.

**Conjecture.** *K-polystable limits of cubic hypersurfaces are cubic hypersurfaces.*

The simplest case in higher dimensions where the expected agreement between GIT moduli and K-moduli fails, or equivalently the conjecture above is invalid for, is the moduli space of quartic 3-folds. Here it is known that their K-moduli space contains elements that are not in the GIT moduli.

**1.1. K-moduli of quartic 3-folds.** Let  $\mathcal{V}_4$  denote the family of *smooth* Fano 3-folds with Picard rank 1 and anticanonical volume 4. Members of  $\mathcal{V}_4$  are either:

- (a) smooth quartic hypersurfaces in  $\mathbb{P}^4$ , simply called *smooth quartic 3-folds*; or
- (b) double covers of the smooth quadric 3-fold with branch divisor of degree 8, often called *hyperelliptic*.

By [17, 26, 29] each smooth Fano 3-fold in the family  $\mathcal{V}_4$  is K-stable. It is not known how to characterise singular quartic 3-folds which are K-(poly/semi)stable. Since not every member of  $\mathcal{V}_4$  is a quartic 3-fold, the connected component of the K-moduli space  $M_{3,4}^{\text{Kps}}$  containing the members of  $\mathcal{V}_4$  does not coincide with the GIT moduli space of quartic 3-folds. However, every smooth member of  $\mathcal{V}_4$  is a complete intersections

---

2020 *Mathematics Subject Classification.* 14J10 (Primary); 32Q20, 14J45, 14J30, 14J70 (Secondary).

*Key words and phrases.* K-stability, Quartic threefolds, K-moduli, Fano hypersurfaces.

of type  $(2, 4)$  in  $\mathbb{P}(1^5, 2)$ ; this can be seen as follows (see also [50, Example 4.3], the introduction of [52], and [7, Example 3.5.2]). Let  $x_0, \dots, x_4, y$  be the homogeneous coordinates of  $\mathbb{P}(1^5, 2)$  with degrees  $1, \dots, 1, 2$  respectively. A  $(2, 4)$ -complete intersection  $X$  is given by the vanishing of

$$f_4(x_0, \dots, x_4) + f_2(x_0, \dots, x_4)y + ay^2 \quad \text{and} \quad g_2(x_0, \dots, x_4) + by,$$

where  $f_2$  and  $g_2$  are quadrics,  $f_4$  is a quartic, and  $a, b \in \mathbb{C}$ . If  $X$  is smooth then  $a \neq 0$  or  $b \neq 0$ . If  $b \neq 0$ , from the second equation we can express  $y$  in terms of the  $x_i$ , therefore  $y$  is a redundant variable. This implies that  $X$  is a quartic hypersurface in  $\mathbb{P}^4$ . If  $b = 0$  then  $a \neq 0$  so, up to scaling,  $a = 1$ . By completing the square (i.e. applying the automorphism of  $\mathbb{P}(1^5, 2)$  given by  $y \mapsto y + \frac{1}{2}f_2(x)$ ) we can assume that  $f_2 = 0$ ; in this case  $X$  is the double cover of the quadric 3-fold  $\{g_2(x) = 0\} \subset \mathbb{P}^4$  branched along the surface  $\{f_4(x) = g_2(x) = 0\}$ .

It is natural to ask whether the connected component of the K-moduli space  $M_{3,4}^{\text{Kps}}$  containing the members of  $\mathcal{V}_4$  parametrises  $(2, 4)$ -complete intersections in  $\mathbb{P}(1^5, 2)$ . If this were true, it might suggest that this component of K-moduli can be constructed via some form of non-reductive GIT by taking the moduli space of such complete intersections. We first prove that this is not the case.

**Theorem A** (cf. Proposition 2.1, Theorem 3.1, and Theorem 4.1). There exist K-polystable non-smooth Fano 3-folds that are smoothable to a quartic 3-fold and that are not  $(2, 4)$ -complete intersections in  $\mathbb{P}(1^5, 2)$ .

An immediate consequence is:

**Corollary.** Let  $M$  be the connected component of the K-moduli space  $M_{3,4}^{\text{Kps}}$  containing the members of  $\mathcal{V}_4$ . Then in  $M$  there are points corresponding to K-polystable Fano 3-folds that are not  $(2, 4)$ -complete intersections in  $\mathbb{P}(1^5, 2)$ .

We provide two types of examples of Fano 3-folds which satisfy Theorem A: three toric examples that we study in §2, and an infinite family that we study in §3. These examples are all  $(2, 2, 4)$ -complete intersections in the weighted projective 6-space  $\mathbb{P}(1^5, 2^2)$ .

We have seen that some K-polystable degenerations of smooth quartic 3-folds are hyperelliptics; in other words, K-moduli limits of quartics include hyperelliptics. In Theorem A we have seen that K-moduli limits of hyperelliptics include Fano 3-folds which are  $(2, 2, 4)$ -complete intersections in  $\mathbb{P}(1^5, 2^2)$  which are not  $(2, 4)$ -complete intersections in  $\mathbb{P}(1^5, 2)$ . We call such 3-folds *pure*  $(2, 2, 4)$ -complete intersections. It is natural to wonder whether pure  $(2, 2, 4)$ -complete intersections degenerate further in the K-moduli space. Surprisingly, the K-moduli limits of pure  $(2, 2, 4)$ -complete intersections are pure  $(2, 2, 4)$ -complete intersections:

**Theorem B** (cf. Proposition 4.3 and Corollary 4.4). In the K-moduli space  $M_{3,4}^{\text{Kps}}$ , the locus of pure  $(2, 2, 4)$ -complete intersections in  $\mathbb{P}(1^5, 2^2)$  is closed.

This is proved in §4 by showing that the K-moduli locus of pure  $(2, 2, 4)$ -complete intersections in  $\mathbb{P}(1^5, 2^2)$  is determined by the K-moduli space of pairs  $(S, \frac{1}{16}\Delta)$ , where  $S$  is a degree 4 del Pezzo surface and  $\Delta \sim -4K_S$ . In §5 we provide a complete study of the wall-crossing for the pairs  $(S, c\Delta)$ , as  $c$  is a rational number in  $(0, \frac{1}{16}]$ .

**1.2. Degenerations of quartic 3-folds inside  $\mathbb{P}(1^5, 2^2)$ .** Denote the variables of the weighted projective space  $\mathbb{P}(1^5, 2^2)$  by  $x_0, \dots, x_4, y_0, y_1$ , where  $y_0$  and  $y_1$  are the two variables of weight 2. Let  $X \subset \mathbb{P}(1^5, 2^2)$  be a complete intersection of two quadrics and a quartic. If  $X$  is not a cone then, after a suitable change of coordinates, it is defined by the equations

$$\begin{aligned} y_0 y_1 &= f(x_0, x_1, x_2, x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) &= a y_0 + b y_1 \\ h(x_0, x_1, x_2, x_3, x_4) &= c y_0 + d y_1 \end{aligned}$$

where  $f$  has degree 4,  $g$  and  $h$  have degree 2, and  $a, b, c, d \in \mathbb{C}$ . There are three possibilities depending on the rank of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If  $\text{rk } A = 2$  then  $X$  is a quartic hypersurface in  $\mathbb{P}^4$ . If  $\text{rk } A = 1$  then  $X$  is hyperelliptic; i.e. a  $(2, 4)$ -complete intersection in  $\mathbb{P}(1^5, 2)$ . If  $A$  is the null matrix then  $X$  has two singular points at  $p_0 = [0 : 0 : 0 : 0 : 0 : 1 : 0]$

and  $p_1 = [0 : 0 : 0 : 0 : 0 : 0 : 1]$ . Let us now assume that we are in the latter case, so that  $X \subset \mathbb{P}(1^5, 2^2)$  is defined by

$$\begin{aligned} y_0 y_1 &= f(x_0, x_1, x_2, x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) &= 0 \\ h(x_0, x_1, x_2, x_3, x_4) &= 0 \end{aligned}$$

where  $f$  is a quartic and  $g$  and  $h$  are quadrics. In §2 we consider special binomials for  $f, g, h$  and we study certain toric varieties  $X$ . In §3 we study the infinite family of varieties  $X$  obtained by picking general  $f, g, h$ . In §4 we first show that if the surface  $\{g = h = 0\} \subset \mathbb{P}^4$  and the curve  $\{f = g = h = 0\} \subset \mathbb{P}^4$  are smooth then  $X$  is K-polystable. We then prove that K-polystable degenerations of these complete intersection  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$  are also complete intersections of these type.

**1.3. Overview of the proof.** There are several methods available to verify K-(poly)stability of a given Fano variety. For the toric examples, we use the most natural tool: a toric variety is K-polystable if and only if the barycentre of its anticanonical polytope is the origin. This translates the algebro-geometric condition into a combinatorial one amenable to computer-assisted investigation. The toric examples which satisfy Theorem A were found via a computer search amongst Fano polytopes using techniques from Mirror Symmetry [3, 20, 21]. This is explained in §1.4; the examples are studied in §2.

For the infinite family of examples satisfying Theorem A, we first use estimates on stability thresholds. Two of the most useful tools in K-stability are the Fujita–Li valuative criterion [30, 38] and the stability threshold, also called the  $\delta$ -invariant, introduced in [31]. It follows that a Fano variety  $X$  is K-stable if and only if  $\delta(X) > 1$ . Although  $\delta(X)$  is extremely difficult to compute in general, a method to find lower bounds for  $\delta(X)$  is described in [1]. Roughly speaking, one chooses a flag over  $X$  and computes the refinement of the anticanonical linear system with respect to this flag [1, §2]; the refinement provides a lower bound for  $\delta(X)$ . With a little care, one can choose the flag so that the stability threshold of the refinement is greater than 1, implying K-stability. We do this for the infinite family in §3. In §4, we use techniques of cyclic covers, cone construction, degeneration and interpolation to obtain two results: we first prove that all complete intersections  $X_{2,2,4}$  as described above are K-polystable as long as  $\{g = h = 0\} \subset \mathbb{P}^4$  and  $\{f = g = h = 0\} \subset \mathbb{P}^4$  are smooth; then we prove that the closure in the K-moduli space of the locus of complete intersections  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$  which are not complete intersections  $X_{2,4} \subset \mathbb{P}(1^5, 2)$  contains only points which correspond to complete intersections  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ .

**1.4. Connection to Mirror Symmetry.** The three toric 3-folds presented in Proposition 2.1 and in Remark 2.2 were found using a computer-assisted search guided by expectations arising from Mirror Symmetry for Fano varieties [3, 20, 21]. It is expected that deformation families of smooth (or maybe mildly singular, i.e. with orbifold terminal singularities) Fano varieties of dimension  $n$  are in one-to-one correspondence with mutation-equivalence classes of certain ‘special’ Laurent polynomials in  $n$  variables. We need to specify: (i) the meaning of the word *mutation*; (ii) the meaning of the adjective ‘special’; and (iii) how the correspondence works.

- (i) By *mutation* we mean a generalisation, introduced in [4], of the notion of mutation in cluster algebra theory [28]. We do not give the definition here: see [4, Definition 2] for details. It is sufficient to know that, under certain circumstances, a Laurent polynomial  $f \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  can be mutated to another Laurent polynomial  $g \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ . Two Laurent polynomials  $f$  and  $g$  are said to be mutation-equivalent if there exists a finite sequence of mutations transforming  $f$  to  $g$ .
- (ii) In place of ‘special’ we should have written *rigid maximally mutable*. The class of rigid maximally mutable Laurent polynomials (or *rigid MMLPs* for short) was introduced in [3] for dimension 2, and in [23] for any dimension. We remark that if  $f \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  is a rigid MMLP in  $n$  variables, then its Newton polytope  $P := \text{Newt}(f) \subset \mathbb{R}^n$  is an  $n$ -dimensional lattice polytope such that the origin lies in the interior of  $P$ , and each vertex of  $P$  is a primitive lattice vector:  $P$  is an example of a *Fano polytope*. One can consider the *spanning fan* (or *face fan*) whose cones are generated by the faces of  $P$ , and the (possibly singular) Fano toric variety  $X_P$  associated to this spanning fan.
- (iii) The correspondence between deformation families of Fano varieties and mutation-equivalence classes of rigid MMLPs is described in [3, 22, 23]. Briefly, we expect it to work as follows: given a rigid MMLP  $f$  with Newton polytope  $P$ , we consider the Fano toric variety  $X_P$  and associate a (partial) smoothing of  $X_P$ . The reason why this assignment should be well-defined is the following result due to Ilten [35]: if  $f$  and  $g$

are Laurent polynomials related via a mutation, then the Fano toric varieties  $X_P$  and  $X_Q$  are deformation equivalent (here  $Q$  is the Newton polytope of  $g$ ); i.e. there exists a flat proper family  $\mathcal{X} \rightarrow \mathbb{P}^1$  such that the fibre over  $0 \in \mathbb{P}^1$  is  $X_P$  and the fibre over  $\infty \in \mathbb{P}^1$  is  $X_Q$ . Notice that a singular Fano toric variety can have many different smoothings (e.g. [37, Theorem 3.1] or [54]); one needs to select the smoothing of  $X_P$  which is compatible with the mutations of  $f$ .

Now we explain how we found the toric examples. We start from the polytope  $P'$  whose vertices are the vectors in (2.2) in Remark 2.3. The toric variety  $X_{P'}$  associated to the spanning fan of  $P'$  is the singular quartic hypersurface  $\{x_1x_2x_3x_4 = x_0^4\} \subset \mathbb{P}^4$ , which clearly deforms to members of  $\mathcal{V}_4$ . There exists a unique rigid MMLP  $f$  such that  $P' = \text{Newt}(f)$ , namely

$$f = \frac{(1+x+y+z)^4}{xyz} - 24.$$

We used the computer algebra system MAGMA [16] to construct many Laurent polynomials  $g$  which are mutation-equivalent to  $f$ . Each of these  $g$ 's gives a Fano toric variety  $X_Q$ , where  $Q := \text{Newt}(g)$ , which, according to the Mirror Symmetry expectations described above, should be a degeneration of quartic 3-folds. We then filter for those  $g$  such that the polar of  $Q$  has barycentre at the origin; this is equivalent to requiring that the Fano toric variety  $X_Q$  is K-polystable. In this way we found three Laurent polynomials, one of which is given by:

$$\begin{aligned} g = & x^3y^2z^4 + 2x^2y^3z^2 + 4x^2y^2z^2 + 8x^2yz^3 + 2x^2yz^2 + xy^4 + 4xy^3 + 8xy^2z + 6xy^2 \\ & + 16xyz + 4xy + 28xz^2 + 8xz + x + 12y + 56z/y + 12/y + 8/(xz) + 16/(xyz) + 70/(xy^2) + 8/(xy^2z) \\ & + 2/(x^2yz^2) + 4/(x^2y^2z^2) + 56/(x^2y^3z) + 2/(x^2y^3z^2) + 28/(x^3y^4z^2) + 8/(x^4y^5z^3) + 1/(x^5y^6z^4). \end{aligned}$$

The Newton polytope  $P := \text{Newt}(g)$  is the Fano polytope given in Proposition 2.1. The Newton polytopes of the remaining two Laurent polynomials found using this method are described in Remark 2.2.

**Notation and conventions.** We work over an algebraically closed field of characteristic zero, denoted by  $\mathbb{C}$ . Every toric variety or toric singularity is assumed to be normal.

## 2. THE TORIC EXAMPLES

We begin by analysing a toric Fano 3-fold satisfying Theorem A.

**Proposition 2.1.** *Let  $P$  be the polytope with vertices*

$$\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -6 \\ -4 \end{pmatrix} \quad (2.1)$$

in the lattice  $N = \mathbb{Z}^3$  and let  $X$  be the toric variety associated to the spanning fan of  $P$ . Then:

- (1)  $X$  is a  $\mathbb{Q}$ -factorial K-polystable Fano 3-fold;
- (2)  $\text{Pic}(X) \simeq \mathbb{Z}$  and  $\text{Cl}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ ;
- (3)  $X$  is the quotient  $\mathbb{P}^3/(\mu_2 \times \mu_8)$ , where  $\mu_2$  acts linearly with weights  $(0, 1, 0, 1)$  and  $\mu_8$  acts linearly with weights  $(0, 5, 1, 6)$ ;
- (4) the singular locus of  $X$  consists of six rational curves generically along which  $X$  has transverse  $A_1$ ,  $A_3$  or  $A_7$  singularities;
- (5) there are exactly two non-Gorenstein points on  $X$  and both of them have Gorenstein index 2 and are not canonical;
- (6)  $X$  is not a  $(2, 4)$ -complete intersection in  $\mathbb{P}(1^5, 2)$ ;
- (7)  $X$  is the  $(2, 2, 4)$ -complete intersection in  $\mathbb{P}(1^5, 2^2)$  given by the equations

$$\begin{aligned} x_0^4 - y_0y_1 &= 0 \\ x_2^2 - x_0x_3 &= 0 \\ x_3^2 - x_1x_4 &= 0 \end{aligned}$$

where  $x_0, \dots, x_4, y_0, y_1$  are the homogeneous coordinates of  $\mathbb{P}(1^5, 2^2)$  with degrees  $1, \dots, 1, 2, 2$  respectively;

- (8)  $X$  deforms to a quartic 3-fold.

*Proof of (1).* Let  $\Sigma$  be the spanning fan of  $P$ . It is clear that  $X$  is a Fano 3-fold. Since  $P$  is a tetrahedron, each cone of  $\Sigma$  is simplicial, therefore  $X$  is  $\mathbb{Q}$ -factorial. Let  $M$  be the lattice dual to  $N$  and let  $\langle \cdot, \cdot \rangle: M \times N \rightarrow \mathbb{Z}$  be the dual pairing. We denote by the same symbol its extension to the associated real vector spaces, i.e.  $M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ . Consider the polar  $P^\circ$  of  $P$ :

$$P^\circ := \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq -1 \text{ for all } v \in P\}.$$

This is the polytope associated to the toric boundary of  $X$ , which is an anticanonical divisor. One can show that  $P^\circ$  is the rational polytope in  $M_{\mathbb{R}}$  with vertices  $(-1, 0, \frac{3}{2})$ ,  $(-1, 1, 0)$ ,  $(3, -1, -2)$ ,  $(-1, 0, \frac{1}{2})$ . Since the barycentre of  $P^\circ$  is the origin,  $X$  is K-polystable by [11].  $\square$

*Proof of (2) and of (3).* Consider the linear map  $\rho: \mathbb{Z}^4 \rightarrow N = \mathbb{Z}^3$  which maps the  $i$ th standard basis vector to the  $i$ th vertex of  $P$  in (2.1). Consider the transpose  ${}^t\rho: M = \mathbb{Z}^3 \rightarrow \mathbb{Z}^4$ . This is injective and its cokernel is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$  via the homomorphism  $\mathbb{Z}^4 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$  given by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \bar{0}^2 & \bar{1}^2 & \bar{0}^2 & \bar{1}^2 \\ \bar{0}^8 & \bar{5}^8 & \bar{1}^8 & \bar{6}^8 \end{pmatrix},$$

where  $\bar{\cdot}^2$  (respectively,  $\bar{\cdot}^8$ ) denotes the reduction modulo 2 (respectively, 8). The short exact sequence

$$0 \longrightarrow M \xrightarrow{{}^t\rho} \mathbb{Z}^4 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \longrightarrow 0$$

is the divisor sequence [25, Theorem 4.1.3], hence the divisor class group of  $X$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ . By [25, Proposition 4.2.5] the Picard group of  $X$  is free; moreover, since  $X$  is  $\mathbb{Q}$ -factorial, the Picard rank of  $X$  coincides with the rank of  $\text{Cl}(X)$ , which is 1. The presentation of  $X$  as a finite abelian quotient of  $\mathbb{P}^3$  follows from [25, Exercise 5.1.13].  $\square$

*Proof of (4).* Let  $\rho_1, \rho_2, \rho_3, \rho_4$  be the elements in  $N$  which appear in (2.1). For  $1 \leq i < j \leq 4$ , let  $\sigma_{ij} \in \Sigma$  the 2-dimensional cone with rays  $\mathbb{R}_{\geq 0}\rho_i$  and  $\mathbb{R}_{\geq 0}\rho_j$ . Let  $C_{ij}$  be the closure of the torus orbit on  $X$  associated to the cone  $\sigma_{ij}$ : it is a smooth rational curve. The shape of the cone  $\sigma_{ij}$  determines the singularities of  $X$  generically along  $C_{ij}$ . We write down the analysis for  $C_{12}$  and we omit the other cases.

We consider the 2-dimensional lattice  $N_{12} = N \cap (\mathbb{R}\rho_1 + \mathbb{R}\rho_2)$ . This is the orthogonal of  $(8, -2, -5) \in M$ . The vectors  $\rho_1$  and  $\rho_2$  form an  $\mathbb{R}$ -basis of  $\mathbb{R}\rho_1 + \mathbb{R}\rho_2 = N_{12} \otimes_{\mathbb{Z}} \mathbb{R}$ , but not a  $\mathbb{Z}$ -basis of  $N_{12}$ . The finite abelian group  $N_{12}/(\mathbb{Z}\rho_1 + \mathbb{Z}\rho_2)$  has order 2 and its generator is the class of

$$\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \in N_{12}.$$

This implies that the toric variety associated to the cone  $\sigma_{12}$  in the lattice  $N_{12}$  is the  $A_1$  surface singularity  $\text{Spec } \mathbb{C}[x, y, z]/(xy - z^2)$ . This implies that  $X$  has transverse  $A_1$  singularities generically along  $C_{12}$ .  $\square$

*Proof of (5).* By (4) the non-Gorenstein locus of  $X$  is contained in the set of the 4 torus-fixed points:  $p_{123}, p_{124}, p_{134}$ , and  $p_{234}$ . Here  $p_{ijk}$  is the torus-fixed point on  $X$  corresponding to the 3-dimensional cone  $\sigma_{ijk}$  with rays  $\mathbb{R}_{\geq 0}\rho_i, \mathbb{R}_{\geq 0}\rho_j$ , and  $\mathbb{R}_{\geq 0}\rho_k$ . We need to analyse the singularities of  $X$  at these points.

Let us start from  $p_{123}$ . The three vectors  $\rho_1, \rho_2$ , and  $\rho_3$  lie on the affine plane

$$H_{(2,0,-1),2} := \{v \in N_{\mathbb{R}} \mid \langle (2, 0, -1), v \rangle = 2\}.$$

Since the lattice vector  $(2, 0, -1) \in M$  is primitive, we get that the Gorenstein index of  $X$  at  $p_{123}$  is 2. Moreover, the singularity  $p_{123} \in X$  is not canonical because the lattice vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4}\rho_1 + \frac{1}{8}\rho_2 + \frac{1}{8}\rho_3$$

lies in the interior of the polytope with vertices  $0, \rho_1, \rho_2, \rho_3$  (see [25, Proposition 11.4.12b]).

Now consider  $p_{124}$ . The three vectors  $\rho_1, \rho_2$ , and  $\rho_4$  lie on the affine plane

$$H_{(-3,1,2),1} := \{v \in N_{\mathbb{R}} \mid \langle (-3, 1, 2), v \rangle = 1\}.$$

Since the lattice vector  $(-3, 1, 2) \in M$  is primitive, we get that the Gorenstein index of  $X$  at  $p_{123}$  is 1. By [25, Proposition 11.4.11] the singularity  $p_{124} \in X$  is canonical.

In an analogous way we can prove that  $p_{134}$  is a Gorenstein canonical singularity, whereas  $p_{234}$  is a non-canonical singularity with Gorenstein index 2.  $\square$

*Proof of (6).* For brevity, set  $\mathbb{P} := \mathbb{P}(1^5, 2)$ . The singular locus of  $\mathbb{P}$  consists of a single point  $p = [0 : 0 : 0 : 0 : 0 : 1]$ . For a contradiction assume that  $X$  is a  $(2, 4)$ -complete intersection in  $\mathbb{P}$  and consider the corresponding closed embedding  $X \hookrightarrow \mathbb{P}$ . We have that  $X \setminus \{p\} \hookrightarrow \mathbb{P} \setminus \{p\}$  is a regular closed embedding, because it is locally defined by the vanishing of the dehomogenisations of the quartic and the quadric that define  $X$  inside  $\mathbb{P}$ . This implies that  $X \setminus \{p\}$  is lci, and in particular Gorenstein. Therefore the non-Gorenstein locus of  $X$  is contained in  $\{p\}$ , so it is either empty or consists of one point. This contradicts (5).  $\square$

*Proof of (7).* In the lattice  $M \oplus \mathbb{Z}$  we consider the cone  $\tau$  whose apex is at the origin and which is spanned by  $P^\circ \times \{1\}$ . In other words,  $\tau$  is the cone over  $P^\circ$  placed at height 1. The primitive generators of  $\tau$  are

$$\begin{aligned} y_1 &= (-2, 0, 3, 2), \\ x_1 &= (-1, 1, 0, 1), \\ x_4 &= (3, -1, -2, 1), \\ y_0 &= (-2, 0, 1, 2). \end{aligned}$$

The Hilbert basis (i.e. the minimal set of generators) of the monoid  $\tau \cap (M \oplus \mathbb{Z})$  is made up of  $y_1, x_1, x_4, y_0, x_0 = (-1, 0, 1, 1), x_2 = (0, 0, 0, 1), x_3 = (1, 0, -1, 1)$ .

Since  $P^\circ$  is the moment polytope of the toric boundary of  $X$ , which is anticanonical and hence ample, we have that  $X = \text{Proj } \mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$ , where the  $\mathbb{N}$ -grading is given by the projection  $M \oplus \mathbb{Z} \rightarrow \mathbb{Z}$ . This shows that  $X$  is a closed subvariety of the weighted projective space  $\mathbb{P}(1^5, 2^2)$ , equipped with homogeneous coordinates  $x_0, \dots, x_4, y_0, y_1$ . It is easy to see that the equations of  $X$  in  $\mathbb{P}(1^5, 2^2)$  are

$$\begin{aligned} x_0^4 - y_0 y_1 &= 0, \\ x_2^2 - x_0 x_3 &= 0, \\ x_3^2 - x_1 x_4 &= 0. \end{aligned}$$

Thus  $X$  is a  $(2, 2, 4)$ -complete intersection in  $\mathbb{P}(1^5, 2^2)$ .  $\square$

*Proof of (8).* By using an argument similar to the one that appears in the introduction, one can see that the general  $(2, 2, 4)$ -complete intersection in  $\mathbb{P}(1^5, 2^2)$  is a quartic 3-fold.  $\square$

**Remark 2.2.** In addition to the example presented in Proposition 2.1, we found two additional Fano toric 3-folds which satisfy Theorem A.

- (1) Let  $X$  be the toric 3-fold associated to the spanning fan of the polytope in  $\mathbb{Z}^3$  with vertices  $(1, 3, 2), (1, 3, 0), (1, 0, 2), (1, 0, 0), (-1, -1, 2), (-1, -1, -4), (-1, -2, 2), (-1, -2, -4)$ . There are isomorphisms  $\text{Cl}(X) \simeq \mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z}$  and  $\text{Pic}(X) \simeq \mathbb{Z}$ . One can prove that  $X$  is the closed subvariety of  $\mathbb{P}(1^5, 2^2)$  defined by the equations

$$\begin{aligned} x_0^3 x_3 - y_0 y_1 &= 0, \\ x_2^2 - x_0 x_3 &= 0, \\ x_2 x_3 - x_1 x_4 &= 0. \end{aligned}$$

- (2) Let  $X$  be the toric 3-fold associated to the spanning fan of the polytope in  $\mathbb{Z}^3$  with vertices  $(3, 4, 4), (3, 2, 4), (1, 2, 0), (1, 0, 0), (-1, 0, 0), (-1, -2, 0), (-3, -2, -4), (-3, -4, -4)$ . There are isomorphisms  $\text{Cl}(X) \simeq \mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$  and  $\text{Pic}(X) \simeq \mathbb{Z}$ . One can prove that  $X$  is the closed subvariety of  $\mathbb{P}(1^5, 2^2)$  defined by the equations

$$\begin{aligned} x_0^2 x_4^2 - y_0 y_1 &= 0, \\ x_2^2 - x_0 x_4 &= 0, \\ x_1 x_3 - x_0 x_4 &= 0. \end{aligned}$$

Each of these examples is a  $(2, 2, 4)$ -complete intersection in  $\mathbb{P}(1^5, 2^2)$  and has two singular points with Gorenstein index 2.

**Remark 2.3.** In addition to the three examples presented in Proposition 2.1 and in Remark 2.2 we know a further K-polystable Fano toric 3-fold which deforms to quartic 3-folds: this is the toric variety associated to the spanning fan of the polytope with vertices

$$\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad (2.2)$$

in  $N = \mathbb{Z}^3$ , and it is the  $\mathbb{Q}$ -factorial hypersurface  $\{x_1x_2x_3x_4 - x_0^4 = 0\}$  in  $\mathbb{P}^4$ . We conjecture that these four toric varieties are the only K-polystable Fano toric 3-folds that deform to quartic 3-folds.

**Remark 2.4.** A Fano polytope  $P$  is called *symmetric* if the only point which is fixed by every automorphism of  $P$  is the origin. According to [33, Conjecture 1.6] it is expected that if  $P$  is a 2-dimensional Fano polytope such that its polar  $P^\circ$  has barycentre at the origin then either  $P$  is symmetric or  $P$  is a triangle (i.e. simplex). The examples in Remark 2.2 show that this is not true in higher dimension

**Remark 2.5.** According to [32] it is expected that in every mutation-equivalence class of 2-dimensional Fano polytopes there is at most one polytope  $P$  such that its polar  $P^\circ$  has barycentre at the origin. This is not true in higher dimension; for instance, the four polytopes presented in Proposition 2.1, Remark 2.2, and Remark 2.3 are mutation-equivalent and the polar of each of them has barycentre at the origin.

### 3. THE INFINITE FAMILY

3.1. **Statement.** The infinite family of examples satisfying Theorem A is given by the following:

**Theorem 3.1.** *Let  $x_0, \dots, x_4, y_0, y_1$  be the homogeneous coordinates of the weighted projective space  $\mathbb{P}(1^5, 2^2)$  with degrees  $1, \dots, 1, 2, 2$  respectively. Consider  $x_0, \dots, x_4$  also as the homogeneous coordinates of  $\mathbb{P}^4$ . Let  $f$  be a quartic in  $x_0, \dots, x_4$  and let  $g$  and  $h$  be two quadrics in  $x_0, \dots, x_4$ . Consider the zero-loci*

$$\begin{aligned} \Delta &= \{f = 0, g = h = 0\} \subset \mathbb{P}^4, \\ S &= \{g = h = 0\} \subset \mathbb{P}^4, \\ X &= \{y_0y_1 - f = 0, g = h = 0\} \subset \mathbb{P}(1^5, 2^2). \end{aligned} \quad (3.1)$$

Assume that  $\Delta$  is a smooth curve and that  $S$  is a smooth surface. Then the following statements hold.

- (1) The surface  $S$  is a smooth del Pezzo of degree 4 and contains exactly 16 lines; their intersection points form a finite subset  $\Sigma \subset S$  consisting of 40 points.
- (2)  $X$  is a klt Fano 3-fold and the group  $G = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$  acts faithfully on  $X$ .
- (3) If  $\Sigma \cap \Delta = \emptyset$ , then  $X$  is K-polystable.

**Remark 3.2.** The last condition  $\Sigma \cap \Delta = \emptyset$  is satisfied for general choices of  $f, g, h$ . Therefore a general  $X$  in Theorem 3.1 is K-polystable. We believe that the condition  $\Sigma \cap \Delta = \emptyset$  can be removed from Theorem 3.1(3).

**Remark 3.3.** Note that if  $X \subset \mathbb{P}(1^5, 2)$  is a hyperelliptic cone, i.e. the term  $y^2$  does not appear in the degree 4 equation, or if  $X \subset \mathbb{P}(1^5, 2^2)$  is a cone, i.e. the quadratic part in the variables  $y_0, y_1$  of the degree 4 equation  $f$  does not have rank 2, then  $X$  is unstable. This can be checked rather easily using [59, Theorem 1.4] by setting  $r = 1/2, c = 0$ , and  $n = 3$  to obtain  $\delta(X) \leq 5/12$ .

The remaining of the section is devoted to giving a proof of Theorem 3.1, hence we always work in the setting of Theorem 3.1. We begin with an explanation of the construction of the models, their symmetries, and the generality condition on them. We then present the proof of the main claim about their K-stability; the proof uses the theory of refinements introduced in [1], which provides a lower bound for stability thresholds. Indeed, some formulae are readily available in [7, §1.7] which compute the refinement for a flag, if the flag is geometrically realised on a Mori Dream Space birational model of the Fano variety. We will use them in the proof of Theorem 3.1(3).

#### 3.2. The models and their symmetries.

*Proof of Theorem 3.1(1).* All of this is very classical. □

Conversely, if  $S$  is a smooth del Pezzo surface of degree 4, then we can view  $S$  as the complete intersection  $\{g = h = 0\} \subset \mathbb{P}^4$  where  $g$  and  $h$  are quadrics in  $x_0, \dots, x_4$ . We denote by  $\Sigma$  the set consisting of the 40 intersection points of the lines on  $S$ .

**Remark 3.4.** Let  $p$  be a point in  $S$ .

- (1) If  $p$  is not contained in any line in  $S$  then there are exactly ten smooth conics in  $S$  that contain  $p$ ; moreover, five of these ten conics can be chosen to intersect pairwise transversally at  $p$ .
- (2) If  $p$  is contained in a line in  $S$  and  $p \notin \Sigma$ , then there are exactly five smooth conics in  $S$  that contain  $p$ , and any two of them intersect transversally at  $p$ .
- (3) If  $p \in \Sigma$ , then it is contained in a unique smooth conic in  $S$ .

*Proof of Theorem 3.1(2).* We study the singularities of  $X$  by looking at the affine charts of  $\mathbb{P}(1^5, 2^2)$ .

Let us consider the chart  $x_0 \neq 0$ ; this is isomorphic to  $\mathbb{A}^6$  with affine coordinates  $x_1, \dots, x_4, y_0, y_1$ . Inside this  $\mathbb{A}^6$  the variety  $X$  is given by the equations

$$\begin{aligned} y_0 y_1 - \bar{f} &= 0 \\ \bar{g} &= 0 \\ \bar{h} &= 0 \end{aligned}$$

where  $\bar{f} = f|_{x_0=1}$  and similarly for  $\bar{g}$  and  $\bar{h}$ . The jacobian matrix of the equations of  $X$  in  $\mathbb{A}^6$  is

$$J = \begin{pmatrix} -\partial_{x_1} \bar{f} & -\partial_{x_2} \bar{f} & -\partial_{x_3} \bar{f} & -\partial_{x_4} \bar{f} & y_1 & y_0 \\ \partial_{x_1} \bar{g} & \partial_{x_2} \bar{g} & \partial_{x_3} \bar{g} & \partial_{x_4} \bar{g} & 0 & 0 \\ \partial_{x_1} \bar{h} & \partial_{x_2} \bar{h} & \partial_{x_3} \bar{h} & \partial_{x_4} \bar{h} & 0 & 0 \end{pmatrix}.$$

Since  $S \subset \mathbb{P}^4$  is a smooth surface, the bottom-left  $2 \times 4$  submatrix of  $J$  has rank 2 in all points of  $X$ . Therefore it is clear that  $J$  has rank 3 at all points of  $X$  which satisfy  $y_0 \neq 0$  or  $y_1 \neq 0$ . The points of  $X$  which satisfy  $y_0 = y_1 = 0$  also satisfy  $f = 0$ , and from the smoothness of  $\Delta$  it follows that the rank of  $J$  is 3 at these points. Therefore the intersection of  $X$  with the chart  $x_0 \neq 0$  is smooth. Similarly, this holds also for the other charts  $x_i \neq 0$  for  $i \in \{0, \dots, 4\}$ . In other words, we have proven that the singular points of  $X$  must satisfy  $x_0 = \dots = x_4 = 0$ . There are exactly two such points:

$$p_0 = [0 : 0 : 0 : 0 : 0 : 1 : 0] \quad \text{and} \quad p_1 = [0 : 0 : 0 : 0 : 0 : 0 : 1].$$

We need to study the singularity type of these two points on  $X$ .

The chart  $y_0 \neq 0$  on  $\mathbb{P}(1^5, 2^2)$  gives the quotient singularity  $\frac{1}{2}(1, 1, 1, 1, 1, 0)$  with orbifold coordinates  $x_0, \dots, x_4, y_1$ . It is easy to see that  $X \cap \{y_0 \neq 0\}$  is isomorphic to  $\{g = h = 0\} \subset \frac{1}{2}(1, 1, 1, 1, 1)$ . Since  $\frac{1}{2}(1, 1, 1, 1, 1)$  is the affine cone over  $(\mathbb{P}^4, \mathcal{O}(2))$ , we have that  $\{g = h = 0\} \subset \frac{1}{2}(1, 1, 1, 1, 1)$  is the affine cone over  $(S, -2K_S)$ , which is a klt singularity. This shows that  $p_0$  and  $p_1$  are klt singularities of  $X$ .

By adjunction  $-K_X = \mathcal{O}_{\mathbb{P}(1^5, 2^2)}(1)|_X$ , so  $X$  is a klt Fano 3-fold. Its degree is  $(-K_X)^3 = \frac{1^3 \cdot 4 \cdot 2 \cdot 2}{2^2} = 4$ .

Now we need to construct an effective action of  $G = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$  on  $X$ . Consider the  $\mathbb{C}^*$ -action given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto [x_0 : x_1 : x_2 : x_3 : x_4 : \lambda y_0 : \lambda^{-1} y_1]$$

for  $\lambda \in \mathbb{C}^*$ . Furthermore, the group  $\text{Aut}(X)$  also contains an involution  $\sigma$  that is given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto [x_0 : x_1 : x_2 : x_3 : x_4 : y_1 : y_0].$$

Together, they generate a subgroup  $G \subset \text{Aut}(X)$  that is isomorphic to  $\mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ . Note that the two singular points  $p_0$  and  $p_1$  on  $X$  are swapped by the action of  $\sigma$  and fixed by the action of  $\mathbb{C}^*$ . This concludes the proof of Theorem 3.1(2).  $\square$

Let  $\rho: X \dashrightarrow S$  be the rational map given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto [x_0 : x_1 : x_2 : x_3 : x_4].$$

Then  $\rho$  is undefined precisely at the points  $p_0$  and  $p_1$ , resulting in the following  $G$ -equivariant commutative diagram:

$$\begin{array}{ccc} & \widetilde{X} & \\ \pi \swarrow & & \searrow \eta \\ X & \overset{\rho}{\dashrightarrow} & S \end{array} \quad (3.2)$$

where  $\widetilde{X}$  is a smooth projective 3-fold,  $\pi$  is a birational morphism that contracts two irreducible smooth surfaces  $E_0$  and  $E_1$  to the points  $p_0$  and  $p_1$ , respectively, and  $\eta$  is a  $G$ -equivariant conic bundle. Furthermore,



the surfaces  $E_0$  and  $E_1$  are sections of the conic bundle  $\eta$ , so that  $\eta$  induces isomorphisms  $E_0 \cong S$  and  $E_1 \cong S$ . We have  $E_0|_{E_0} \sim -2K_{E_0}$  and  $E_1|_{E_1} \sim -2K_{E_1}$ , which gives

$$-K_{\tilde{X}} \sim_{\mathbb{Q}} \pi^*(-K_X) + \frac{1}{2}(E_0 + E_1) \sim_{\mathbb{Q}} \eta^*(-K_S) + E_0 + E_1.$$

**Corollary 3.5.** *The 3-fold  $\tilde{X}$  is a Mori Dream Space.*

*Proof.* Let  $\varepsilon$  be a sufficiently small positive rational number. Then  $(\tilde{X}, \frac{1+\varepsilon}{2}(E_0 + E_1))$  has Kawamata log terminal singularities, and  $-(K_{\tilde{X}} + \frac{1+\varepsilon}{2}(E_0 + E_1))$  is ample. The claim follows from [12, Corollary 1.3.1].  $\square$

Let  $R_0$  and  $R_1$  be the surfaces in  $X$  that are cut out by  $y_0 = f = 0$  and  $y_1 = f = 0$ , respectively. Denote by  $\tilde{R}_0$  and  $\tilde{R}_1$  the strict transforms of  $R_0$  and  $R_1$  on  $\tilde{X}$ , respectively. Then  $\eta^*(\Delta) = \tilde{R}_0 + \tilde{R}_1$ , and  $\Delta$  is the discriminant curve of the conic bundle  $\eta$ . We have the following non- $G$ -equivariant commutative diagram

$$\begin{array}{ccccc} & & U & \xleftarrow{\eta_0} & \tilde{X} & \xrightarrow{\eta_1} & U & & \\ & \psi \swarrow & & & \downarrow \eta & & & \searrow \psi & \\ V & & & & & & & & V \\ & \tau \swarrow & & & \downarrow \phi & & & \searrow \phi & \\ & & & & S & & & & \end{array} \quad (3.3)$$

where  $U = \mathbb{P}(\mathcal{O}_S(2K_S) \oplus \mathcal{O}_S)$ ,  $\phi$  is the  $\mathbb{P}^1$ -bundle given by the projection  $\mathbb{P}(\mathcal{O}_S(2K_S) \oplus \mathcal{O}_S) \rightarrow S$ , the morphism  $\psi$  is a contraction of the negative section of the  $\mathbb{P}^1$ -bundle  $\phi$ ,  $V$  is a cone over  $S$ , the map  $\tau$  is the projection from the vertex of the cone, and morphisms  $\eta_0$  and  $\eta_1$  are birational contractions of the surfaces  $\tilde{R}_0$  and  $\tilde{R}_1$ , respectively.

**Remark 3.6.** In the left hand side of (3.3), we have  $\phi^*(\Delta) = \eta_0(R_1)$ , the morphism  $\psi$  contracts  $\eta_0(E_0)$ , and  $\psi \circ \eta_0(E_1)$  is a smooth hyperplane section of the cone  $V \subset \mathbb{P}^{13}$ . In the right hand side of (3.3), we have  $\phi^*(\Delta) = \eta_1(R_0)$ ,  $\psi$  contracts  $\eta_1(E_1)$ , and  $\psi \circ \eta_1(E_0)$  is a hyperplane section of the cone  $V$ . The involution  $\sigma$  swaps the left and the right hand sides of the diagram (3.3).

Set  $Z = R_0 \cap R_1$  and  $\tilde{Z} = \tilde{R}_0 \cap \tilde{R}_1$ . Then  $Z = \pi(\tilde{Z})$ ,  $Z \cong \tilde{Z} \cong \Delta$ , and  $Z = \{y_0 = y_1 = f = 0\} \subset X$ . Observe that  $p_0 \notin Z$ ,  $p_1 \notin Z$ ,  $\tilde{Z} \cap E_0 = \tilde{Z} \cap E_1 = \emptyset$ , and every point in  $Z$  and  $\tilde{Z}$  is  $G$ -invariant.

**Lemma 3.7.** *Let  $q$  be a  $G$ -invariant point in  $\tilde{X}$ , let  $C$  be a  $G$ -invariant irreducible curve in  $\tilde{X}$ , and let  $B$  be a  $G$ -invariant irreducible surface in  $\tilde{X}$ . Then the following assertions hold:*

- (1)  $q \in \tilde{Z}$ ;
- (2) either  $C = \tilde{Z}$  or  $C$  is a smooth fibre of the conic bundle  $\eta$ ;
- (3)  $B = \eta^*(\mathcal{C})$  for some irreducible curve  $\mathcal{C} \subset S$ .

In particular, we have that  $B \cap \tilde{Z} \neq \emptyset$ .

*Proof.* Left to the reader.  $\square$

**3.3. K-stability of the general models.** The proof of Theorem 3.1(3) is quite involved. However, it is less difficult to produce specific examples of  $X$  with large symmetries and prove they are K-polystable by taking advantage of the group of symmetries. Here, for illustration, we present one such case. Then we proceed by proving that a general  $X$  is K-polystable (see Theorem 3.1(3)).

*Special case of Theorem 3.1(3): diagonal models.* Suppose that

$$\begin{aligned} f &= \alpha_0 x_0^4 + \alpha_1 x_1^4 + \alpha_2 x_2^4 + \alpha_3 x_3^4 + \alpha_4 x_4^4, \\ g &= x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ h &= \epsilon_0 x_0^2 + \epsilon_1 x_1^2 + \epsilon_2 x_2^2 + \epsilon_3 x_3^2 + \epsilon_4 x_4^2, \end{aligned}$$

where  $\alpha_0, \dots, \alpha_4$  are sufficiently general complex numbers and  $\epsilon_0, \dots, \epsilon_4$  are pairwise distinct complex numbers. Then the group  $\text{Aut}(X)$  is larger than the group  $G$  in Theorem 3.1(2), because  $\text{Aut}(X)$  also contains 15 additional involutions given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto [x_0 : (-1)^a x_1 : (-1)^b x_2 : (-1)^c x_3 : (-1)^d x_4 : y_0 : y_1]$$

for  $a, b, c, d \in \{0, 1\}$ . Let  $\Gamma$  be the subgroup in  $\text{Aut}(X)$  generated by  $G$  and these 15 involutions. Then  $\Gamma \cong \mathbb{C}^* \rtimes (\mathbb{Z}/2\mathbb{Z})^5$ , and (3.2) is  $\Gamma$ -equivariant. Note that  $X$  does not contain  $\Gamma$ -invariant points. Furthermore, using the diagram (3.2) and [27, Theorem 6.9], we conclude that  $\text{Cl}^G(X) = \mathbb{Z}[-K_X]$ . Now, arguing as in the proof of [7, Theorem 1.4.11], we conclude that

$$\alpha_\Gamma(X) \geq 1,$$

where  $\alpha_\Gamma(X)$  is the  $\Gamma$ -invariant  $\alpha$ -invariant of Tian for the 3-fold  $X$ , see [55] for the analytic definition and [19] for an algebraic definition of  $\alpha$ . It follows that  $X$  is  $K$ -polystable by [55] as

$$\alpha_\Gamma(X) > \frac{3}{4} = \frac{\dim X}{1 + \dim \bar{X}}.$$

This concludes the claim that  $X$  is  $K$ -polystable.  $\square$

*Proof of Theorem 3.1(3).* Suppose that  $X$  is not  $K$ -polystable. By [62, Corollary 4.14], there exists a  $G$ -invariant prime divisor  $F$  over  $X$  for which  $\beta(F) \leq 0$ . For the definition of  $\beta(F)$ , see [30, 38]. Let  $\mathfrak{C}$  be the center of the divisor  $F$  on  $X$ . Then, for every point  $P \in \mathfrak{C}$ , we must have  $\delta_P(X) \leq 1$ , where  $\delta_P(X)$  is the local stability threshold of  $X$  at  $p$ . For the precise definition of  $\delta_P(X)$ , see [1, Definition 2.5] and [7, § 1.5]. On the other hand, by Lemma 3.7, we have the following four possibilities:

- (1)  $\mathfrak{C}$  is a  $G$ -invariant point in  $Z$ ;
- (2)  $\mathfrak{C}$  is the curve  $Z$ ;
- (3)  $\mathfrak{C}$  is an irreducible fibre of the rational map  $\rho: X \dashrightarrow S$ ;
- (4)  $\mathfrak{C}$  is a  $G$ -invariant surface and  $\mathfrak{C} \cap Z \neq \emptyset$ .

Hence, we see that either  $\mathfrak{C}$  is an irreducible curve that is a fibre of the rational map  $\rho: X \dashrightarrow S$ , or the curve  $Z$  contains a point  $x$  such that  $\delta_x(X) \leq 1$ . We will show that both cases are impossible, by first lifting the problem to the level of  $\tilde{X}$ , then using the results obtained in [1] and [7, § 1.7] to derive a contradiction.

Since  $\tilde{X}$  is smooth and it is a Mori Dream Space, it is more convenient to work on  $\tilde{X}$  than on  $X$ . Namely, we let  $L = \pi^*(-K_X)$ , and we let  $\tilde{\mathfrak{C}}$  be the center of the divisor  $F$  on the 3-fold  $\tilde{X}$ . Then  $\tilde{\mathfrak{C}}$  is the strict transform of the center  $\mathfrak{C}$ , and we have the following four possibilities:

- (1)  $\tilde{\mathfrak{C}}$  is a  $G$ -invariant point in  $\tilde{Z}$ ;
- (2)  $\tilde{\mathfrak{C}}$  is the curve  $\tilde{Z}$ ;
- (3)  $\tilde{\mathfrak{C}}$  is a smooth fibre of the conic bundle  $\eta: \tilde{X} \rightarrow S$ ;
- (4)  $\tilde{\mathfrak{C}}$  is a  $G$ -invariant surface and  $\tilde{\mathfrak{C}} \cap \tilde{Z} \neq \emptyset$ .

Note that  $\delta_x(\tilde{X}, L) = \delta_{\pi(x)}(X)$  for every point  $x \in \tilde{Z}$ . For the definition of  $\delta_x(\tilde{X}, L)$ , see [1, Definition 2.5] and [7, § 1.5]. Therefore, we conclude that one of the following two cases hold:

- ( $\diamond$ )  $\tilde{\mathfrak{C}}$  a smooth fibre of the conic bundle  $\eta: \tilde{X} \rightarrow S$ ,
- ( $\heartsuit$ ) there exists a point  $x \in \tilde{Z}$  such that  $\delta_x(\tilde{X}, L) \leq 1$ .

In both cases, let us introduce a new curve  $\mathcal{C}$  in the 3-fold  $\tilde{X}$  as follows:

- in case ( $\diamond$ ), we let  $\mathcal{C} = \tilde{\mathfrak{C}}$ ,
- in case ( $\heartsuit$ ), we let  $\mathcal{C}$  be the (singular) fibre of the conic bundle  $\eta$  that contains  $x$ .

Let  $p = \eta(\mathcal{C})$ . It follows from Remark 3.4 that  $S$  contains a smooth conic  $\mathcal{C}$  such that  $p \in \mathcal{C}$ . Moreover, if  $p \in \Delta$ , it follows from Remark 3.4 that we can also choose  $\mathcal{C}$  such that it intersects the curve  $\Delta$  transversally at the point  $p$ , because  $\Sigma \cap \Delta = \emptyset$  by assumption.

Let  $B = \eta^*(\mathcal{C})$ , and let  $B' = \eta^*(Z)$  for a general conic  $Z \in |-K_S - \mathcal{C}|$ . Then  $B$  is normal, it has at most Du Val singularities of type  $\mathbb{A}$ , it is smooth along the curve  $\mathcal{C}$ , and  $B'$  is smooth.

Let us compute  $\beta(B)$ . We have  $\beta(B) = A_X(B) - S_L(B) = 1 - S_L(B)$ , where

$$S_L(B) = \frac{1}{L^3} \int_0^\infty \text{vol}(L - uB) du.$$

For  $u \in \mathbb{R}_{\geq 0}$ , the divisor  $L - uB$  is pseudo-effective  $\iff u \leq 1$ , because

$$L - uB \sim_{\mathbb{Q}} (1 - u)B + B' + \frac{1}{2}(E_0 + E_1).$$

For  $u \in [0, 1]$ , let  $P(u)$  be the positive part of the Zariski decomposition of the divisor  $L - uB$ , and let  $N(u)$  be the negative part of the Zariski decomposition of the divisor  $L - uB$ . Then

$$P(u) = L - uB - \frac{u}{2}(E_0 + E_1)$$

and  $N(u) = \frac{u}{2}(E_0 + E_1)$  for every  $u \in [0, 1]$ . This gives

$$S_L(B) = \frac{1}{L^3} \int_0^\infty \text{vol}(L - uB) du = \frac{1}{4} \int_0^1 (P(u))^3 du = \frac{1}{4} \int_0^1 (2u^3 - 6u + 4) du = \frac{3}{8},$$

which implies that  $\beta(B) = \frac{5}{8}$ . Here, we used the following intersections on the 3-fold  $\widetilde{X}$ :

$$\begin{aligned} B^3 = 0, (B')^3 = 0, E_0^3 = 16, E_1^3 = 16, E_0^2 \cdot E_1 = 0, E_0 \cdot E_1^2 = 0, E_0 \cdot B \cdot E_1 = 0, \\ E_0 \cdot B' \cdot E_1 = 0, E_0 \cdot B \cdot B' = 2, B \cdot B' \cdot E_1 = 2, E_0 \cdot B^2 = 0, B^2 \cdot B' = 0, E_0 \cdot (B')^2 = 0, B \cdot (B')^2 = 0, \\ B^2 \cdot E_1 = 0, E_0^2 \cdot B = -4, B \cdot E_1^2 = -4, E_0^2 \cdot B' = -4, B' \cdot E_1^2 = -4, (B')^2 \cdot E_1 = 0. \end{aligned}$$

We set  $\mathbf{e}_0 = E_0|_B$  and  $\mathbf{e}_1 = E_1|_B$ . Then  $\mathbf{e}_0$  and  $\mathbf{e}_1$  are smooth irreducible rational disjoint curves, and  $B$  is smooth along  $\mathbf{e}_0$  and  $\mathbf{e}_1$ . On  $B$ , we have  $\mathbf{e}_0^2 = \mathbf{e}_1^2 = -4$ ,  $\mathbf{e}_0 \cdot \mathbf{e}_1 = 0$ ,  $\mathbf{e}_0 \cdot C = \mathbf{e}_1 \cdot C = 1$ .

Suppose that case  $(\diamond)$  holds. Then  $\widetilde{C}$  is a smooth fibre of the conic bundle  $\eta$ , and  $C = \widetilde{C}$ . Set

$$S(W_{\bullet, \bullet}^B; C) = \frac{3}{L^3} \int_0^1 \int_0^\infty \text{vol}(P(u)|_B - vC) dv du.$$

Using [7, Corollary 1.7.26], we get  $S(W_{\bullet, \bullet}^B; C) \geq 1$ , since  $\beta(\mathbf{F}) \leq 0$ ,  $\beta(B) > 0$  and  $C \not\subset \text{Supp}(N(u))$ . On the other hand, it is easy to compute  $S(W_{\bullet, \bullet}^B; C)$ . Indeed, take  $v \in \mathbb{R}_{\geq 0}$ . Then

$$P(u)|_B - vC \sim_{\mathbb{R}} (2 - v)C + \frac{1 - u}{2}(\mathbf{e}_0 + \mathbf{e}_1).$$

Therefore, the divisor  $P(u)|_B - vC$  is nef for  $v \leq 2u$ , and it is not pseudo-effective for  $v > 2$ . Moreover, if  $v \in [2u, 2]$ , the positive part of its Zariski decomposition is  $\frac{2-v}{4}(4C + \mathbf{e}_0 + \mathbf{e}_1)$ . Then

$$\text{vol}(P(u)|_B - vC) = \begin{cases} 2 - 2u^2 + 2uv - 2v & \text{if } 0 \leq v \leq 2u, \\ \frac{(v-2)^2}{2} & \text{if } 2u \leq v \leq 2, \\ 0 & \text{if } v \geq 2. \end{cases}$$

Integrating, we get  $S(W_{\bullet, \bullet}^B; C) = \frac{3}{4}$ , which is a contradiction. Thus, case  $(\diamond)$  does not hold.

Now, we assume that the case  $(\heartsuit)$  holds, and derive a contradiction. Recall that this means that  $\delta_x(\widetilde{X}, L) \leq 1$  for some point  $x \in \widetilde{Z}$ . In this case, the curve  $C$  is singular. Namely, we have  $C = C_0 + C_1$ , where  $C_0$  and  $C_1$  are smooth irreducible rational curves that intersect each other transversally at  $x$ . Without loss of generality, we may assume that  $C_0 \cap E_0 \neq \emptyset$ . Moreover, since the surface  $B$  is smooth along  $C$ , the numerical intersections of the curves  $C_0, C_1, \mathbf{e}_0, \mathbf{e}_1$  on the surface  $B$  are given in the following table:

	$C_0$	$C_1$	$\mathbf{e}_0$	$\mathbf{e}_1$
$C_0$	-1	1	1	0
$C_1$	1	-1	0	1
$\mathbf{e}_0$	1	0	-4	0
$\mathbf{e}_1$	0	1	0	-4

Let us estimate  $\delta_x(\widetilde{X}, L)$  using [7, Theorem 1.7.30]. We have  $N(u)|_B = \frac{u}{2}(\mathbf{e}_0 + \mathbf{e}_1)$  and  $x \notin \mathbf{e}_0 \cup \mathbf{e}_1$ . For every  $u \in [0, 1]$ , we let

$$t(u) = \inf \left\{ v \in \mathbb{R}_{\geq 0} \mid \text{the divisor } P(u)|_B - vC_0 \text{ is pseudo-effective} \right\}.$$

For  $v \in [0, t(u)]$ , we let  $P(u, v)$  be the positive part of the Zariski decomposition of  $P(u)|_B - vC_0$ , and we let  $N(u, v)$  be its negative part. Then we set

$$S(W_{\bullet, \bullet}^B; C_0) = \frac{3}{L^3} \int_0^1 \int_0^{t(u)} \text{vol}(P(u)|_B - vC_0) dv du.$$

Note that  $C_0 \not\subset \text{Supp}(N(u, v))$  for every  $u \in [0, 1)$  and  $v \in (0, t(u))$ . Thus, we can let

$$F_x(W_{\bullet, \bullet, \bullet}^B; C_0) = \frac{6}{L^3} \int_0^1 \int_0^{t(u)} (P(u, v) \cdot C_0) \cdot \text{ord}_x(N(u, v)|_{C_0}) dv du.$$

Finally, we let

$$S(W_{\bullet, \bullet, \bullet}^{B, C_0}; x) = \frac{3}{L^3} \int_0^1 \int_0^{t(u)} (P(u, v) \cdot C_0)^2 dv du + F_x(W_{\bullet, \bullet, \bullet}^{B, C_0}).$$

Then, since  $C_0 \notin \text{Supp}(N(u))$ , it follows from [7, Theorem 1.7.30] that

$$1 \geq \delta_x(\tilde{X}, L) \geq \min \left\{ \frac{1}{S(W_{\bullet, \bullet, \bullet}^{B, C_0}; x)}, \frac{1}{S(W_{\bullet, \bullet, \bullet}^B; C_0)}, \frac{1}{S_L(B)} \right\}.$$

Recall that  $S_L(B) = \frac{3}{8}$ . Thus, either  $S(W_{\bullet, \bullet, \bullet}^{B, C_0}; x) \geq 1$  or  $S(W_{\bullet, \bullet, \bullet}^B; C_0) \geq 1$  (or both).

Let us compute  $S(W_{\bullet, \bullet, \bullet}^{B, C_0}; x)$  and  $S(W_{\bullet, \bullet, \bullet}^B; C_0)$ . As above, take  $v \in \mathbb{R}_{\geq 0}$ . Then

$$P(u)|_B - vC_0 \sim_{\mathbb{R}} (2-v)C_0 + 2C_1 + \frac{1-u}{2}(\mathbf{e}_0 + \mathbf{e}_1).$$

Therefore, since the intersection form of the curves  $C_1, \mathbf{e}_0, \mathbf{e}_1$  on the surface  $B$  is negative definite, we see that  $P(u)|_B - vC_0$  is pseudoeffective  $\iff v \leq 2$ , so  $t(u) = 2$ . Moreover, if  $0 \leq u \leq \frac{1}{5}$ , then

$$P(u, v) = \begin{cases} (2-v)C_0 + 2C_1 + \frac{1}{2}(1-u)(\mathbf{e}_0 + \mathbf{e}_1) & \text{if } 0 \leq v \leq 2u, \\ \frac{1}{4}(2-v)(4C_0 + \mathbf{e}_0) + 2C_1 + \frac{1}{2}(1-u)\mathbf{e}_1 & \text{if } 2u \leq v \leq \frac{1}{2}(1-u), \\ \frac{1}{4}(2-v)(4C_0 + \mathbf{e}_0) + \frac{1}{2}(5-2v-u)C_1 + \frac{1}{2}(1-u)\mathbf{e}_1 & \text{if } 1-u \leq 2v \leq 1+3u, \\ \frac{1}{12}(2-v)(12C_0 + 16C_1 + 3\mathbf{e}_0 + 4\mathbf{e}_1) & \text{if } \frac{1}{2}(1+3u) \leq v \leq 2, \end{cases}$$

and,

$$N(u, v) = \begin{cases} 0 & \text{if } 0 \leq v \leq 2u, \\ \frac{1}{4}(v-2u)\mathbf{e}_0 & \text{if } 2u \leq v \leq \frac{1}{2}(1-u), \\ \frac{1}{4}(v-2u)\mathbf{e}_0 + \frac{1}{2}(2v+u-1)C_1 & \text{if } 1-u \leq 2v \leq 1+3u, \\ \frac{1}{4}(v-2u)\mathbf{e}_0 + \frac{1}{3}(4v-2)C_1 + \frac{1}{6}(2v-3u-1)\mathbf{e}_1 & \text{if } \frac{1}{2}(1+3u) \leq v \leq 2, \end{cases}$$

which compute

$$P(u, v) \cdot C_0 = \begin{cases} \frac{1}{2}(1-u+2v) & \text{if } 0 \leq v \leq 2u, \\ \frac{1}{4}(2+3v) & \text{if } 2u \leq v \leq \frac{1}{2}(1-u), \\ \frac{1}{4}(4-2u-v) & \text{if } 1-u \leq 2v \leq 1+3u, \\ \frac{1}{12}(14-7v) & \text{if } \frac{1+3u}{2} \leq v \leq 2, \end{cases}$$

$$\text{vol}(P(u)|_B - vC_0) = \begin{cases} 2-2u^2+uv-v^2-v & \text{if } 0 \leq v \leq 2u, \\ 2-v-u^2-\frac{3}{4}v^2 & \text{if } 2u \leq v \leq \frac{1}{2}(1-u), \\ \frac{1}{4}(9-3u^2+4uv+v^2-2u-8v) & \text{if } 1-u \leq 2v \leq 1+3u, \\ \frac{7}{12}(2-v)^2 & \text{if } \frac{1}{2}(1+3u) \leq v \leq 2. \end{cases}$$

Similarly, if  $\frac{1}{5} \leq u \leq 1$ , then

$$P(u, v) = \begin{cases} (2-v)C_0 + 2C_1 + \frac{1}{2}(1-u)(\mathbf{e}_0 + \mathbf{e}_1) & \text{if } 0 \leq v \leq \frac{1}{2}(1-u), \\ (2-v)C_0 + \frac{1}{2}(5-2v-u)C_1 + \frac{1}{2}(1-u)(\mathbf{e}_0 + \mathbf{e}_1) & \text{if } \frac{1}{2}(1-u) \leq v \leq 2u, \\ \frac{1}{4}(2-v)(4C_0 + \mathbf{e}_0) + \frac{1}{2}(5-2v-u)C_1 + \frac{1}{2}(1-u)\mathbf{e}_1 & \text{if } 1-u \leq 2v \leq 1+3u, \\ \frac{1}{12}(2-v)(12C_0 + 16C_1 + 3\mathbf{e}_0 + 4\mathbf{e}_1) & \text{if } \frac{1}{2}(1+3u) \leq v \leq 2, \end{cases}$$

$$N(u, v) = \begin{cases} 0 & \text{if } 0 \leq v \leq \frac{1}{2}(1-u), \\ \frac{1}{2}(2v+u-1)C_1 & \text{if } \frac{1}{2}(1-u) \leq v \leq 2u, \\ \frac{1}{4}(v-2u)\mathbf{e}_0 + \frac{1}{2}(2v+u-1)C_1 & \text{if } 1-u \leq 2v \leq 1+3u, \\ \frac{1}{4}(v-2u)\mathbf{e}_0 + \frac{1}{3}(4v-2)C_1 + \frac{1}{6}(2v-3u-1)\mathbf{e}_1 & \text{if } \frac{1}{2}(1+3u) \leq v \leq 2, \end{cases}$$

$$P(u, v) \cdot C_0 = \begin{cases} \frac{1}{2}(1-u+2v) & \text{if } 0 \leq v \leq \frac{1}{2}(1-u), \\ 1-u & \text{if } \frac{1}{2}(1-u) \leq v \leq 2u, \\ \frac{1}{4}(4-2u-v) & \text{if } 1-u \leq 2v \leq 1+3u, \\ \frac{1}{12}(14-7v) & \text{if } \frac{1}{2}(1+3u) \leq v \leq 2, \end{cases}$$

$$\text{vol}(P(u)|_B - vC_0) = \begin{cases} 2-2u^2+uv-v^2-v & \text{if } 0 \leq v \leq \frac{1}{2}(1-u), \\ \frac{1}{4}(1-u)(7u-8v+9) & \text{if } \frac{1}{2}(1-u) \leq v \leq 2u, \\ \frac{1}{4}(9-3u^2+4uv+v^2-2u-8v) & \text{if } 1-u \leq 2v \leq 1+3u, \\ \frac{7}{12}(2-v)^2 & \text{if } \frac{1}{2}(1+3u) \leq v \leq 2. \end{cases}$$

Now, integrating  $\text{vol}(P(u)|_B - vC_0)$  and  $(P(u, v) \cdot C_0)^2$ , we obtain  $S(W_{\bullet, \bullet}^B; C_0) = \frac{13}{16}$  and

$$S(W_{\bullet, \bullet, \bullet}^{B, C_0}; x) = \frac{77}{320} + F_x(W_{\bullet, \bullet, \bullet}^{B, C_0}).$$

Furthermore, since  $x \notin \mathbf{e}_0 \cup \mathbf{e}_1$  and the curves  $C_0$  and  $C_1$  intersect transversally at  $x$ , we have

$$\begin{aligned} F_x(W_{\bullet, \bullet, \bullet}^{B, C_0}) &= \frac{3}{2} \int_0^1 \int_0^2 \text{ord}_x(N(u, v)|_{C_0}) \times (P(u, v) \cdot C_0) dv du \\ &= \frac{3}{2} \int_0^{\frac{1}{5}} \int_{\frac{1-u}{2}}^{\frac{1+3u}{2}} \frac{2v+u-1}{2} \times (P(u, v) \cdot C_0) dv du + \frac{3}{2} \int_0^{\frac{1}{5}} \int_{\frac{1+3u}{2}}^2 \frac{4v-2}{3} \times (P(u, v) \cdot C_0) dv du \\ &+ \frac{3}{2} \int_{\frac{1}{5}}^1 \int_{\frac{1-u}{2}}^{\frac{1+3u}{2}} \frac{2v+u-1}{2} \times (P(u, v) \cdot C_0) dv du + \frac{3}{2} \int_{\frac{1}{5}}^1 \int_{\frac{1+3u}{2}}^2 \frac{4v-2}{3} \times (P(u, v) \cdot C_0) dv du \\ &= \frac{3}{2} \int_0^{\frac{1}{5}} \int_{\frac{1-u}{2}}^{\frac{1+3u}{2}} \frac{2v+u-1}{2} \times \frac{4-2u-v}{4} dv du + \frac{3}{2} \int_0^{\frac{1}{5}} \int_{\frac{1+3u}{2}}^2 \frac{4v-2}{3} \times \frac{14-7v}{12} dv du \\ &+ \frac{3}{2} \int_{\frac{1}{5}}^1 \int_{\frac{1-u}{2}}^{2u} \frac{2v+u-1}{2} \times (1-u) dv du + \frac{3}{2} \int_{\frac{1}{5}}^1 \int_{2u}^{\frac{1+3u}{2}} \frac{2v+u-1}{2} \times \frac{4-2u-v}{4} dv du \\ &+ \frac{3}{2} \int_{\frac{1}{5}}^1 \int_{\frac{1+3u}{2}}^2 \frac{4v-2}{3} \times \frac{14-7v}{12} dv du = \frac{183}{320}, \end{aligned}$$

which gives  $S(W_{\bullet, \bullet, \bullet}^{B, C_0}; x) = \frac{13}{16}$ . Thus, we see that  $S(W_{\bullet, \bullet, \bullet}^{B, C_0}; x) = S(W_{\bullet, \bullet}^B; C_0) = \frac{13}{16} < 1$ , which is a contradiction. This completes the proof of Theorem 3.1.  $\square$

#### 4. K-MODULI COMPACTIFICATION

As in previous sections, let  $X$  be a pure complete intersection  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ , i.e. a  $(2, 2, 4)$ -complete intersection in  $\mathbb{P}(1^5, 2^2)$  which is not a  $(2, 4)$ -complete intersection in  $\mathbb{P}(1^5, 2)$ . For convenience, assume at no cost that  $X$  is given by  $y_1^2 = y_0^2 - f(x_0, \dots, x_4)$ ,  $g(x_0, \dots, x_4) = 0$  and  $h(x_0, \dots, x_4) = 0$  where  $f$ ,  $g$ , and  $h$  have degree 4, 2, and 2 respectively. First, we make a slight improvement on the previous results using techniques of cyclic covers, cone construction, degeneration and interpolation.

**Theorem 4.1.** *Let  $\Delta = \{f = 0, g = h = 0\} \subset \mathbb{P}^4$  and  $S = \{g = h = 0\} \subset \mathbb{P}^4$ . Assume  $\Delta$  is a smooth curve and  $S$  is a smooth surface. Then  $X$  is K-polystable.*

*Proof.* By forgetting  $y_1$  in the defining equation of  $X$  we obtain a double cover  $X \rightarrow (Y, \frac{1}{2}D)$  where  $Y = \{g = h = 0\} \subset \mathbb{P}(1^5, 2)$  and  $D = \{y_0^2 = f\}|_Y$ . In other words,  $Y \cong C_p(S, \mathcal{O}_S(2))$ . By rescaling the  $y_0$  coordinate only, it is easy to see that  $(Y, D)$  admits an isotrivial degeneration to  $(Y, D_0)$  where  $D_0 = \{y_0^2 = 0\}|_Y$  is twice of the section at infinity, which we denote by  $S_\infty = \frac{1}{2}D_0$ . Since  $S$  is a smooth del Pezzo surface of degree 4, we know that  $S$  is K-stable. Hence by the cone construction [39, 41, 59] we know that  $(Y, (1 - \frac{r}{3})S_\infty)$  is K-polystable for  $\mathcal{O}_S(2) = -r^{-1}K_S$ , i.e.  $r = \frac{1}{2}$ . Hence  $(Y, \frac{5}{12}D_0)$  is K-polystable which implies that  $(Y, \frac{5}{12}D)$  is K-semistable by openness. Since  $D \sim \mathcal{O}_Y(4)$  and  $-K_Y \sim \mathcal{O}_Y(3)$ , we know that  $(Y, \frac{3}{4}D)$  is a klt log Calabi-Yau pair as  $D$  is smooth and contained in the smooth locus of  $Y$ . Therefore by interpolation of K-stability [8, Proposition 2.13] we know that  $(Y, cD)$  is K-stable for every  $c \in (\frac{5}{12}, \frac{3}{4})$ . In particular, taking  $c = \frac{1}{2}$  yields K-stability of  $(Y, \frac{1}{2}D)$  which implies K-polystability of  $X$  by the cyclic cover result on K-polystability [26, 48, 62].  $\square$

Next we study the K-moduli compactification of pure  $(2, 2, 4)$ -complete intersections in  $\mathbb{P}(1^5, 2^2)$ . The following result is inspired by [10, Theorem 5.2].

**Proposition 4.2.** *Let  $S$  be a log del Pezzo surface. Let  $\Delta \sim -4K_S$  be an effective  $\mathbb{Q}$ -Cartier Weil divisor given by  $\Delta = \{f = 0\}$  for  $f \in H^0(S, -4K_S)$ . Let  $Y = C_p(S, -2K_S)$  be the projective cone. Let  $D = \{y^2 = f\}$  be a divisor in  $Y$ , where  $\{y = 0\}$  is the section at infinity of  $Y$ . Let  $c \in (0, \frac{1}{4}) \cap \mathbb{Q}$ . If  $(S, c\Delta)$  is K-semistable (resp. K-polystable), then  $(Y, \frac{5+16c}{12}D)$  is also K-semistable (resp. K-polystable).*

*Proof.* The proof is similar to [10, Proof of Theorem 5.2]. Denote by  $a = \frac{5+16c}{12}$ . Denote by  $\tau : Y \rightarrow Y$  the involution preserving each ruling given by  $\tau^*y = -y$ . Let  $Z = C_p(S, -4K_S)$  be a new projective cone where  $S_1 = \{z = 0\}$  is the section at infinity. Then the quotient map of  $\tau$  yields a double cover  $\pi : Y \rightarrow Z$  branched along  $S_1$ . Moreover, it is clear that  $D$  is  $\tau$ -invariant whose quotient  $S_2 = D/\tau$  is a section of  $Z$  such that  $S_2|_{S_1} = \Delta$ . Thus we have a crepant finite Galois morphism  $\pi : (Y, aD) \rightarrow (Z, \frac{1}{2}S_1 + aS_2)$ . By [48, 62], it suffices to show K-semistability (resp. K-polystability) for  $(Z, \frac{1}{2}S_1 + aS_2)$ .

We first treat K-semistability. The natural  $\mathbb{G}_m$ -action on  $(Z, S_1)$  degenerates  $S_2$  to  $S_{2,0}$  as the cone over  $\Delta$ . Let  $r \in \mathbb{Q}_{>0}$  be chosen such that  $-4K_S \sim_{\mathbb{Q}} r^{-1}(-K_S - c\Delta)$ , i.e.  $r = \frac{1-4c}{4}$ . It follows from the cone construction [41, Proposition 5.3] (see also [39]) that  $(Z, (1 - \frac{r}{3})S_1 + cS_{2,0})$  is K-semistable. Since  $1 - \frac{r}{3} = \frac{11+4c}{12}$ , we know that  $(Z, \frac{11+4c}{12}S_1 + cS_2)$  is K-semistable by openness of K-semistability [14, 56]. By symmetry between  $S_1$  and  $S_2$ , we know that  $(Z, cS_1 + \frac{11+4c}{12}S_1)$  is also K-semistable. Since  $(\frac{1}{2}, a)$  is a convex combination of  $(c, \frac{11+4c}{12})$  and  $(\frac{11+4c}{12}, c)$ , we know that  $(Z, \frac{1}{2}S_1 + aS_2)$  is K-semistable.

The K-polystable part follows from the exact same argument as in [10, Proof of Theorem 5.2].  $\square$

From Proposition 4.2, we see that if  $c = \frac{1}{16}$  then  $\frac{5+16c}{12} = \frac{1}{2}$  which precisely gives K-moduli of the double cover  $X$  of  $(Y, \frac{1}{2}D)$ . In other words, let  $\mathfrak{M}_c$  be the K-moduli space of  $(S, c\Delta)$  where  $S$  is a  $\mathbb{Q}$ -Gorenstein smoothable del Pezzo surface of degree 4 and  $\Delta \sim -4K_S$ . In the convention of [8], the K-moduli space  $\mathfrak{M}_c := KM_{\chi_0, 4, c}$  where  $\chi_0$  is the Hilbert polynomial of an anti-canonically polarized smooth del Pezzo surface of degree 4. Then by similar arguments to [10, Theorem 5.9 and Proposition 6.12], Proposition 4.2 defines a finite morphism  $\psi : \mathfrak{M}_{\frac{1}{16}} \rightarrow M_{3,4}^{\text{Kps}}$ , whose image is precisely the closure of the locus parametrising pure  $(2, 2, 4)$ -complete intersections  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ . Thus to describe the compactification of the K-moduli of pure  $(2, 2, 4)$ -complete intersections in  $\mathbb{P}(1^5, 2^2)$ , it suffices to describe the K-moduli space  $\mathfrak{M}_{\frac{1}{16}}$  of  $(S, \frac{1}{16}\Delta)$  consisting of a del Pezzo surface  $S$  of degree 4 and  $\Delta \sim -4K_S$ . As a consequence, the converse of Proposition 4.2 also holds.

The next result bounds the singularities of surfaces appearing in  $\mathfrak{M}_c$ .

**Proposition 4.3.** *Let  $[(S, c\Delta)] \in \mathfrak{M}_c$  be a K-polystable pair for some  $c \in (0, \frac{1}{16}]$ . Then either  $S$  is smooth, or  $S$  has only  $A_1, A_2$ , or  $A_3$ -singularities. If in addition  $S$  has an  $A_3$ -singularity, then we have*

$$c = \frac{1}{16}, S \cong \{y_3^2 = y_2y_4\} \subset \mathbb{P}(1, 2, 3, 4)_{[y_1, y_2, y_3, y_4]}, \text{ and } \Delta = \{y_4^4 = 0\}.$$

Note that this surface  $S$  is No25 from [18, Big Table] (see also Table 1).

*Proof.* We know that

$$(-K_S - c\Delta)^2 = (1 - 4c)^2(-K_S)^2 = 4(1 - 4c)^2.$$

Let  $x \in S$  be a singular point with local orbifold group  $G_x$ . The local-global volume comparison [39] implies

$$\frac{4}{|G_x|} = \widehat{\text{vol}}(x, S) \geq \widehat{\text{vol}}(x, S, c\Delta) \geq \frac{4}{9}(-K_S - c\Delta)^2 = \frac{16}{9}(1 - 4c)^2 \geq 1. \quad (4.1)$$

Thus we have that  $|G_x| \leq 4$ . This, together with the  $T$ -singularity condition, implies that  $x \in S$  is  $A_1, A_2, A_3$  or of type  $\frac{1}{4}(1, 1)$ .

Next, we rule out the case of type  $\frac{1}{4}(1, 1)$ . Assume to the contrary that  $x \in S$  has type  $\frac{1}{4}(1, 1)$ . Since  $|G_x| = 4$ , every inequality from (4.1) is an equality. Thus we have  $x \notin \text{Supp}(\Delta)$ ,  $c = \frac{1}{16}$ , and the inequality in the local-global volume comparison from [39, 42] is an equality. By [42, Remark 38(1)], we have that  $S \cong \mathbb{P}^2/\mu_4$  where the  $\mu_4$ -action on  $\mathbb{P}^2$  has weight  $(1, 1, 0)$ , and  $\frac{1}{16}\Delta$  corresponds to the orbifold divisor of  $\mathbb{P}^2/\mu_4$ . In particular, we have  $S \cong \mathbb{P}(1, 1, 4)$  which implies that  $(K_S^2) = 9 \neq 4$ , a contradiction. Thus singularities of type  $\frac{1}{4}(1, 1)$  cannot appear.

Finally, we consider the case where  $x \in S$  has type  $A_3$ . Similar to the previous case of type  $\frac{1}{4}(1, 1)$ , we have  $x \notin \text{Supp}(\Delta)$ ,  $c = \frac{1}{16}$ . Moreover,  $S \cong \mathbb{P}^2/\mu_4$  where the  $\mu_4$ -action on  $\mathbb{P}^2$  has weight  $(1, 3, 0)$ , and  $\frac{1}{16}\Delta$  corresponds to the orbifold divisor of  $\mathbb{P}^2/\mu_4$ . Suppose a generator  $g$  of  $\mu_4$  acting on  $\mathbb{P}_{[x,y,z]}^2$  as

$$g \cdot [x, y, z] = [ix, -iy, z] = [-x, y, iz].$$

Denote by  $R = \bigoplus_{m=0}^{\infty} H^0(\mathbb{P}^2, \mathcal{O}(m))$  the graded section ring of  $(\mathbb{P}^2, \mathcal{O}(1))$ . If we pick the latter  $\mu_4$ -linearization on  $(\mathbb{P}^2, \mathcal{O}(1))$  of weight  $(2, 0, 1)$ , then the  $\mu_4$ -invariant subring  $R^{\mu_4}$  is generated by  $y, x^2, xz^2, z^4$ . Thus we have

$$\mathbb{P}^2/\mu_4 \cong \text{Proj } R^{\mu_4} \cong \{(y_3^2 = y_2y_4) \subset \mathbb{P}(1, 2, 3, 4)\},$$

where the generators corresponds to  $y_1 = y, y_2 = x^2, y_3 = xz^2$  and  $y_4 = z^4$ . Since the  $\mu_4$ -action has one orbifold curve given by  $(z = 0)$  of order 2 whose reduced image in  $\mathbb{P}^2/\mu_4$  corresponds  $\frac{1}{2}(y_4 = 0)$ , we have  $\frac{1}{16}\Delta = \frac{1}{2} \cdot \frac{1}{2}(y_4 = 0)$  which implies that  $\Delta = \{y_4^4 = 0\}$ . It is easy to check that  $(K_S^2) = 4$  and  $\Delta \sim -4K_S$ , hence  $[(S, \frac{1}{16}\Delta)] \in \mathfrak{M}_{\frac{1}{16}}$ . The proof is finished.  $\square$

**Corollary 4.4.** *The K-moduli compactification of pure  $(2, 2, 4)$ -complete intersections  $\mathbb{P}(1^5, 2^2)$  only consists of complete intersections of the same type.*

*Proof.* By Proposition 4.3, we see that every pair  $(S, \Delta)$  in  $\mathfrak{M}_{\frac{1}{16}}$  satisfies that  $S$  has du Val singularities. Hence  $S = \{g = h = 0\}$  is always a complete intersection of two quadrics in  $\mathbb{P}^4$  and  $\Delta = \{f = 0\}|_S$ , which implies that  $Y = C_p(S, -2K_S) = \{g = h = 0\}$  is the complete intersection in  $\mathbb{P}(1^5, 2)$ . Hence  $\psi(S, \Delta)$  is a complete intersection  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$  given by  $\{y_0y_1 = f\}$  and  $\{g = h = 0\}$ .  $\square$

## 5. WALL CROSSING

In this section, we study the explicit wall crossings for the K-moduli spaces  $\mathfrak{M}_c$  where  $c \in (0, \frac{1}{16}]$ . Similar problems were studied in [8–10, 53, 60].

Let us recall the convention from [8]. A wall  $c = c_i$  of the K-moduli spaces  $\mathfrak{M}_c$  is a rational value such that K-polystability changes from  $c = c_i - \epsilon$  to  $c = c_i + \epsilon$  for  $0 < \epsilon \ll 1$ . By [8, Theorems 1.1 and 1.2], there are finitely many walls  $0 < c_1 < c_2 < \dots < c_k \leq \frac{1}{16}$  for the K-moduli spaces  $\mathfrak{M}_c$  with  $c \in (0, \frac{1}{16}]$ . Moreover, we have wall crossing diagrams of birational morphisms

$$\mathfrak{M}_{c_i-\epsilon} \rightarrow \mathfrak{M}_{c_i} \leftarrow \mathfrak{M}_{c_i+\epsilon}.$$

A K-polystable pair  $(S, c_i\Delta)$  in  $\mathfrak{M}_{c_i}$  is called a *new K-polystable pair on the wall  $c_i$*  if  $(S, c\Delta)$  is K-unstable whenever  $c \neq c_i$ . By [8] we know that every new K-polystable pair on a wall admits an effective  $\mathbb{G}_m$ -action, hence is not K-stable; on the other hand, every wall admits a new K-polystable pair.

In addition to walls in  $(0, \frac{1}{16}]$ , there is an initial wall crossing at  $c_0 = 0$ , where we will have a fibre-type contraction  $\mathfrak{M}_\epsilon \rightarrow \mathfrak{M}_0$ . Here  $\mathfrak{M}_0$  is the K-moduli space of  $\mathbb{Q}$ -Gorenstein smoothable del Pezzo surfaces of degree 4, which is literally the K-moduli space of  $(S, 0\Delta)$ . By [49, 51] we know that  $\mathfrak{M}_0$  is isomorphic to the GIT moduli space of complete intersections of two quadric hypersurfaces in  $\mathbb{P}^4$ . Note that this initial wall crossing was not explicitly stated in [8], although the techniques therein are sufficient to deduce such a result.

**Theorem 5.1.** *There are four walls for  $c \in (0, \frac{1}{16}]$  for the K-moduli spaces  $\mathfrak{M}_c$ :  $c_1 = \frac{1}{28}, c_2 = \frac{1}{22}, c_3 = \frac{1}{19}, c_4 = \frac{1}{16}$ . The new K-polystable pairs  $(S, c_i\Delta)$  on the walls are completely classified in Proposition 4.3, Theorems 5.2 and 5.8 where  $S$  is isomorphic to  $\text{No}25, 29$ , and  $33$  in Table 1 respectively. Moreover, the initial wall crossing at  $c_0 = 0$  gives a surjective morphism  $\mathfrak{M}_\epsilon \rightarrow \mathfrak{M}_0$  with connected fibres whose general fibre is isomorphic to  $\mathbb{P}^{40}/\mu_2^4$ .*

To find all walls for the K-moduli spaces  $\mathfrak{M}_c$  where  $c \in (0, \frac{1}{16}]$ , we need to classify new K-polystable pairs  $(S, c\Delta)$ , i.e. it is K-polystable but not K-stable, and  $S$  is K-unstable. This way we can obtain a complete list of walls and new K-polystable pairs on the wall.

By Proposition 4.3, we know that either  $S$  is a  $K$ -unstable del Pezzo surface of degree 4 with at worst  $A_2$  singularities that admits a  $\mathbb{G}_m$ -action, or  $S$  is isomorphic to No25. The following table from [18, Big Table] classifies del Pezzo surface of degree 4 with at worst  $A_2$  singularities that admits a  $\mathbb{G}_m$ -action, with an additional row of No25 which shows up in  $\mathfrak{M}_{\frac{1}{16}}$  only.

TABLE 1. Del Pezzo surfaces of degree 4 with  $\mathbb{G}_m$ -actions and at worst  $A_2$ -singularities, together with No25

No	$\rho$	Sing.	Aut <sup>0</sup>	Equation & Ambient space
29	2	$A_2A_1$	$\mathbb{G}_m^2$	$\{y_2y_2' = y_1^3y_1'\} \subset \mathbb{P}(1, 1, 2, 2)$
30	2	$4A_1$	$\mathbb{G}_m^2$	$\{y_2y_2' = y_1^2y_1'\} \subset \mathbb{P}(1, 1, 2, 2)$
33	3	$A_2A_1$	$\mathbb{G}_m$	$\{x_0x_1 - x_2x_3 = x_1x_2 + x_2x_4 + x_3x_4 = 0\} \subset \mathbb{P}^4$
34	3	$3A_1$	$\mathbb{G}_m$	$\{y_2y_2' = y_1^2y_1'(y_1 + y_1')\} \subset \mathbb{P}(1, 1, 2, 2)$
35	4	$2A_1$	$\mathbb{G}_m$	$\{y_2y_2' = y_1y_1'(y_1' - y_1)(y_1' - \lambda y_1)\} \subset \mathbb{P}(1, 1, 2, 2)$ for $\lambda \in \mathbb{C} \setminus \{0, 1\}$
25	1	$A_32A_1$	$\mathbb{B}_2 \times \mathbb{G}_m$	$\{y_3^2 = y_2y_4\} \subset \mathbb{P}(1, 2, 3, 4)$

From Table 1 and [49, 51], we see that No25, 29, 33 are  $K$ -unstable, while No30, 35 are  $K$ -polystable, and No34 is  $K$ -semistable but not  $K$ -polystable.

From now on, we study the walls from  $\mathfrak{M}_\epsilon$  to  $\mathfrak{M}_{\frac{1}{16}}$  which involve only No25, 29 or 33. Since there is only one wall at  $c = \frac{1}{16}$  involving No25 by Proposition 4.3, we will focus on walls involving No29 or 33.

5.1. **No29.** Here  $S = \{y_2y_2' = y_1^3y_1'\} \subset \mathbb{P}(1, 1, 2, 2)_{[y_1, y_1', y_2, y_2']}$ .

**Theorem 5.2.** *There are precisely four walls in  $(0, \frac{1}{16}]$  involving  $S$  from No29 such that  $(S, c_i\Delta_i)$  is a new  $K$ -polystable pair for the following choices of  $c_i$  and  $\Delta_i$ .*

- (1)  $c_1 = \frac{1}{28}$ ,  $\Delta_1 = \{y_1^8 = 0\}$ .
- (2)  $c_2 = \frac{1}{22}$ ,  $\Delta_2 = \{y_1y_1'^7 = 0\}$ .
- (3)  $c_3 = \frac{1}{19}$ ,  $\Delta_3 = \{(y_2 + y_2')y_1^6 = 0\}$ .
- (4)  $c_4 = \frac{1}{16}$ ,  $\Delta_4 = \Delta_{4,\mathbf{a}}$  or  $\Delta_{4,\infty}$ , where

$$\Delta_{4,\mathbf{a}} = \{y_2y_1'^6 + a_1y_1^2y_1'^6 + a_2y_1y_2'y_1'^5 + a_3y_2^2y_1'^4 = 0\} \text{ for } \mathbf{a} = (a_1, a_2, a_3) \in \mathbb{C} \times (\mathbb{C}^2 \setminus \{(0, 0)\}),$$

$$\text{and } \Delta_{4,\infty} = \{y_1^2y_1'^6 = 0\}.$$

We first show that these pairs are  $K$ -polystable. The fact that  $S$  is toric can be seen by

$$(s, t) \cdot [y_1, y_1', y_2, y_2'] = [sy_1, s^{-3}y_1', ty_2, t^{-1}y_2']$$

It is clear that  $\Delta_i$  for  $i \leq 2$  is toric. Thus it suffices to show the Futaki character  $\text{Fut}_{(S, c_i\Delta_i)} = 0$ . In addition,  $S$  admits an involution  $\tau : S \rightarrow S$  given by  $\tau([y_1, y_1', y_2, y_2']) = [y_1, y_1', y_2', y_2]$ .

Denote by  $C = \{y_1' = y_2 = 0\}$ ,  $C' = \{y_1' = y_2' = 0\}$ ,  $B = \{y_1 = y_2 = 0\}$ ,  $B' = \{y_1 = y_2' = 0\}$ . The intersection numbers of these curves are summarized in Table 2.

TABLE 2. Intersection numbers of the surface No29

$\bullet$	$C$	$C'$	$B$	$B'$
$C$	$-\frac{1}{2}$	1	$\frac{1}{2}$	0
$C'$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$
$B$	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{3}$
$B'$	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

**Lemma 5.3.** *The Mori cone  $\overline{NE}(S)$  is generated by  $[C]$  and  $[C']$ .*



*Proof.* From the torus action on  $S$ , we see that there are precisely four torus invariant divisors:  $C, C', B$ , and  $B'$ . Thus  $\overline{NE}(S)$  is generated by these four curves. From the intersection table above we see that both  $B$  and  $B'$  are nef and big, while both  $C$  and  $C'$  are extremal. Thus the proof is finished.  $\square$

We first compute  $S$ -invariant of these curves.

**Proposition 5.4.** *Under the above notation, we have*

$$S_S(C) = S_S(C') = \frac{5}{6}, \quad S_S(B) = S_S(B') = \frac{7}{6}.$$

*Proof.* First of all, the involution  $\tau$  satisfies  $\tau(C) = C'$  and  $\tau(B) = B'$ . Thus we have equalities  $S_S(C) = S_S(C')$  and  $S_S(B) = S_S(B')$ .

Next, we compute  $S_S(C)$ . It is clear that  $-K_S = \mathcal{O}_S(2) = 2(C + C')$ . This gives

$$S_S(C) = \frac{1}{4} \int_0^\infty \text{vol}_S((2-t)C + 2C') dt.$$

It follows from Table 2 and Lemma 5.3 that  $(2-t)C + 2C'$  is nef if  $0 \leq t \leq 1$ , and not big if  $t \geq 2$ . For  $1 \leq t \leq 2$ , the Zariski decomposition is

$$(2-t)C + 2C' = P + N = (2-t)(C + 2C') + (2t-2)C'.$$

Thus computation shows

$$\text{vol}((2-t)C + 2C') = \begin{cases} 4 - 2t - \frac{t^2}{2} & \text{if } 0 \leq t \leq 1; \\ \frac{3}{2}(2-t)^2 & \text{if } 1 \leq t \leq 2. \end{cases}$$

Thus

$$S_S(C) = \frac{1}{4} \left( \int_0^1 (4 - 2t - \frac{t^2}{2}) dt + \int_1^2 \frac{3}{2}(2-t)^2 dt \right) = \frac{5}{6}.$$

Finally, we compute  $S_S(B)$ . It is clear that  $-K_S = \mathcal{O}_S(2) = 2(B + B')$ . This gives

$$S_S(B) = \frac{1}{4} \int_0^\infty \text{vol}_S((2-t)B + 2B') dt.$$

Moreover, by Table 2 and Lemma 5.3, we know that  $(2-t)B + 2B'$  is nef if  $0 \leq t \leq 2$ , and not big if  $t \geq 3$ . For  $2 \leq t \leq 3$ , the Zariski decomposition is

$$(2-t)B + 2B' = P + N = (6-2t)B' + (t-2)C.$$

Thus computation shows

$$\text{vol}((2-t)B + 2B') = \begin{cases} 4 - 2t + \frac{t^2}{6} & \text{if } 0 \leq t \leq 2; \\ \frac{2}{3}(3-t)^2 & \text{if } 2 \leq t \leq 3. \end{cases}$$

Thus

$$S_S(B) = \frac{1}{4} \left( \int_0^2 (4 - 2t + \frac{t^2}{6}) dt + \int_2^3 \frac{2}{3}(3-t)^2 dt \right) = \frac{7}{6}.$$

$\square$

**Proposition 5.5.** *The pairs  $(S, c_i \Delta_i)$  from Theorem 5.2 are  $K$ -polystable and not  $K$ -stable.*

*Proof.* We first look at cases (1) and (2). For these two cases,  $(S, c_i \Delta_i)$  is toric hence not  $K$ -stable. Thus we just need to show that the Futaki character  $\text{Fut}_{(S, c_i \Delta_i)} = 0$ . Since the 1-PS induced by  $\text{ord}_C, \text{ord}_{C'}, \text{ord}_B$ , and  $\text{ord}'_B$  generates the torus, it suffices to show that  $\beta_{(S, c_i \Delta_i)}(v) = 0$  where  $v$  is one of the four valuations. By symmetry of  $\tau$ , it suffices to show that  $\beta_{(S, c_i \Delta_i)}(C) = \beta_{(S, c_i \Delta_i)}(B) = 0$ .

Since  $\Delta_i \equiv -4K_S$ , we know that  $A_{(S, c_i \Delta_i)}(v) = A_S(v) - c_i v(\Delta_i)$  and  $S_{(S, c_i \Delta_i)}(v) = (1 - 4c_i)S_S(v)$ . Thus

$$\beta_{(S, c_i \Delta_i)}(v) = A_S(v) - c_i v(\Delta_i) - (1 - 4c_i)S_S(v)$$

(1) Since  $c_1 = \frac{1}{28}$  and  $\Delta_1 = (y_1^8 = 0)$ , we have

$$\begin{aligned} \beta_{(S, c_1 \Delta_1)}(C) &= 1 - \frac{8}{28} - (1 - \frac{4}{28})\frac{5}{6} = 0, \\ \beta_{(S, c_1 \Delta_1)}(B) &= 1 - (1 - \frac{4}{28})\frac{7}{6} = 0. \end{aligned}$$

(2) Since  $c_2 = \frac{1}{22}$  and  $\Delta_2 = (y_1 y_1'^7 = 0)$ , we have

$$\begin{aligned}\beta_{(S, c_2 \Delta_2)}(C) &= 1 - \frac{7}{22} - \left(1 - \frac{4}{22}\right) \frac{5}{6} = 0, \\ \beta_{(S, c_2 \Delta_2)}(B) &= 1 - \frac{1}{22} - \left(1 - \frac{4}{22}\right) \frac{7}{6} = 0.\end{aligned}$$

Next, we look at case (3). In this case,  $(S, \Delta_3)$  admits a  $\mathbb{G}_m$ -action  $\sigma$  given by

$$s \cdot [y_1, y_1', y_2, y_2'] = [s y_1, s^{-3} y_1', y_2, y_2'].$$

Thus  $(S, c_3 \Delta_3)$  is not K-stable. In the affine chart  $U := \{y_1 = 1\} \subset S$ , we have an isomorphism  $U \cong \mathbb{A}^2$  given by  $[1, y_1', y_2, y_2'] \mapsto (y_2, y_2')$  where  $C|_U$  and  $C'|_U$  corresponds to  $(y_2 = 0)$  and  $(y_2' = 0)$  respectively. If we denote by  $v_0 = \text{ord}_{[1,0,0,0]}$ , then  $v_0$  is the quasi-monomial combination of  $\text{ord}_C$  and  $\text{ord}_{C'}$  of weight  $(1, 1)$ . Thus we have

$$\beta_S(v_0) = \beta_S(C) + \beta_S(C') = 2\beta_S(C) = 2(1 - S_S(C)) = \frac{1}{3}.$$

It is clear that  $v_0$  induces the  $\mathbb{G}_m$ -action  $\sigma$ . Thus by [44, Theorem 3.2] (cf. [7, Theorem 1.3.9] and [34]) it suffices to show that  $\beta_{(S, c_3 \Delta_3)}(v_0) = 0$ , and that  $\beta_{(S, c_3 \Delta_3)}(D) > 0$  for every vertical prime divisor  $D$  on  $S$ .

We first compute  $\beta$ -invariant of  $v_0$ . Since  $\beta_S(v_0) = \frac{1}{3}$  and  $A_S(v_0) = 2$ , we know that  $S_S(v_0) = \frac{5}{3}$ . Thus

$$\beta_{(S, c_3 \Delta_3)}(v_0) = 2 - c_3 v_0(\Delta_3) - (1 - 4c_3) S_S(v_0) = 2 - \frac{13}{19} - \left(1 - \frac{4}{19}\right) \frac{5}{3} = 0.$$

Next we compute  $\beta$ -invariant of vertical divisors  $D$ . A simple analysis of the  $\sigma$ -action on  $S$  shows that  $D$  is one of the following:

- (i)  $C$  or  $C'$ ;
- (ii)  $B$  or  $B'$ ;
- (iii)  $D_a = \{y_2 + a y_2' = 0\}$  for  $a \neq 0$ .

Next, we split into these three cases.

(i) Since  $\Delta_3$  is invariant under the involution  $\tau$ , we have

$$\beta_{(S, c_3 \Delta_3)}(C) = \beta_{(S, c_3 \Delta_3)}(C') = 1 - \frac{6}{19} - \left(1 - \frac{4}{19}\right) \frac{5}{6} = \frac{1}{38} > 0.$$

(ii) Similarly, we have

$$\beta_{(S, c_3 \Delta_3)}(B) = \beta_{(S, c_3 \Delta_3)}(B') = 1 - \left(1 - \frac{4}{19}\right) \frac{7}{6} = \frac{3}{38} > 0.$$

(iii) We know that  $\text{ord}_{D_a}(\Delta_3) \leq 1$ . Moreover, we know that  $-K_S - t D_a = \mathcal{O}_S(2 - 2t)$  which is nef if  $0 \leq t \leq 1$  and not big if  $t \geq 1$ . Hence

$$S_S(D_a) = \frac{1}{4} \int_0^1 (2 - 2t)^2 dt = \frac{1}{3}.$$

Thus

$$\beta_{(S, c_3 \Delta_3)}(D_a) \geq 1 - \frac{1}{19} - \left(1 - \frac{4}{19}\right) \frac{1}{3} = \frac{13}{19} > 0.$$

Finally, we look at case (4). We first treat the case of  $\Delta_{4, \infty}$ . It is clear that  $\Delta_{4, \infty} = \{y_1^2 y_1'^6 = 0\}$  is toric. Hence  $(S, c_4 \Delta_{4, \infty})$  is not K-stable. Since  $c_4 = \frac{1}{16}$ , we have

$$\begin{aligned}\beta_{(S, c_4 \Delta_{4, \infty})}(C) &= 1 - \frac{6}{16} - \left(1 - \frac{4}{16}\right) \frac{5}{6} = 0, \\ \beta_{(S, c_4 \Delta_{4, \infty})}(B) &= 1 - \frac{2}{16} - \left(1 - \frac{4}{16}\right) \frac{7}{6} = 0.\end{aligned}$$

This shows that  $(S, c_4 \Delta_{4, \infty})$  is K-polystable.

Next, we treat the case of  $\Delta_{4, \mathbf{a}}$  for  $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{C} \times (\mathbb{C}^2 \setminus \{(0, 0)\})$ . Since  $\Delta_{4, \mathbf{a}} = \{y_2 y_1'^6 + a_1 y_1^2 y_1'^6 + a_2 y_1 y_2' y_1'^5 + a_3 y_2^2 y_1'^4 = 0\}$ , we know that  $(S, \Delta_{4, \mathbf{a}})$  admits a  $\mathbb{G}_m$ -action  $\sigma'$  given by

$$t \cdot [y_1, y_1', y_2, y_2'] = [t y_1, y_1', t^2 y_2, t y_2'].$$

Thus  $(S, c_4 \Delta_{4, \mathbf{a}})$  is not K-stable. In the affine chart  $U' := \{y_1' = 1\} \subset S$ , we have an isomorphism  $U' \cong \{y_1^3 = y_2 y_2'\} \subset \mathbb{A}_{(y_1, y_2, y_2')}^3$ . Let  $v_1$  be the monomial valuation on  $U'$  of weight  $(1, 2, 1)$  in  $(y_1, y_2, y_2')$  centered at  $[0, 1, 0, 0]$ . Then  $v_1$  is a quasi-monomial combination of  $\text{ord}_B$  and  $\text{ord}_{B'}$  of weight  $(\frac{2}{3}, \frac{1}{3})$ . Thus we have

$$\beta_S(v_1) = \frac{2}{3} \beta_S(B) + \frac{1}{3} \beta_S(B') = \beta_S(B) = -\frac{1}{6}.$$

It is clear that  $v_1$  induces the  $\mathbb{G}_m$ -action  $\sigma'$ . Thus by [44, Theorem 3.2] (cf. [7, Theorem 1.3.9] and [34]) it suffices to show that  $\beta_{(S, c_4 \Delta_{4, \mathbf{a}})}(v_1) = 0$ , and that  $\beta_{(S, c_4 \Delta_{4, \mathbf{a}})}(D) > 0$  for every vertical prime divisor  $D$  on  $S$ .

We first compute  $\beta$ -invariant of  $v_1$ . Since  $A_S(v_1) = \frac{2}{3}A_S(B) + \frac{1}{3}A_S(B') = 1$ , we know that  $S_S(v_1) = A_S(v_1) - \beta_S(v_1) = \frac{7}{6}$ . Moreover, we have  $\Delta_{4, \mathbf{a}}|_{U'} = \{y_2 + a_1 y_1^2 + a_2 y_1 y_2' + a_3 y_2'^2 = 0\}$  which implies  $v_1(\Delta_{4, \mathbf{a}}) = 2$ . Thus

$$\beta_{(S, c_4 \Delta_{4, \mathbf{a}})}(v_1) = 1 - c_4 v_1(\Delta_{4, \mathbf{a}}) - (1 - 4c_4)S_S(v_1) = 1 - \frac{2}{16} - (1 - \frac{4}{16})\frac{7}{6} = 0.$$

We also observe that  $C'$  is the other horizontal divisor of  $\sigma'$ , whose  $\beta$ -invariant is automatically zero as  $\beta = \text{Fut}$  is linear on the space of cocharacters.

Next we compute  $\beta$ -invariant of vertical divisors  $D$ . A simple analysis of the  $\sigma'$ -action on  $S$  shows that  $D$  is one of the following:

- (i)  $C$ ;
- (ii)  $B$  or  $B'$ ;
- (iii)  $D_b = (y_2 + b y_1^2 = 0) - 2B$  for  $b \neq 0$ .

Next, we split into these three cases.

- (i) Since  $\text{ord}_C(\Delta_{4, \mathbf{a}}) = 4$  or  $5$ , we have

$$\beta_{(S, c_4 \Delta_{4, \mathbf{a}})}(C) \geq 1 - \frac{5}{16} - (1 - \frac{4}{16})\frac{5}{6} = \frac{1}{16} > 0.$$

- (ii) Since  $\text{ord}_{B'}(\Delta_{4, \mathbf{a}}) = 0$  and  $\text{ord}_B(\Delta_{4, \mathbf{a}}) = 0$  or  $1$ , we have

$$\beta_{(S, c_4 \Delta_{4, \mathbf{a}})}(B') \geq \beta_{(S, c_4 \Delta_{4, \mathbf{a}})}(B) \geq 1 - \frac{1}{16} - (1 - \frac{4}{16})\frac{7}{6} = \frac{1}{16} > 0.$$

- (iii) In the affine chart  $U = \{y_1 = 1\} \cong \mathbb{A}_{(y_2, y_2')}^2$ , we have  $\Delta_{4, \mathbf{a}}|_U = \{y_2^4 y_2'^6 (y_2^3 + a_1 y_2^2 + a_2 y_2 + a_3) = 0\}$  and  $D_b|_U = \{y_2 + b = 0\}$ . Hence  $\text{ord}_{D_b}(\Delta_{4, \mathbf{a}}) \leq 3$  as  $b \neq 0$ . Moreover, we know that  $D_b \equiv \mathcal{O}_S(2) - 2B \equiv 2B'$ . Hence

$$S_S(D_b) = \frac{1}{4} \int_0^\infty \text{vol}_S(-K_S - 2tB') dt = \frac{1}{2} S_S(B') = \frac{7}{12}.$$

Thus

$$\beta_{(S, c_4 \Delta_{4, \mathbf{a}})}(D_b) \geq 1 - \frac{3}{16} - (1 - \frac{4}{16})\frac{7}{12} = \frac{3}{8} > 0. \quad \square$$

**Proposition 5.6.** *Let  $S$  be the surface No29 from Table 1. Suppose  $\Delta \in |-4K_S|$  and  $c \in (0, \frac{1}{16}]$  satisfy that  $(S, c\Delta)$  is  $K$ -polystable but not  $K$ -stable. Then  $(S, c\Delta) \cong (S, c_i \Delta_i)$  for some  $1 \leq i \leq 4$  and  $c_i, \Delta_i$  described in Theorem 5.2.*

*Proof.* Let  $f(y_1, y_1', y_2, y_2')$  be the defining polynomial of  $\Delta$ . Since  $S = \{y_2 y_2' = y_1^3 y_1'\} \subset \mathbb{P}(1, 1, 2, 2)$ , we may assume that no monomial in  $f$  is divisible by  $y_2 y_2'$ . We split into the following cases.

(a)  $y_1'^8$  appears in  $f$ . Then  $\Delta$  admits a special degeneration to  $\{y_1'^8 = 0\} = \Delta_1$  under the  $\mathbb{G}_m$ -action  $\sigma$ . Thus  $(S, c_1 \Delta)$  is  $K$ -semistable whose  $K$ -polystable degeneration is  $(S, c_1 \Delta_1)$ . This shows that  $(S, \Delta) \cong (S, \Delta_1)$  and  $c = c_1$  by [8, Proposition 3.18].

(b)  $y_1'^8$  does not appear in  $f$ , while  $y_1 y_1'^7$  appears in  $f$ . Then  $\Delta$  admits a special degeneration to  $\{y_1 y_1'^7 = 0\} = \Delta_2$  under the  $\mathbb{G}_m$ -action  $\sigma$ . By similar argument as in case (a), we conclude that  $(S, \Delta) \cong (S, \Delta_2)$  and  $c = c_2$ .

(c) Both  $y_1'^8$  and  $y_1 y_1'^7$  do not appear in  $f$ , while  $y_1'^6$  divides some monomial in  $f$ . Let  $(a_1 y_1^2 + a_0 y_2 + a_0' y_2') y_1'^6$  be the sum of all terms in  $f$  divisible by  $y_1'^6$ .

If  $a_0 \neq 0$  and  $a_0' \neq 0$ , then  $\Delta$  admits a special degeneration to  $\Delta' := \{(a_0 y_2 + a_0' y_2') y_1'^6 = 0\}$  under the  $\mathbb{G}_m$ -action  $\sigma$ . After rescaling we have  $(S, \Delta') \cong (S, \Delta_3)$ . By similar argument as in case (a), we conclude that  $(S, \Delta) \cong (S, \Delta_3)$  and  $c = c_3$ .

If either  $a_0 = 0$  or  $a_0' = 0$  but not both, we will show that  $c = c_4$  and  $(S, \Delta) \cong (S, \Delta_{4, \mathbf{a}})$  for some  $\mathbf{a} \in \mathbb{C} \times (\mathbb{C}^2 \setminus \{(0, 0)\})$ . By symmetry we may assume that  $a_0 \neq 0$  and  $a_0' = 0$ . Hence after rescaling we may assume that  $a_0 = 1$ . Under the  $\mathbb{G}_m$ -action  $\sigma'$  induced by the valuation  $v_1$ , then we see that  $v_1(\Delta) = 2$  and  $\Delta$  admits a special degeneration to  $\Delta' = \{y_2 y_1'^6 + a_1 y_1^2 y_1'^6 + a_2 y_1 y_2' y_1'^5 + a_3 y_2'^2 y_1'^4 = 0\}$ . Moreover, we compute

$$\beta_{(S, c\Delta)}(v_1) = 1 - c v_1(\Delta) - (1 - 4c)S_S(v_1) = 1 - 2c - (1 - 4c)\frac{7}{6}.$$

Since  $(S, c\Delta)$  is  $K$ -polystable, we have  $\beta_{(S, c\Delta)}(v_1) \geq 0$  which implies that  $c \geq \frac{1}{16}$ . Thus we have  $c = c_4 = \frac{1}{16}$  and  $\beta_{(S, c\Delta)}(v_1) = 0$ . By  $K$ -polystability, we have  $(S, c\Delta) \cong (S, c_4 \Delta')$ . If  $(a_2, a_3) \neq (0, 0)$ , then  $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{C} \times (\mathbb{C}^2 \setminus \{(0, 0)\})$  and  $(S, \Delta) \cong (S, \Delta') \cong (S, \Delta_{4, \mathbf{a}})$ . If  $a_2 = a_3 = 0$  and  $a_1 \neq 0$ , then clearly  $\Delta'$  admits a special

degeneration to  $\{y_1^2 y_1'^6 = 0\} = \Delta_{4,\infty}$  under  $\sigma^{-1}$ , which contradicts K-polystability. If  $a_2 = a_3 = a_1 = 0$ , then  $\Delta' = \{y_2 y_1'^6 = 0\}$  is toric. Then we have

$$\beta_{(S,c\Delta')}(B) = 1 - \frac{3}{16} - (1 - \frac{4}{16})\frac{7}{6} = -\frac{1}{16} < 0.$$

This implies that  $(S, c\Delta')$  is K-unstable, which contradicts our assumption that  $(S, c\Delta') \cong (S, c\Delta)$  is K-polystable.

If  $a_0 = a'_0 = 0$ , then  $a_1 \neq 0$ . Hence  $\Delta$  admits a special degeneration to  $\{y_1^2 y_1'^6 = 0\} = \Delta_{4,\infty}$  under the  $\mathbb{G}_m$ -action  $\sigma$ . By similar argument as in case (a), we conclude that  $(S, \Delta) \cong (S, \Delta_{4,\infty})$  and  $c = c_4$ .

(d) If no monomial in  $f$  is divisible by  $y_1'^6$ , then we will show that  $(S, c\Delta)$  is K-unstable for every  $c \in (0, \frac{1}{16}]$ . Recall that the affine chart  $U' = \{y_1' = 1\}$  is isomorphic to  $\{y_1^3 = y_2 y_2'\} \subset \mathbb{A}^3$ . Let  $v_2$  be the monomial valuation on  $U$  of weight  $(2, 3, 3)$  in  $(y_1, y_2, y_2')$  centred at  $[0, 1, 0, 0]$ . Then  $v_2$  is a quasi-monomial combination of  $\text{ord}_B$  and  $\text{ord}_{B'}$  of weight  $(1, 1)$ . Thus we have

$$\beta_S(v_2) = \beta_S(B) + \beta_S(B') = 2\beta_S(B) = -\frac{1}{3}.$$

Moreover,  $A_S(v_2) = A_S(B) + A_S(B') = 2$  and hence  $S_S(v_2) = A_S(v_2) - \beta_S(v_2) = \frac{7}{3}$ . Since no monomial in  $f$  is divisible by  $y_1'^6$ , we have

$$v_2(\Delta) \geq \min\{v_2(y_1^3), v_2(y_1 y_2), v_2(y_1 y_2'), v_2(y_2^2), v_2(y_2'^2)\} = 5.$$

Thus

$$\beta_{(S,c\Delta)}(v_2) = 2 - cv_2(\Delta) - (1 - 4c)\frac{7}{3} \leq 2 - 5c - (1 - 4c)\frac{7}{3} = \frac{13c - 1}{3} < 0,$$

as  $c \leq \frac{1}{16} < \frac{1}{13}$ . This implies that  $(S, c\Delta)$  is K-unstable for every  $c \in (0, \frac{1}{16}]$ .  $\square$

*Proof of Theorem 5.2.* This follows directly from Propositions 5.5 and 5.6.  $\square$

5.2. **N $\circ$ 33.** Here  $S = \{x_0 x_1 - x_2 x_3 = x_1 x_2 + x_2 x_4 + x_3 x_4 = 0\} \subset \mathbb{P}^4$ .

**Lemma 5.7.** *Let  $S$  be the surface from N $\circ$ 33. Then  $S$  is isomorphic to the ordinary blow-up of  $\mathbb{P}(1, 2, 3)$  at two smooth points  $p_1 = [1, 0, 0]$  and  $p_2 = [1, 1, 0]$ .*

*Proof.* It follows from [18, Appendix A] or from [24] that the surface  $S$  is a blow up of a del Pezzo surface of degree 6 with the same singularities as  $S$  at two smooth points. It is well-known that such a sextic del Pezzo surface is isomorphic to  $\mathbb{P}(1, 2, 3)$  and its automorphism group is the Borel subgroup of the group  $\text{PGL}_3(\mathbb{C})$ , see [18, Big Table]. Since the surface  $S$  is also unique [18, Big Table], we see that  $S$  can be obtained from  $\mathbb{P}(1, 2, 3)$  by blowing up the points  $[1 : 0 : 0]$  and  $[1 : 1 : 0]$  — up to the action of  $\text{Aut}(\mathbb{P}(1, 2, 3))$  this choice of blow up points is unique.  $\square$

We denote by  $\pi : S \rightarrow \mathbb{P}(1, 2, 3)$  the blow-up map. Let  $\sigma$  be the  $\mathbb{G}_m$ -action on  $S$  that is a lifting of the following  $\mathbb{G}_m$ -action on  $\mathbb{P}(1, 2, 3)_{[x,y,z]}$ :

$$t \cdot [x, y, z] = [tx, t^2 y, z].$$

Let  $E_1$  and  $E_2$  be  $\pi$ -exceptional curves over  $p_1$  and  $p_2$ , respectively. Let  $C_1 := \pi_*^{-1}(y = 0)$  and  $C_2 := \pi_*^{-1}(y = x^2)$ . Let  $F := \pi^*(x = 0)$  and  $B := \pi_*^{-1}(z = 0)$ . Let  $Q_a := \pi^*(y = ax^2)$  for  $a \in \mathbb{C} \setminus \{0, 1\}$ . By analysing the  $\sigma$ -action on  $S$ , it is clear that  $B$  is the only horizontal divisor on  $S$ , while  $C_1, C_2, E_1, E_2, F$ , and  $Q_a$  give all vertical divisors on  $S$ .

The intersection numbers of these curves are summarized in Table 3.

TABLE 3. Intersection numbers of the surface N $\circ$ 33

$\bullet$	$C_1$	$C_2$	$E_1$	$E_2$	$B$
$C_1$	$-\frac{1}{3}$	$\frac{2}{3}$	1	0	0
$C_2$	$\frac{2}{3}$	$-\frac{1}{3}$	0	1	0
$E_1$	1	0	-1	0	1
$E_2$	0	1	0	-1	1
$B$	0	0	1	1	$-\frac{1}{2}$

**Theorem 5.8.** *There is precisely one wall in  $(0, \frac{1}{16}]$  involving  $S$  from Ne33 such that  $(S, c\Delta)$  is a new K-polystable pair:  $c = \frac{1}{16}$  and  $\Delta = 6B + 2E_1 + 2E_2 + \pi^*\Gamma$  where  $\Gamma = \{y^3 + b_1x^2y^2 + b_2x^4y + b_3x^6 = 0\}$  for  $(b_1, b_2, b_3) \in \mathbb{C}^3$  in the projective coordinates  $[x, y, z]$  of  $\mathbb{P}(1, 2, 3)$ .*

**Lemma 5.9.** *The Mori cone  $\overline{NE}(S)$  is generated by  $[C_1], [C_2], [E_1], [E_2]$ , and  $[B]$ .*

*Proof.* From the  $\mathbb{G}_m$ -action  $\sigma$  on  $S$ , we see that the list of  $\sigma$ -invariant divisors are:  $C_1, C_2, E_1, E_2, F, Q_a$ , and  $B$ . Thus  $\overline{NE}(S)$  is generated by these curves. From the intersection table above we see that  $C_1, C_2, E_1, E_2$ , and  $B$  are all extremal. Besides,  $F \sim \pi^*\mathcal{O}(1)$  and  $Q_a \sim \pi^*\mathcal{O}(2)$  are both nef and big. Thus the proof is finished.  $\square$

**Proposition 5.10.** *Under the above notation, we have*

$$S_S(C_1) = S_S(C_2) = \frac{7}{8}, \quad S_S(E_1) = S_S(E_2) = \frac{7}{24}, \quad S_S(B) = \frac{5}{6}, \quad S_S(F) = \frac{7}{6}, \quad S_S(Q_a) = \frac{7}{12}.$$

*Proof.* First of all, consider the involution  $\tau : S \rightarrow S$  induced by  $[x, y, z] \mapsto [x, x^2 - y, z]$  on  $\mathbb{P}(1, 2, 3)$ . It is clear that  $\tau(C_1) = C_2$  and  $\tau(E_1) = E_2$ . Thus we have equalities  $S_S(C_1) = S_S(C_2)$  and  $S_S(E_1) = S_S(E_2)$ . Besides, since  $Q_a \sim 2F$ , we have  $S_S(Q_a) = \frac{1}{2}S_S(F)$ .

Next, we compute  $S_S(C_1)$ . It is clear that  $-K_S = \pi^*\mathcal{O}(6) - E_1 - E_2$ . It follows from Table 3 and Lemma 5.9 that  $-K_S - tC_1$  is nef if  $0 \leq t \leq 1$ , and not big if  $t \geq 2$ . For  $1 \leq t \leq 2$ , the Zariski decomposition  $-K_S - tC_1 = P(t) + N(t)$  gives the nef part

$$P(t) = \begin{cases} \pi^*\mathcal{O}(6 - 2t) - E_2 & \text{if } 1 \leq t \leq \frac{3}{2}; \\ (4 - 2t)(\pi^*\mathcal{O}(3) - E_2) & \text{if } \frac{3}{2} \leq t \leq 2. \end{cases}$$

Thus computation shows

$$\text{vol}(-K_S - tC_1) = \begin{cases} 4 - 2t - \frac{t^2}{3} & \text{if } 0 \leq t \leq 1; \\ 5 - 4t + \frac{2t^2}{3} & \text{if } 1 \leq t \leq \frac{3}{2}; \\ 2(2 - t)^2 & \text{if } \frac{3}{2} \leq t \leq 2. \end{cases}$$

Thus

$$S_S(C_1) = \frac{1}{4} \left( \int_0^1 (4 - 2t - \frac{t^2}{3}) dt + \int_1^{\frac{3}{2}} (5 - 4t + \frac{2t^2}{3}) dt + \int_{\frac{3}{2}}^2 2(2 - t)^2 dt \right) = \frac{7}{8}.$$

Next, we compute  $S_S(E_1)$ . It follows from Table 3 and Lemma 5.9 that  $-K_S - tE_1$  is nef if  $0 \leq t \leq 1$ , and not big if  $t \geq \frac{3}{2}$ . For  $1 \leq t \leq \frac{3}{2}$ , the Zariski decomposition  $-K_S - tE_1 = P(t) + N(t)$  gives the nef part

$$P(t) = (3 - 2t)(\pi^*\mathcal{O}(6) - 2E_1 - E_2).$$

Thus computation shows

$$\text{vol}(-K_S - tE_1) = \begin{cases} 4 - 2t - t^2 & \text{if } 0 \leq t \leq 1; \\ (3 - 2t)^2 & \text{if } 1 \leq t \leq \frac{3}{2}. \end{cases}$$

Thus

$$S_S(E_1) = \frac{1}{4} \left( \int_0^1 (4 - 2t - t^2) dt + \int_1^{\frac{3}{2}} (3 - 2t)^2 dt \right) = \frac{17}{24}.$$

Next, we compute  $S_S(B)$ . It follows from Table 3 and Lemma 5.9 that  $-K_S - tB$  is nef if  $0 \leq t \leq 1$ , and not big if  $t \geq 2$ . For  $1 \leq t \leq 2$ , the Zariski decomposition  $-K_S - tB = P(t) + N(t)$  gives the nef part

$$P(t) = \pi^*\mathcal{O}(6 - 3t).$$

Thus computation shows

$$\text{vol}(-K_S - tB) = \begin{cases} 4 - 2t - \frac{t^2}{2} & \text{if } 0 \leq t \leq 1; \\ \frac{3}{2}(2 - t)^2 & \text{if } 1 \leq t \leq 2. \end{cases}$$

Thus

$$S_S(B) = \frac{1}{4} \left( \int_0^1 (4 - 2t - \frac{t^2}{2}) dt + \int_1^2 \frac{3}{2}(2 - t)^2 dt \right) = \frac{5}{6}.$$

Finally, we compute  $S_S(F)$ . It follows from Table 3 and Lemma 5.9 that  $-K_S - tF$  is nef if  $0 \leq t \leq 2$ , and not big if  $t \geq 3$ . For  $2 \leq t \leq 3$ , the Zariski decomposition  $-K_S - tF = P(t) + N(t)$  gives the nef part

$$P(t) = (3 - t)(\pi^*\mathcal{O}(4) - E_1 - E_2).$$

Thus computation shows

$$\text{vol}(-K_S - tF) = \begin{cases} 4 - 2t + \frac{t^2}{6} & \text{if } 0 \leq t \leq 2; \\ \frac{2}{3}(3-t)^2 & \text{if } 2 \leq t \leq 3. \end{cases}$$

Thus

$$S_S(F) = \frac{1}{4} \left( \int_0^2 (4 - 2t + \frac{t^2}{6}) dt + \int_2^3 \frac{2}{3}(3-t)^2 dt \right) = \frac{7}{6}.$$

□

**Proposition 5.11.** *Let  $S$  be the surface  $\text{No}33$  from Table 1. Suppose  $\Delta \in |-4K_S|$  and  $c \in (0, \frac{1}{16}]$  satisfy that  $(S, c\Delta)$  is  $K$ -polystable but not  $K$ -stable. Then  $c = \frac{1}{16}$  and  $\text{ord}_B(\Delta) = 6$ .*

*Proof.* Assume that  $(S, c\Delta)$  is  $K$ -polystable but not  $K$ -stable. Since  $S$  admits a  $\mathbb{G}_m$ -action  $\sigma$  but is not toric, we may assume that  $\Delta$  is  $\mathbb{G}_m$ -invariant. It is clear that  $-K_S - 2B \sim E_i + E_j$  is at the boundary of  $\overline{NE}(S)$ . Thus  $\text{ord}_B(\Delta) \leq 8$ .

First, we show that  $\text{ord}_B(\Delta) \neq 8$ . Suppose not, then  $-4K_S - 8B \sim 4(E_1 + E_2)$  is exceptional over  $\mathbb{P}(1, 2, 3)$ , which implies that  $\Delta = 8B + 4E_1 + 4E_2$ . By  $K$ -polystability and Proposition 5.10, we have

$$0 = \beta_{(S, c\Delta)}(B) = 1 - 8c - (1 - 4c)\frac{5}{6} = 0,$$

which is equivalent to  $c = \frac{1}{28}$ . On the other hand,

$$0 < \beta_{(S, c\Delta)}(F) = 1 - (1 - 4c)\frac{7}{6} = 0,$$

which is a contradiction.

Next, we show that  $\text{ord}_B(\Delta) \neq 7$ . Suppose not, then  $((-4K_S - 7B) \cdot E_i) = -3$  which implies that  $3(E_1 + E_2)$  is in the base component of  $|-4K_S - 7B|$ . Thus we may write  $\Delta = 7B + 3E_1 + 3E_2 + \pi^*\Gamma_1$ , where  $\pi^*\Gamma_1 \sim -4K_S - 7B - 3E_1 - 3E_2 \sim \pi^*\mathcal{O}(3)$ . Since  $\Gamma_1$  is  $\mathbb{G}_m$ -invariant and does not contain  $\pi_*B = (z = 0)$ , it must contain  $\pi_*F = (x = 0)$  which implies that  $\text{ord}_F(\pi^*\Gamma_1) \geq 1$ . By  $K$ -polystability and Proposition 5.10, we have

$$0 = \beta_{(S, c\Delta)}(B) = 1 - 7c - (1 - 4c)\frac{5}{6} = 0,$$

which is equivalent to  $c = \frac{1}{22}$ . On the other hand,

$$0 < \beta_{(S, c\Delta)}(F) \leq 1 - c - (1 - 4c)\frac{7}{6} = 0,$$

which is a contradiction.

Next, we show that if  $\text{ord}_B(\Delta) = 6$ , then  $c = \frac{1}{16}$ . By  $K$ -polystability and Proposition 5.10, we have

$$0 = \beta_{(S, c\Delta)}(B) = 1 - 6c - (1 - 4c)\frac{5}{6} = 0,$$

which is equivalent to  $c = \frac{1}{16}$ .

Finally, we show that  $\text{ord}_B(\Delta) > 5$ . Suppose not, then we have

$$0 = \beta_{(S, c\Delta)}(B) \geq 1 - 5c - (1 - 4c)\frac{5}{6},$$

which implies that  $c \geq \frac{1}{10}$ , a contradiction. □

*Proof of Theorem 5.8.* Assume that  $(S, c\Delta)$  is  $K$ -polystable but not  $K$ -stable where  $S$  is the surface  $\text{No}33$  from Table 1,  $\Delta \in |-4K_S|$  and  $c \in (0, \frac{1}{16}]$ . By Proposition 5.11, we must have  $c = \frac{1}{16}$  and  $\text{ord}_B(\Delta) = 6$ . Since  $((-4K_S - 6B) \cdot E_i) = -2$ , we know that  $2(E_1 + E_2)$  is in the base component of  $|-4K_S - 6B|$ . Thus we may write  $\Delta = 6B + 2E_1 + 2E_2 + \pi^*\Gamma$ , where  $\pi^*\Gamma \sim -4K_S - 6B - 2E_1 - 2E_2 \sim \pi^*\mathcal{O}(6)$  is  $\sigma$ -invariant and does not contain  $B$  in its support.

Next, we classify all possible  $\Gamma$  such that  $(S, \frac{1}{16}\Delta)$  is  $K$ -polystable and not  $K$ -stable. Since this pair is invariant under the  $\mathbb{G}_m$ -action  $\sigma$  but not toric, by [44, Theorem 3.2] (cf. [7, Theorem 1.3.9] and [34]) the pair being  $K$ -polystable is equivalent to  $\beta_{(S, c\Delta)}(D) > 0$  for every vertical divisor  $D$  on  $S$ . Here we use the fact that  $\beta_{(S, c\Delta)}(B) = 0$  by the proof of Proposition 5.11 and  $B$  is horizontal. A simple analysis of the  $\sigma$ -action on  $S$  shows that  $D$  is one of the following:

- (i)  $F$ ;
- (ii)  $E_1$  or  $E_2$ ;
- (iii)  $C_1$  or  $C_2$ ;

(iv)  $Q_a$  for  $a \in \mathbb{C} \setminus \{0, 1\}$ .

Next, we split into these four cases.

(i) By computation,

$$\beta_{(S,c\Delta)}(F) = 1 - \frac{1}{16} \text{ord}_F(\pi^*\Gamma) - \left(1 - \frac{4}{16}\right) \frac{7}{6}.$$

Thus  $\beta_{(S,c\Delta)}(F) > 0$  is equivalent to  $\text{ord}_F(\pi^*\Gamma) < 2$ . Therefore, we have  $\text{ord}_F(\pi^*\Gamma) = 0$  as it is always even.

(ii) By computation,

$$\beta_{(S,c\Delta)}(E_i) = 1 - \frac{1}{16}(2 + \text{ord}_{E_i}(\pi^*\Gamma)) - \left(1 - \frac{4}{16}\right) \frac{7}{24}.$$

Thus  $\beta_{(S,c\Delta)}(E_i) > 0$  is equivalent to  $\text{ord}_{E_i}(\pi^*\Gamma) < \frac{21}{2}$ . This is always true as  $\text{ord}_{E_i}(\pi^*\Gamma) \leq 3$ .

(iii) By computation,

$$\beta_{(S,c\Delta)}(C_i) = 1 - \frac{1}{16} \text{ord}_{C_i}(\pi^*\Gamma) - \left(1 - \frac{4}{16}\right) \frac{7}{8}.$$

Thus  $\beta_{(S,c\Delta)}(C_i) > 0$  is equivalent to  $\text{ord}_{C_i}(\pi^*\Gamma) < \frac{11}{2}$ . This is always true as  $\text{ord}_{C_i}(\pi^*\Gamma) \leq 3$ .

(iv) By computation,

$$\beta_{(S,c\Delta)}(Q_a) = 1 - \frac{1}{16} \text{ord}_{Q_a}(\pi^*\Gamma) - \left(1 - \frac{4}{16}\right) \frac{7}{12}.$$

Thus  $\beta_{(S,c\Delta)}(Q_a) > 0$  is equivalent to  $\text{ord}_{Q_a}(\pi^*\Gamma) < \frac{11}{2}$ . This is always true as  $\text{ord}_{Q_a}(\pi^*\Gamma) \leq 3$ .

To summarise,  $(S, \frac{1}{16}\Delta)$  is K-polystable and not K-stable if and only if  $\Gamma$  is  $\mathbb{G}_m$ -invariant and does not contain  $\pi_*B = \{z = 0\}$  or  $\pi_*F = \{x = 0\}$  in its support. This is the same as saying that  $\Gamma = \{y^3 + b_1x^2y^2 + b_2x^4y + b_3x^6 = 0\}$  for  $(b_1, b_2, b_3) \in \mathbb{C}^3$ .  $\square$

*Proof of Theorem 5.1.* Let  $(S, c\Delta)$  be a new K-polystable pair on a wall  $c \in (0, \frac{1}{16}]$ . In particular,  $(S, c\Delta)$  is K-polystable with an effective  $\mathbb{G}_m$ -action, and  $S$  is K-unstable. By Proposition 4.3, we know that  $S$  has at worst  $A_3$  singularities.

If  $S$  has an  $A_3$ -singularity, the classification of walls and new K-polystable pairs is completed in Proposition 4.3, where we must have  $c = \frac{1}{16}$  and  $(S, \Delta) \cong \{(y_3^2 = y_2y_4) \subset \mathbb{P}(1, 2, 3, 4), \{y_4^4 = 0\}\}$ . In particular,  $S$  is isomorphic to the surface No25.

If  $S$  has no  $A_3$ -singularities, then Proposition 4.3 implies that  $S$  is a K-unstable del Pezzo surface of degree 4 with at worst  $A_2$ -singularities admitting an effective  $\mathbb{G}_m$ -action. Thus by Table 1 we know that  $S$  is isomorphic to either No29 or No33. The classifications of walls and new K-polystable pairs for these two surfaces are completed in Theorems 5.2 and 5.8.

It remains to prove the statement on the initial wall  $c_0 = 0$ . Let  $\mathcal{M}_\epsilon$  and  $\mathcal{M}_0$  be the corresponding K-moduli stacks whose good moduli spaces are  $\mathfrak{M}_\epsilon$  and  $\mathfrak{M}_0$  respectively. Since K-semistability for coefficients is a closed condition, we know that every  $[(S, \Delta)] \in \mathcal{M}_\epsilon$  satisfies that  $S$  is K-semistable. In particular, the forgetful map  $[(S, \Delta)] \mapsto [S]$  gives a morphism between Artin stacks  $\mathcal{M}_\epsilon \rightarrow \mathcal{M}_0$ . Since good moduli spaces are initial among maps to algebraic spaces [5, Theorem 6.6], the forgetful map induces a morphism  $\phi_0 : \mathfrak{M}_\epsilon \rightarrow \mathfrak{M}_0$  between K-moduli spaces.

Finally we analyse the map  $\phi_0$ . Let  $[S] \in \mathfrak{M}_0$  be a general point, then we may assume that  $S$  is a general K-stable smooth del Pezzo surface of degree 4. Moreover, we may assume that  $\text{Aut}(S) \cong \mu_2^4$  by [27, Section 6.4]. By [61], we know that there exists  $\epsilon_0 > 0$  such that  $(S, \epsilon\Delta)$  is K-stable for every  $\Delta \in |-4K_S|$  and every  $\epsilon \in (0, \epsilon_0)$ . This implies that the finite quotient  $\mathbb{P}(H^0(S, -4K_S))/\text{Aut}(S)$  admits a closed immersion into the fibre  $\phi_0^{-1}([S])$ . On the other hand, if  $[(S', \Delta')] \in \phi_0^{-1}([S])$ , then  $S'$  and  $S$  are S-equivalent K-semistable Fano varieties. Since  $S$  is K-stable, by [15, 40] we know that  $S' \cong S$  which implies that  $[(S', \Delta')]$  is contained in the image of  $\mathbb{P}(H^0(S, -4K_S))/\text{Aut}(S)$ . Thus we have  $\mathbb{P}(H^0(S, -4K_S))/\text{Aut}(S) \cong \phi_0^{-1}([S])$ . By a Riemann-Roch computation, we have  $h^0(S, -4K_S) = 41$  which implies that  $\mathbb{P}(H^0(S, -4K_S))/\text{Aut}(S) \cong \mathbb{P}^{40}/\mu_2^4$ . Since both  $\mathfrak{M}_\epsilon$  and  $\mathfrak{M}_0$  are normal by [9, Theorem 2.21], the statement follows by Zariski's main theorem.  $\square$

**Remark 5.12.** In §2, we found three K-polystable toric Fano 3-folds of the form  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$  defined by  $y_0y_1 = f(x_0, \dots, x_4)$ ,  $g(x_0, \dots, x_4) = 0$  and  $h(x_0, \dots, x_4) = 0$ , where  $f$ ,  $g$ , and  $h$  are of the following:

- (1)  $f = x_0^4$ ,  $g = x_2^2 - x_0x_3$ , and  $h = x_3^2 - x_1x_4$ ;
- (2)  $f = x_0^3x_3$ ,  $g = x_2^2 - x_0x_3$ , and  $h = x_2x_3 - x_1x_4$ ;
- (3)  $f = x_0^2x_4^2$ ,  $g = x_2^2 - x_0x_4$ , and  $h = x_1x_3 - x_0x_4$ .

By Proposition 4.2 and Corollary 4.4, each of the above toric Fano 3-folds corresponds to a K-polystable pair  $(S, \frac{1}{16}\Delta)$  in  $\mathfrak{M}_{\frac{1}{16}}$  where  $S = \{g = h = 0\} \subset \mathbb{P}^4$  and  $\Delta = \{f = 0\}$ . It turns out that these three surface pairs are precisely the toric ones in  $\mathfrak{M}_{\frac{1}{16}}$ :

- (1) corresponds to the surface No25 and  $\Delta = \{y_4^4 = 0\}$  via  $[x_0, x_1, x_2, x_3, x_4] = [y_4, y_2^2, y_1 y_3, y_1^2 y_2, y_1^4]$ ;
- (2) corresponds to the surface No29 and  $\Delta = \Delta_{4,\infty}$  via  $[x_0, x_1, x_2, x_3, x_4] = [y_1^2, y_2, y_1 y_1', y_1^2, y_2']$ ;
- (3) corresponds to the surface No30 and  $\Delta = \{y_1^4 y_1'^4 = 0\}$  via  $[x_0, x_1, x_2, x_3, x_4] = [y_1^2, y_2, y_1 y_1', y_2', y_1'^2]$ .

Here K-polystability of  $(S, \frac{1}{2}\Delta)$  from (1) or (2) follows from Theorem 5.1, while (3) follows by interpolation [8, Proposition 2.13] as No30 is K-polystable, and  $\frac{1}{4}\Delta$  is the reduced toric boundary.

**5.3. Further discussions.** Let  $M$  be the K-moduli component of quartic 3-folds. While our results provide a decisive statement about the global nature of objects in  $M$ , several questions remain open. For instance, it would be interesting to determine all the pairs parametrised by  $\mathfrak{M}_{\frac{1}{16}}$  via wall crossings. Although Theorem 5.1 gives a complete description of walls and new K-polystable pairs, one would need to describe the exceptional loci for K-moduli wall crossings, which is similar to the VGIT analysis on basins and attractions. We leave this for future studies.

It would also be handy to have an explicit description of GIT-stable quartic 3-folds. Indeed, we think that the locus in  $M$  parametrising pure  $(2, 2, 4)$ -complete intersections  $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$  should correspond to the locus of the union of two quadric hypersurfaces  $\{gh = 0\} \subset \mathbb{P}^4$  in the GIT moduli space of quartic 3-folds via wall crossing. On the one hand,  $\{gh = 0\} \subset \mathbb{P}^4$  degenerates to  $\{y_0 y_1 = 0, g = h = 0\} \subset \mathbb{P}(1^5, 2^2)$  via iterated degeneration to normal cone. On the other hand,  $X_{2,2,4} = \{y_0 y_1 = f, g = h = 0\} \subset \mathbb{P}(1^5, 2^2)$  also degenerates to  $\{y_0 y_1 = 0, g = h = 0\} \subset \mathbb{P}(1^5, 2^2)$  via a suitable one parameter subgroup. Note that  $\dim M = 45$ , and the dimension of the locus parametrising pure  $(2, 2, 4)$ -complete intersections in  $\mathbb{P}(1^5, 2^2)$  is equal to  $\dim \mathfrak{M}_{\frac{1}{16}}$  which is 42.

In a different direction, the following example from [10, Remark 6.13] gives another closed locus in  $M$ .

**Example 5.13.** In [10], the K-moduli compactification  $\overline{\mathfrak{M}}_{\frac{3}{4}}^K$  of  $(\mathbb{P}^3, \frac{3}{4}D)$  where  $S$  is a quartic surface was studied. If  $(X, \frac{3}{4}D)$  is a K-polystable 3-fold pair in  $\overline{\mathfrak{M}}_{\frac{3}{4}}^K$ , then we can take the quadruple cyclic cover  $Y \rightarrow X$  branched along  $D$ . By [48, 62] we know that  $Y$  is K-polystable. Since a quadruple cyclic cover of  $\mathbb{P}^3$  branched along a quartic surface  $\{f(x_0, x_1, x_2, x_3) = 0\}$  is precisely the quartic 3-fold  $\{x_4^4 = f(x_0, x_1, x_2, x_3)\} \subset \mathbb{P}^4$ , we know that  $Y$  is a K-polystable limit of a family of smooth quartic 3-folds. This cyclic cover construction can be made fibrewise (see [10, §6.3] for the case of quartic double solids), which yields a finite morphism  $\iota : \overline{\mathfrak{M}}_{\frac{3}{4}}^K \rightarrow M$ . It is clear that the image of  $\iota$  has dimension 19 and is a K-moduli compactification of smooth quartic 3-folds of the form  $\{x_4^4 = f(x_0, x_1, x_2, x_3)\} \subset \mathbb{P}^4$ . By [10, Theorem 5.16], we know that there are four of the Fano 3-fold  $X$  appearing in  $\overline{\mathfrak{M}}_{\frac{3}{4}}^K$ :  $\mathbb{P}^3, X_h, \mathbb{P}(1, 1, 2, 4)$ , or  $X_u$ . Here  $X_h = \{x_0 x_1 = x_2 x_3\} \subset \mathbb{P}(1^4, 2)$  which is isomorphic to the projective cone over  $\mathbb{P}^1 \times \mathbb{P}^1$  with polarisation  $\mathcal{O}(2, 2)$ ;  $X_u$  is a 2-step birational modification of a  $\mathbb{P}(1, 2, 3)$ -bundle over  $\mathbb{P}^1$  constructed in [10, §4.2]. One can check that the quadruple cyclic cover of  $X_h$  is a  $(2, 4)$ -complete intersection of  $\mathbb{P}(1^5, 2)$ , while the quadruple cyclic cover of  $\mathbb{P}(1, 1, 2, 4)$  is a weighted hypersurface of degree 8 in  $\mathbb{P}(1, 1, 2, 2, 4)$ . However, we expect that the quadruple cyclic cover of  $X_u$  will not be a weighted complete intersection. Such quadruple cyclic covers of  $X_u$  form an 18-dimensional closed locus in the K-moduli space  $M$ .

**Acknowledgements.** We thank Thomas Hall and Anne-Sophie Kaloghiros for fruitful conversations.

This project was in-part carried out whilst HA visited the Institut de Mathématiques de Toulouse, supported by LabEx CIMI; IC visited the Institut des Hautes Études Scientifiques; YL visited the University of Edinburgh; and AP visited Osaka University and Kumamoto University. The authors would like to thank the members of these institutions for creating stimulating mathematical environments and excellent scientific activities. HA is supported by EPSRC Grant EP/V048619. IC is supported by EPSRC Grant EP/V054597. AK is supported by EPSRC Fellowship EP/N022513/1. YL is partially supported by NSF grant DMS-2148266 (formerly DMS-2001317) and an Alfred P. Sloan research fellowship. AP is supported by INdAM GNSAGA “Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni” and PRIN2020 2020KKWT53 “Curves, Ricci flat Varieties and their Interactions”.

## REFERENCES

1. Hamid Abban and Ziquan Zhuang, *K-stability of Fano varieties via admissible flags*, Forum Math. Pi **10** (2022), Paper No. e15.
2. ———, *Seshadri constants and K-stability of Fano manifolds*, Duke Math. J. **172** (2023), no. 6, 1109–1144.
3. Mohammad Akhtar, Tom Coates, Alessio Corti, Liana Heuberger, Alexander M. Kasprzyk, Alessandro Oneto, Andrea Petracci, Thomas Prince, and Ketil Tveiten, *Mirror symmetry and the classification of orbifold del Pezzo surfaces*, Proc. Amer. Math. Soc. **144** (2016), no. 2, 513–527.



4. Mohammad Akhtar, Tom Coates, Sergey Galkin, and Alexander M. Kasprzyk, *Minkowski polynomials and mutations*, SIGMA Symmetry Integrability Geom. Methods Appl. **8** (2012), Paper 094, 17.
5. Jarod Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402.
6. Jarod Alper, Harold Blum, Daniel Halpern-Leistner, and Chenyang Xu, *Reductivity of the automorphism group of  $K$ -polystable Fano varieties*, Invent. Math. **222** (2020), no. 3, 995–1032.
7. Carolina Araujo, Ana-Maria Castravet, Ivan Cheltsov, Kento Fujita, Anne-Sophie Kaloghiros, Jesus Martinez-Garcia, Constantin Shramov, Hendrik Süß, and Nivedita Viswanathan, *The Calabi problem for Fano threefolds*, 2021, Book draft.
8. Kenneth Ascher, Kristin DeVleming, and Yuchen Liu, *Wall crossing for  $K$ -moduli spaces of plane curves*, [arXiv:1909.04576 \[math.AG\]](https://arxiv.org/abs/1909.04576), 2019.
9. Kenneth Ascher, Kristin DeVleming, and Yuchen Liu,  *$K$ -moduli of curves on a quadric surface and  $K3$  surfaces*, J. Inst. Math. Jussieu **22** (2023), no. 3, 1251–1291.
10. Kenneth Ascher, Kristin DeVleming, and Yuchen Liu,  *$K$ -stability and birational models of moduli of quartic  $K3$  surfaces*, Invent. Math. **232** (2023), no. 2, 471–552.
11. Robert J. Berman,  *$K$ -polystability of  $\mathbb{Q}$ -Fano varieties admitting Kähler-Einstein metrics*, Invent. Math. **203** (2016), no. 3, 973–1025.
12. Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.
13. Harold Blum, Daniel Halpern-Leistner, Yuchen Liu, and Chenyang Xu, *On properness of  $K$ -moduli spaces and optimal degenerations of Fano varieties*, Selecta Math. (N.S.) **27** (2021), no. 4, Paper No. 73, 39.
14. Harold Blum, Yuchen Liu, and Chenyang Xu, *Openness of  $K$ -semistability for Fano varieties*, Duke Math. J. **171** (2022), no. 13, 2753–2797.
15. Harold Blum and Chenyang Xu, *Uniqueness of  $K$ -polystable degenerations of Fano varieties*, Ann. of Math. (2) **190** (2019), no. 2, 609–656.
16. Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3–4, 235–265, Computational algebra and number theory (London, 1993).
17. Ivan Cheltsov, *Log canonical thresholds on hypersurfaces*, Mat. Sb. **192** (2001), no. 8, 155–172.
18. Ivan Cheltsov and Yuri Prokhorov, *Del Pezzo surfaces with infinite automorphism groups*, Algebr. Geom. **8** (2021), no. 3, 319–357.
19. Ivan Cheltsov and Constantin Shramov, *On exceptional quotient singularities*, Geom. Topol. **15** (2011), no. 4, 1843–1882.
20. Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev, and Alexander M. Kasprzyk, *Mirror symmetry and Fano manifolds*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2013, pp. 285–300.
21. Tom Coates, Alessio Corti, Sergey Galkin, and Alexander M. Kasprzyk, *Quantum periods for 3-dimensional Fano manifolds*, Geom. Topol. **20** (2016), no. 1, 103–256.
22. Tom Coates, Liana Heuberger, and Alexander M. Kasprzyk, *Mirror symmetry, Laurent inversion and the classification of  $\mathbb{Q}$ -Fano threefolds*, [arXiv:2210.07328 \[math.AG\]](https://arxiv.org/abs/2210.07328), 2022.
23. Tom Coates, Alexander M. Kasprzyk, Giuseppe Pitton, and Ketil Tveiten, *Maximally mutable Laurent polynomials*, Proc. Royal Society A **477** (2021), no. 2254, Paper No. 20210584, 21.
24. D. F. Coray and M. A. Tsfasman, *Arithmetic on singular Del Pezzo surfaces*, Proc. London Math. Soc. (3) **57** (1988), no. 1, 25–87.
25. David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.
26. Ruadhaí Dervan, *On  $K$ -stability of finite covers*, Bull. Lond. Math. Soc. **48** (2016), no. 4, 717–728.
27. Igor V. Dolgachev and Vasily A. Iskovskikh, *Finite subgroups of the plane Cremona group*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, Progr. Math., vol. 269, Birkhäuser Boston, Boston, MA, 2009, pp. 443–548.
28. Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.
29. Kento Fujita,  *$K$ -stability of Fano manifolds with not small alpha invariants*, J. Inst. Math. Jussieu **18** (2019), no. 3, 519–530.
30. ———, *A valuative criterion for uniform  $K$ -stability of  $\mathbb{Q}$ -Fano varieties*, J. Reine Angew. Math. **751** (2019), 309–338.
31. Kento Fujita and Yuji Odaka, *On the  $K$ -stability of Fano varieties and anticanonical divisors*, Tohoku Math. J. (2) **70** (2018), no. 4, 511–521.
32. Thomas Hall, *The behaviour of Kähler-Einstein polygons under mutation*, In preparation.
33. DongSeon Hwang and Yeonsu Kim, *On Kähler-Einstein Fano polygons*, [arXiv:2012.13373 \[math.AG\]](https://arxiv.org/abs/2012.13373), 2000.
34. Nathan Ilten and Hendrik Süß,  *$K$ -stability for Fano manifolds with torus action of complexity 1*, Duke Math. J. **166** (2017), no. 1, 177–204.
35. Nathan Owen Ilten, *Mutations of Laurent polynomials and flat families with toric fibers*, SIGMA Symmetry Integrability Geom. Methods Appl. **8** (2012), Paper 047, 7.
36. Chen Jiang, *Boundedness of  $\mathbb{Q}$ -Fano varieties with degrees and alpha-invariants bounded from below*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 5, 1235–1248.
37. Anne-Sophie Kaloghiros and Andrea Petracci, *On toric geometry and  $K$ -stability of Fano varieties*, Trans. Amer. Math. Soc. Ser. B **8** (2021), 548–577.
38. Chi Li,  *$K$ -semistability is equivariant volume minimization*, Duke Math. J. **166** (2017), no. 16, 3147–3218.
39. Chi Li and Yuchen Liu, *Kähler-Einstein metrics and volume minimization*, Adv. Math. **341** (2019), 440–492.
40. Chi Li, Xiaowei Wang, and Chenyang Xu, *Algebraicity of the metric tangent cones and equivariant  $K$ -stability*, J. Amer. Math. Soc. **34** (2021), no. 4, 1175–1214.
41. Chi Li and Chenyang Xu, *Stability of valuations and Kollár components*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 8, 2573–2627.
42. Yuchen Liu, *The volume of singular Kähler-Einstein Fano varieties*, Compos. Math. **154** (2018), no. 6, 1131–1158.
43. ———,  *$K$ -stability of cubic fourfolds*, J. Reine Angew. Math. **786** (2022), 55–77.
44. ———,  *$K$ -stability of Fano threefolds of rank 2 and degree 14 as double covers*, Math. Z. **303** (2023), no. 2, Paper No. 38.
45. Yuchen Liu and Andrea Petracci, *On  $K$ -stability of some del Pezzo surfaces of Fano index 2*, Bull. Lond. Math. Soc. **54** (2022), no. 2, 517–525.
46. Yuchen Liu and Chenyang Xu,  *$K$ -stability of cubic threefolds*, Duke Math. J. **168** (2019), no. 11, 2029–2073.
47. Yuchen Liu, Chenyang Xu, and Ziquan Zhuang, *Finite generation for valuations computing stability thresholds and applications to  $K$ -stability*, Ann. of Math. (2) **196** (2022), no. 2, 507–566.

48. Yuchen Liu and Ziwon Zhu, *Equivariant  $K$ -stability under finite group action*, *Internat. J. Math.* **33** (2022), no. 1, Paper No. 2250007, 21.
49. Toshiki Mabuchi and Shigeru Mukai, *Stability and Einstein-Kähler metric of a quartic del Pezzo surface*, *Einstein metrics and Yang-Mills connections (Sanda, 1990)*, *Lecture Notes in Pure and Appl. Math.*, vol. 145, Dekker, New York, 1993, pp. 133–160.
50. Shigefumi Mori, *On a generalization of complete intersections*, *J. Math. Kyoto Univ.* **15** (1975), no. 3, 619–646.
51. Yuji Odaka, Cristiano Spotti, and Song Sun, *Compact moduli spaces of del Pezzo surfaces and Kähler-Einstein metrics*, *J. Differential Geom.* **102** (2016), no. 1, 127–172.
52. John Christian Ottem and Stefan Schreieder, *On deformations of quintic and septic hypersurfaces*, *J. Math. Pures Appl. (9)* **135** (2020), 140–158.
53. Theodoros Papazachariou,  *$K$ -moduli of log Fano complete intersection pairs*, [arXiv:2212.09332 \[math.AG\]](https://arxiv.org/abs/2212.09332).
54. Andrea Petracci, *An example of mirror symmetry for Fano threefolds*, *Birational geometry and moduli spaces*, Springer INdAM Ser., vol. 39, Springer, Cham, 2020, pp. 173–188.
55. Gang Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with  $C_1(M) > 0$* , *Invent. Math.* **89** (1987), no. 2, 225–246.
56. Chenyang Xu, *A minimizing valuation is quasi-monomial*, *Ann. of Math. (2)* **191** (2020), no. 3, 1003–1030.
57. ———,  *$K$ -stability of Fano varieties: an algebro-geometric approach*, *EMS Surv. Math. Sci.* **8** (2021), no. 1-2, 265–354.
58. Chenyang Xu and Ziquan Zhuang, *On positivity of the CM line bundle on  $K$ -moduli spaces*, *Ann. of Math. (2)* **192** (2020), no. 3, 1005–1068.
59. Kewei Zhang and Chuyu Zhou, *Delta invariants of projective bundles and projective cones of Fano type*, *Math. Z.* **300** (2022), no. 1, 179–207.
60. Junyan Zhao,  *$K$ -moduli of quintic Del Pezzo pairs and moduli of genus six curves*, [arXiv:2212.06992 \[math.AG\]](https://arxiv.org/abs/2212.06992).
61. Chuyu Zhou, *Log  $K$ -stability of GIT-stable divisors on Fano manifolds*, [arXiv:2102.07458 \[math.AG\]](https://arxiv.org/abs/2102.07458).
62. Ziquan Zhuang, *Optimal destabilizing centers and equivariant  $K$ -stability*, *Invent. Math.* **226** (2021), no. 1, 195–223.

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF NOTTINGHAM, NOTTINGHAM, NG7 2RD, UNITED KINGDOM  
Email address: hamid.abban@nottingham.ac.uk

SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, EDINBURGH, EH9 3FD, UNITED KINGDOM  
Email address: i.cheltsov@ed.ac.uk

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF NOTTINGHAM, NOTTINGHAM, NG7 2RD, UNITED KINGDOM  
Email address: a.m.kasprzyk@nottingham.ac.uk

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208, USA  
Email address: yuchenl@northwestern.edu

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI BOLOGNA, PIAZZA DI PORTA SAN DONATO 5, BOLOGNA 40126, ITALY  
Email address: a.petracci@unibo.it